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ADDING ±1 TO THE ARGUMENT OF A HALL–LITTLEWOOD POLYNOMIAL

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Abstract. Shifting by ±1 power sums: $p_i \to p_i \pm 1$ induces a transformation on symmetric functions that we detail in the case of Hall–Littlewood polynomials. By iteration, this gives a description of these polynomials in terms of plane partitions, as well as some generating functions. We recover in particular an identity of Warnaar related to Rogers–Ramanujan identities.

1. Introduction

To free analysis from infinitesimal quantities, and to quieten down the metaphysical anguish of Bishop Berkeley[1], Lagrange [3] proposed to replace differential calculus by the study of the behaviour of a function[2] fx under the addition of an "increment" y to x:

$$fx \to f(x+y) .$$

This works perfectly well, at least under the mild proviso of analyticity in the neighbourhood of x.

One can adopt the same strategy in the realm of symmetric polynomials, adding a letter x to a set of indeterminates (we say *alphabet* A). However, there exists a canonical involution on symmetric functions, and

[1] Il s'est élevé un Docteur ennemi de la Science qui a déclaré la guerre aux Mathématiciens; ce Docteur monte en Chaire pour apprendre aux fidèles que la Géométrie est contraire à la religion; il leur dit d'être en garde contre les Géomètres, ce sont, selon lui, des gens aveugles et indociles qui ne savent ni raisonner, ni croire; des visionnaires qui se refusent aux choses simples et qui donnent tête baissée dans les merveilles.... , préface de Buffon [9]. Other (anonymous) defenses of infinitesimals have been published in England. See, for example, *A defence of free-thinking in Mathematics*, London (1735), and *Geometry no friend to infidelity*, London (1734).

[2] Lagrange, in harmony with D. Knuth, writes fx and $f(x+y)$, forgetting parentheses when the argument is a single letter. We shall follow Lagrange, when no ambiguity is to be feared.

this entails that there exists in fact two versions of the "addition" of x. In terms of λ-rings, one has to study both transformations

$$fA \to f(A+x) \ , \qquad fA \to f(A-x) \, ,$$

which may look rather different when considering explicit polynomials.

Of course, when restricting to homogeneous polynomials, one does not lose any information by specializing x to 1, i.e., by studying instead

$$fA \to f(A+1) \ , \qquad fA \to f(A-1) \, .$$

We consider these two operations in the case of Hall–Littlewood polynomials in Theorem 3.1 and Theorem 4.2.

An interesting outcome is a short proof of an identity of Warnaar (5.2) concerning a generating function of Hall–Littlewood polynomials related to the Rogers–Ramanujan identities.

Recall that the ring of symmetric polynomials $\mathfrak{Sym}(X)$ in an infinite set of indeterminates $X = \{x_1, x_2, \ldots\}$, with coefficients in $\mathbb{Q}[[t]]$, t another indeterminate, admits a linear basis, the *Schur functions* $S_\lambda X$, indexed by all partitions[3] λ.

One can take, for a symmetric function, more general arguments than a set of indeterminates, and we shall need to use, given two sets of indeterminates X,Y, and a symmetric function f, the functions $f(X \pm Y)$, $f(XY)$, $f(X(1-t))$, $f(X \pm 1)$. Since $\mathfrak{Sym}(X)$ is a ring of polynomials in the power sums p_i, the above symmetric functions are induced from the case where f is a power sum, setting

$$p_i(X \pm Y) = p_i X \pm p_i Y \ , \ p_i(XY) = p_i X p_i Y \ ,$$
$$p_i(X(1-t)) = (1-t^i) p_i X \ , \ p_i(X \pm 1) = p_i X \pm 1 \, .$$

For more informations about the flexibility of arguments of symmetric functions, and the use of λ-rings, see [5].

We shall also need the generating function of complete functions:

$$\sigma_1 X := \prod_{x \in X} (1-x)^{-1} = \sum_i S_i X \, .$$

Schur functions occur naturally when decomposing the Cauchy kernel $\sigma_1(XY)$. The Hall–Littlewood polynomials, our present concern, are associated to the kernel $\sigma_1(XY(1-t)) := \prod_{x \in X, y \in Y} (1-txy)(1-xy)^{-1}$.

2. Hall–Littlewood Polynomials

From now on, we fix a positive integer n. $\mathfrak{Part}$ will be the set of partitions of length not more than n, considered as elements of $\mathbb{N}^n$. Schur functions may be defined as determinants of complete functions, and

[3]We follow Macdonald's conventions [8]. Partitions are weakly decreasing sequences of non-negative integers. One identifies two such sequences if they only differ by trailing 0's on the right.

this allows to extend their indexation to any $v \in \mathbb{Z}^n$, n arbitrary. This amounts to introducing the relations[4]

$$S_v = -S_{\dots,v_{i+1}-1,v_i+1,\dots}\ ,\ S_{v_1,\dots,v_n} = 0 \text{ if } v_n < 0\ . \tag{1}$$

A more powerful point of view than using straightening relations is to introduce *symmetrizing operators*, which can be defined as products of isobaric divided differences π_i, $i = 1, 2, \dots$ (operators act on their left):

$$f \to f\,\pi_i := \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}}$$

where s_i acts on functions of X by transposition of x_i, x_{i+1}.

The operator $x^v \to S_v(x_1, \dots, x_n)$, $v \in \mathbb{N}^n$, can be expressed as a product, denoted π_ω, of π_i. It can also be written as a summation over the symmetric group $\mathfrak{S}_n$:

$$f\,\pi_\omega = \sum_{\sigma \in \mathfrak{S}_n} \left(f \prod_{1 \le i < j \le n} (1 - x_j/x_i)^{-1} \right)^{\sigma} .$$

We refer to Chapter 7 of [5] for some of its properties.

In particular, for any i, one has $\pi_i\,\pi_\omega = \pi_\omega$, and the reordering (1) of the indexation of Schur functions comes from the relation $x_{i+1}\,\pi_i = 0$. Indeed, if $v \in \mathbb{Z}^n$ is such that $v_i = v_{i+1}$, and $\alpha, \beta \in \mathbb{Z}$, then

$$x^v \left(x_i^\alpha x_{i+1}^\beta + x_i^\beta x_{i+1}^\alpha \right) x_{i+1}\,\pi_i = x_{i+1}\,\pi_i\,x^v \left(x_i^\alpha x_{i+1}^\beta + x_i^\beta x_{i+1}^\alpha \right) = 0\,,$$

because symmetric functions in x_i, x_{i+1} commute with π_i.

Some care is needed when taking exponents or indices in $\mathbb{Z}^n$, instead of $\mathbb{N}^n$ (this corresponds to the difference between using characters of the symmetric group, or of the linear group).

We first extend the natural order on partitions to elements of $\mathbb{Z}^n$ by

$$v \ge u \quad \text{if and only if for all } k > 0\,, \quad \sum_{i=k}^{n} (v_i - u_i) \ge 0\,.$$

The operator π_ω commutes with multiplication by any power of $x_1 \cdots x_n$. In the case of a positive power, one has $(x_1 \cdots x_n)^k S_\lambda = S_{\lambda + [k,\dots,k]}$. But in the case of a negative power, we also have to obey the rule that $S_v = 0$ when $v_n = 0$. The solution is to combine the symmetrization with a truncation operator:

$$x^v \to x^v \quad \text{if} \quad v \ge 0 \quad , \quad x^v \to 0 \quad \text{otherwise.}$$

Denote by $\Cup$ the operator "truncation followed by π_ω." Now, one has $x^v \Cup = S_v$, for all $v \in \mathbb{Z}^n$, and moreover, one can compute the image of a Laurent series with a finite number of terms of exponents ≥ 0.

[4]For example, $S_{-2,2,0,0} = -S_{1,-1,0,0} = S_{1,-1,0,0,0} = 0$ and $S_{-3,2,1,1} = -S_{1,-2,1,1} = S_{1,0,-1,1} = -S_{1,0,0,0} = -S_1$.

Introducing an extra indeterminate t, and given any $u \in \mathbb{Z}^n$, one defines the *modified Hall–Littlewood polynomial* Q'_u by

$$Q'_u = x^u \prod_{1\le i<j\le n} (1 - tx_i/x_j)^{-1} \Cup \, . \tag{2}$$

This is, in fact, a finite sum of Schur functions, because we first eliminate in the expansion of $x^u \prod (1 - tx_i/x_j)^{-1}$ all the monomials which are not ≥ 0.

The set $\{Q'_\lambda : \lambda \in \mathfrak{Part}\}$ is a basis of $\mathfrak{Sym}$, which specializes to the basis of Schur functions for $t = 0$. Any Q'_u can be expressed in terms of the Q'_λ thanks to the following relations:

$$Q'_{\dots,\alpha,\beta+1,\dots} + Q'_{\dots,\beta,\alpha+1,\dots} - t\, Q'_{\dots,\beta+1,\alpha,\dots} - t\, Q'_{\dots,\alpha+1,\beta,\dots} = 0\, . \tag{3}$$

These relations still result from $x_{i+1}\,\pi_i = 0$, because, when v is such that $v_i = v_{i+1}$, then

$$x^v \left(x_i^\alpha x_{i+1}^\beta + x_i^\beta x_{i+1}^\alpha \right) \frac{(1 - tx_i/x_{i+1})}{\prod_{j<h}(1 - tx_j/x_h)}\, x_{i+1}\,\pi_i = 0\,,$$

the factor on the left of x_{i+1} being symmetrical in x_i, x_{i+1}, and therefore commuting with π_i.

Relations (3), together with $Q'_{\dots,u_n} = 0$ if $u_n < 0$, suffice to express any Q'_u in terms of the $Q'_\lambda : \lambda \in \mathfrak{Part}$.

The other types of Hall–Littlewood polynomials are

$$Q_\lambda X := Q'_\lambda\big(X\,(1-t)\big)\,, \tag{4}$$

$$P_\lambda X := b_\lambda^{-1}\, Q_\lambda X\,, \tag{5}$$

with $b_\lambda = \prod_i \prod_{j=1}^{m_i} (1 - t^j)$, for $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$.

These functions can also be defined by symmetrization, but problems arise when taking general weights instead of only *dominant weights* (i.e., partitions), see the terminal note.

We give in Corollary 3.2 a description of the functions Q'_λ in terms of *plane partitions*. Recall that the Q'_λ admit another combinatorial description, this time in terms of tableaux:

$$Q'_\mu = \sum_T t^{\mathfrak{ch} T}\, S_{\mathfrak{sh} T}\,, \tag{6}$$

where the sum is over all tableaux of weight μ, $\mathfrak{sh}T$ being the shape of T, and $\mathfrak{ch}T$ being the *charge* of T (cf. [8, III.6])[5].

[5]The charge is a *rank function* on the set of all tableaux. More generally, it can be defined as a function on words in letters $1, 2, 3, \dots$, satisfying the following relations in the case of a dominant evaluation, i.e., when $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$:

$$\mathfrak{ch}\,(\cdots 3^{\mu_3} 2^{\mu_2} 1^{\mu_1}) = 0\ ,\ \mathfrak{ch}(w\,i) = \mathfrak{ch}(iw) + 1 \text{ if } i > 1\,, \tag{7}$$

and the invariance with respect to the plactic relations [6]:

$$w \equiv w' \ \Rightarrow\ \mathfrak{ch} w = \mathfrak{ch} w'\ . \tag{8}$$

3. Adding 1

Adding ±1 to the argument of a symmetric function is, as we already said, induced by the transformation

$$p_i \to p_i \pm 1\,,\ i = 1, 2, \ldots$$

of the powers sums p_i.

In terms of Schur functions, for a partition $\lambda \in \mathbb{N}^n$, this amounts to

$$S_\lambda(X-1) = \sum_{v \in \{0,1\}^n} (-1)^{|v|} S_{\lambda - v} X\,, \tag{9}$$

$$S_\lambda(X+1) = \sum_{u \in \mathbb{N}^n} S_{\lambda+u} X\,. \tag{10}$$

Both summations can be reduced to a summation over partitions, erasing vertical or horizontal strips from the diagram of λ [8, I.5].

One can rewrite (9,10) as

$$S_\lambda(X-1) = x^\lambda \prod_{1\le i\le n} (1 - 1/x_i)\, \pi_\omega\,, \tag{11}$$

$$S_\lambda(X+1) = x^\lambda \prod_{1\le i\le n} \frac{1}{1-1/x_i}\, \pi_\omega\,, \tag{12}$$

and therefore,

$$Q'_\lambda(X-1) = x^\lambda \frac{\prod_{1\le i\le n}(1-1/x_i)}{\prod_{1\le i<j\le n}(1-tx_i/x_j)} \pi_\omega = \sum_{v\in\{0,1\}^n} (-1)^{|v|} Q'_{\lambda-v} X\,, \tag{13}$$

$$Q'_\lambda(X+1) = x^\lambda \frac{1}{\prod_{1\le i\le n}(1-1/x_i)} \frac{1}{\prod_{1\le i<j\le n}(1-tx_i/x_j)}\, \pi_\omega\,. \tag{14}$$

The first summation is easy to reduce, using the reordering

$$\sum_v Q'_{k^m - v} = \begin{bmatrix} m \\ \alpha \end{bmatrix} Q'_{k^{m-\alpha},(k-1)^\alpha}\,, \tag{15}$$

where v runs over all permutations of $[1^\alpha,\, 0^{m-\alpha}]$, and where $\begin{bmatrix} m \\ \alpha \end{bmatrix}$ denotes the t-binomial coefficient $(1-t^m)\cdots(1-t^{m-\alpha+1})/(1-t)\cdots(1-t^\alpha)$.

Iterating (15), one gets

$$Q'_\lambda(X-1) = \sum_\mu \prod_i (-1)^{\alpha_i} \begin{bmatrix} m_i \\ \alpha_i \end{bmatrix} Q'_\mu\,, \tag{16}$$

where the sum is over all partitions $\mu = 0^{\alpha_1} 1^{m_1-\alpha_1} 1^{\alpha_2} 2^{m_2-\alpha_2} 2^{\alpha_3} 3^{m_3-\alpha_3} \cdots$ differing from $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$ by a vertical strip.

The second summation is a little more complicated to transform, and contrary to the case of Schur functions, will not restrict to erasing strips.

Let us first introduce *skew Hall–Littlewood polynomials* [8, III.5] $Q'_{\lambda/\mu}$, by

$$Q'_\lambda(X+Y) = \sum_{\mu \in \mathfrak{Part}} Q'_{\lambda/\mu} X \, Q'_\mu Y \ , \tag{17}$$

i.e. $Q'_{\lambda/\mu}X$ is defined as the coefficient of $Q'_\mu Y$ in the expansion of the function Q'_λ evaluated in $X+Y$. Note that this definition makes sense for $\lambda \in \mathbb{Z}^n$, and not only for $\lambda \in \mathfrak{Part}$.

For any pair of partitions λ, μ, let us define

$$\mathbf{n}(\lambda/\mu) = \sum_i (\lambda_i^\sim - \mu_i^\sim)(\lambda_i^\sim - \mu_i^\sim - 1)/2 \, ,$$

where $\lambda^\sim$ and $\mu^\sim$ are the partitions conjugate to λ, μ respectively. In the case where $\mu = 0$, then one writes $\mathbf{n}(\lambda)$ instead of $\mathbf{n}(\lambda/0)$. Moreover, $\mathbf{n}(\lambda) = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + \cdots$.

The main result of this section is the following evaluation of $Q'_{\lambda/\mu}1$.

Theorem 3.1. *Given a partition λ, then*

$$Q'_\lambda(X+1) = \sum_{\mu \subseteq \lambda} Q'_\mu X \, \aleph(\lambda/\mu) \, ,$$

with

$$Q'_{\lambda/\mu}1 = \aleph(\lambda/\mu) := b_\mu^{-1} \, t^{\mathbf{n}(\lambda/\mu)} \, (1-t^{\nu_1-0})(1-t^{\nu_2-1}) \cdots (1-t^{\nu_r-r+1}) \ , \tag{18}$$

$\nu_1, \ldots, \nu_r$ being the parts of index $\mu_1, \ldots, \mu_r$, $r = \ell(\mu)$, of the partition conjugate to λ.

One can visualize the value of $Q'_{\lambda/\mu}1 = \aleph(\lambda/\mu)$ in the Cartesian plane as follows: colour in black the right-most box of each row of the diagram of μ, write $0, 1, 2, \ldots$ in the successive boxes of the same column of λ/μ. Each column with α black boxes, and β boxes above, gives a contribution $t^{\binom{\beta}{2}} \begin{bmatrix} \alpha+\beta \\ \alpha \end{bmatrix}$ to $Q'_{\lambda/\mu}(1)$, which is the product over all columns of these contributions. For example, for $\lambda = [4432221]$, $\mu = [2211]$, then $\lambda^\sim = [7632]$, $\nu = [6677]$, $\mathbf{n}(\lambda/\mu) = 13$, $b_\mu = (1-t)^2(1-t^2)^2$,

$$Q'_{4432221/2211}1 = t^{13}(1-t^6)(1-t^{6-1})(1-t^{7-2})(1-t^{7-3})(1-t)^{-2}(1-t^2)^{-2} \, ,$$

$$\begin{array}{|c|c|c|c|}
\cline{1-1} 2 \\
\cline{1-2} 1 & 3 \\
\cline{1-2} 0 & 2 \\
\cline{1-2} \blacksquare & 1 \\
\cline{1-3} \blacksquare & 0 & 2 \\
\hline & \blacksquare & 1 & 1 \\
\hline & \blacksquare & 0 & 0 \\
\hline
\end{array}
\quad \Rightarrow \quad Q'_{4432221/2211}1 = t^3 \begin{bmatrix} 5 \\ 2 \end{bmatrix} t^6 \begin{bmatrix} 6 \\ 2 \end{bmatrix} t^3 \begin{bmatrix} 3 \\ 0 \end{bmatrix} t^1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} .$$

Proof of the theorem. We shall evaluate $\sum_u Q'_{\lambda-u}X$ by decomposing the set $u \in \mathbb{N}^n$ into two subsets, according to whether $u_1 = 0$ or not.

Let a, j, k be such that

$$\lambda_1 = a = \cdots = \lambda_k > \lambda_{k+1}\ ,\ \ \mu_1 = a = \cdots = \mu_j > \mu_{j+1}\,,$$

and write $\lambda = a^k\zeta$, $\mu = a^j\eta$.

In terms of these parameters, using (15), one wants to show that

$$Q'_{\lambda/\mu}1 = t^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} Q'_{[a^{k-j}\zeta]/\eta}(1)\,. \tag{19}$$

The sub-sum $\sum_{u:\,u_1=0} Q'_{\lambda-u}X$ is equal to

$$Q'_aX \odot Q'_{a^{k-1}\zeta}(X+1)\,,$$

where, by definition, $Q'_a \odot Q'_\nu$ is the concatenation $Q'_{a,\nu}$, and extending by linearity.

By induction on $\ell(\lambda)$, the coefficient of $Q'_\mu X$ in $Q'_aX \odot Q'_{a^{k-1}\zeta}(X+1)$ is equal to

$$t^{\binom{k-j}{2}} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} Q'_{[a^{k-j}\zeta]/\eta}1\ .$$

The terms $Q'_{\lambda-u}X$, $u_1 \geq 1$, can be rewritten

$$\sum_{u\in\mathbb{N}^n} Q_{\lambda^- - u}XQ'_{\lambda^-}(X+1) = t^{k-1}Q'_{a^{k-1},a-1,\zeta}(X+1)\,,$$

with $\lambda^- = [a\text{–}1, \lambda_2, \lambda_3, \ldots]$.

By induction on $|\lambda|$, the coefficient of $Q'_\mu X$ in this function is equal to

$$t^{k-1}t^{\binom{k-1-j}{2}} \begin{bmatrix} k-1 \\ j \end{bmatrix} Q'_{[a^{k-j}\zeta]/\eta}1\ .$$

The identity

$$\begin{bmatrix} k-1 \\ j-1 \end{bmatrix} + t^{k-1+\binom{k-1-j}{2}-\binom{k-j}{2}} \begin{bmatrix} k-1 \\ j \end{bmatrix} = \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} + t^j \begin{bmatrix} k-1 \\ j \end{bmatrix} = \begin{bmatrix} k \\ j \end{bmatrix}$$

allows to sum up the two contributions and finishes the proof. □

For example, $Q'_{221}(X+1)$ is obtained by the following enumeration:

$$t^4 \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline 0 & 0 \\ \hline\end{array} + t^2 \begin{bmatrix}3\\1\end{bmatrix} \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 0 & 1 \\ \hline ■ & 0 \\ \hline\end{array} + t \begin{bmatrix}2\\1\end{bmatrix} \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 0 & 0 \\ \hline & ■ \\ \hline\end{array} + t \begin{bmatrix}3\\2\end{bmatrix} \begin{array}{|c|c|}\cline{1-1} 0 & \multicolumn{1}{c}{} \\ \hline ■ & 1 \\ \hline ■ & 0 \\ \hline\end{array} + \begin{bmatrix}2\\1\end{bmatrix}^2 \begin{array}{|c|c|}\cline{1-1} 0 & \multicolumn{1}{c}{} \\ \hline ■ & 0 \\ \hline & ■ \\ \hline\end{array}$$

$$+t \begin{bmatrix}3\\3\end{bmatrix} \begin{array}{|c|c|}\cline{1-1} ■ & \multicolumn{1}{c}{} \\ \hline ■ & 1 \\ \hline ■ & 0 \\ \hline\end{array} + \begin{bmatrix}2\\2\end{bmatrix} \begin{array}{|c|c|}\cline{1-1} 0 & \multicolumn{1}{c}{} \\ \hline & ■ \\ \hline & ■ \\ \hline\end{array} + \begin{bmatrix}2\\2\end{bmatrix} \begin{bmatrix}2\\1\end{bmatrix} \begin{array}{|c|c|}\cline{1-1} ■ & \multicolumn{1}{c}{} \\ \hline ■ & 0 \\ \hline & ■ \\ \hline\end{array} + \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}2\\2\end{bmatrix} \begin{array}{|c|c|}\cline{1-1} ■ & \multicolumn{1}{c}{} \\ \hline & ■ \\ \hline & ■ \\ \hline\end{array}$$

$$= t^4Q'_0 + t^2(1+t+t^2)Q'_1 + t(1+t)Q'_2 + t(1+t+t^2)Q'_{11} + (1+t)^2Q'_{21}$$
$$+ tQ'_{111} + Q'_{22} + (1+t)Q'_{211} + Q'_{221}\,.$$

Recall that a *plane partition* of shape a partition λ is a filling of the diagram of λ with positive integers such that numbers weakly increase when faring towards the origin. In other words, the successive domains occupied by the letters $1, 2, , 3, \ldots$ are skew partitions λ/λ^1, λ^1/λ^2, λ^2/λ^3, …. Define the *weight* of a plane partition $\boxplus$ to be the product

$$\aleph_x(\boxplus) = \aleph(\lambda/\lambda^1) x_1^{|\lambda/\lambda^1|}\, \aleph(\lambda^1/\lambda^2) x_2^{|\lambda^1/\lambda^2|}\, \aleph(\lambda^2/\lambda^3) x_3^{|\lambda^2/\lambda^3|} \cdots .$$

Iteration of Theorem 3.1 leads to the following description of Q'_λ.

Corollary 3.2. *Let λ be a partition, n be a positive integer. Then*

$$Q'_\lambda(x_1 + \cdots + x_n) = \sum_{\boxplus} \aleph_x(\boxplus) ,$$

where the sum is over all plane partitions in $1, \ldots, n$ of shape λ.

For example, for $\lambda = [21]$, one has

$$\begin{aligned} Q'_{21} &= \sum_i t \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{i}\\\hline i & i\\\hline\end{array} + \sum_{i<j} (1{+}t) \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{i}\\\hline j & i\\\hline\end{array} + t \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{j}\\\hline j & i\\\hline\end{array} + \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{i}\\\hline j & j\\\hline\end{array} \\ &\qquad + \sum_{i<j<k} \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{i}\\\hline k & j\\\hline\end{array} + (1{+}t) \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{j}\\\hline k & i\\\hline\end{array} \\ &= \sum_i t\, x_i^3 + \sum_{i<j} (1{+}t)\, x_i^2 x_j + t\, x_i x_j^2 + x_i x_j^2 + \sum_{i<j<k} (2+t) x_i x_j x_k . \end{aligned}$$

On the other hand, the interpretation in terms of tableaux and charge reads

$$\begin{aligned} Q'_{21} &= S_{21} + t S_3 \\ &= \sum_{i<j} \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{j}\\\hline i & i\\\hline\end{array} + \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{j}\\\hline i & j\\\hline\end{array} + \sum_{i<j<k} \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{j}\\\hline i & k\\\hline\end{array} + \begin{array}{|c|c|}\cline{1-1}\multicolumn{1}{|c|}{k}\\\hline i & j\\\hline\end{array} + \sum_{i\le j\le k} t \begin{array}{|c|c|c|}\hline i & j & k\\\hline\end{array} . \end{aligned}$$

4. Argument $1 - X$

Instead of describing $Q'_\lambda(X-1)$, we prefer taking $Q'_\lambda(1-X)$. Addition and subtraction of alphabets involve rather different properties of symmetric functions. In the case of Hall–Littlewood polynomials, with a hypothesis on λ, we shall find a connection between $Q'_\lambda(1-X)$ and resultants.

Recall that, given two finite alphabets $\mathbb{A}, \mathbb{B}$, of respective cardinalities α, β, then the *resultant* $\prod_{a\in\mathbb{A}, b\in\mathbb{B}} (a-b)$ is equal to the Schur function $S_{\beta^\alpha}(\mathbb{A} - \mathbb{B})$. More generally, the resultant appears as a factor of some Schur functions, thanks to a proposition due to Berele and Regev [5, 1.4.3].

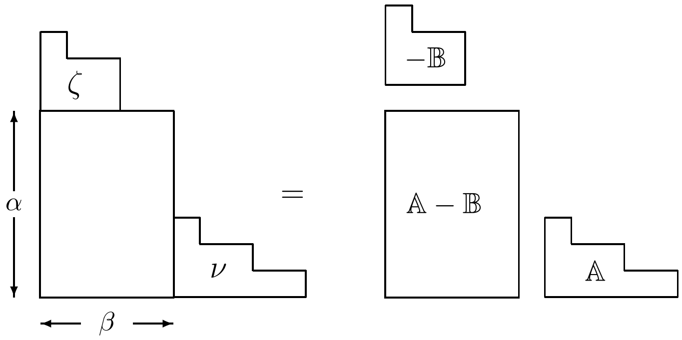
Proposition 4.1. *Let $\mathbb{A}$, $\mathbb{B}$, be of cardinalities α, β, $p \in \mathbb{N}$, $\zeta \in \mathbb{N}^p$, $\nu \in \mathbb{N}^\alpha$. Then, writing $[\beta+\nu_1, \dots, \beta+\nu_\alpha, \zeta_1, \dots, \zeta_p] = [\beta^\alpha+\nu, \zeta]$, one has*

$$S_{\beta^\alpha+\nu,\zeta}(\mathbb{A}-\mathbb{B}) = S_\zeta(-\mathbb{B})\, S_\nu(\mathbb{A}) \prod_{a\in\mathbb{A}, b\in\mathbb{B}} (a-b) \ . \tag{20}$$

Moreover,

$$\nu \in \mathfrak{Part},\ \nu \supseteq (\beta+1)^{\alpha+1} \ \Rightarrow S_\nu(\mathbb{A}-\mathbb{B}) = 0\,. \tag{21}$$

Pictorially, the relation is



The next theorem shows that $Q'_{n^k}(1-X)$ is also a resultant, and more generally, factors out a resultant from $Q'_\lambda(1-X)$ (it is more convenient to take a power of t instead of 1, to simplify exponents).

Theorem 4.2. *Let $n, r \in \mathbb{N}$, X be an alphabet of cardinality n, λ be a partition that is written, with ζ such that $\zeta_1 < n$,*

$$\lambda = [n+\nu_1, \dots, n+\nu_k, \zeta_1, \zeta_2, \dots] = [n^k+\nu, \zeta]\,.$$

Then

$$Q'_\lambda(t^r - X) = t^{\mathbf{n}(\nu)+r|\nu|} \prod_{i=r}^{k+r-1} \prod_{j=1}^{n} (t^i - x_j)\, Q'_\zeta(t^{k+r}-X)\,. \tag{22}$$

Proof. $Q'_\lambda(1-X) = \sum_{\mu\subseteq\lambda} Q'_\mu X\, Q'_{\lambda/\mu} t^r$. The terms $Q'_\mu X$ vanish if $\mu_1 > n$. Eq. (18) shows that $Q'_{\lambda/\mu} t^r$, when $\mu \le n$, is equal to

$$Q'_{[n^k,\zeta]/\mu} t^r\, t^{\mathbf{n}(\nu)+r|\nu|}\,.$$

Thus we need only treat the case where $\nu = 0$, and we shall do it by induction on k.

The function $Q'_{[n^{k-1},\zeta]/\mu}(t^{r+1} - X)$ can be written as as sum of Schur functions , enumerating all tableaux of commutative evaluation

$$2^n \cdots k^n (k+1)^{\zeta_1} (k+2)^{\zeta_2} \cdots\,,$$

of shape not containing $[n+1, n+1]$ (otherwise the Schur function vanishes, according to (21)). Given any such tableau T, of shape ρ, given any horizontal strip ρ/ξ, then there exists a unique pair (u, T) such that $T \equiv uT'$, with T' of shape ξ and u a tableau of row shape. If $\xi_1 \le n$, then

$T'' = T'\,1^n u$ is a tableau of charge $\mathfrak{ch}T + \ell(u)$, which contributes to $Q'_{[n^k,\zeta]}$, and all tableaux of weight $[n^k,\zeta]$, of shape not containing $[n{+}1,n{+}1]$ are obtained in this way.

The contribution of T to $Q'_{[n^{k-1},\zeta]/\mu}(t^{r+1}-X)$ is, writing $R(y,X)$ for $\prod_{1\le i\le n}(y-x_j)$,

$$t^{\mathfrak{ch}T} S_\rho(t^{r+1}-X) = t^{\mathfrak{ch}T} S_{\rho_2,\rho_3,\ldots}(-X)\, R(t^{r+1},X)\, t^{(r+1)(\rho_1-n)}\,,$$

thanks to the factorization (20). Each of its successors $T'' = T'1^n u$ contributes to $Q'_{[n^k,\zeta]}(t^r-X)$ the term

$$t^{\mathfrak{ch}T''} S_{n+\ell(u),\xi}(t^r-X) = t^{\mathfrak{ch}T} t^{\ell(u)} S_\xi(-X) R(t^r,X) t^{r\ell(u)}\,.$$

Summing over all the tableaux T'' whose predecessor is T, one gets the contribution[6]

$$t^{\mathfrak{ch}T}\, R(t^r,X)\, S_\rho(t^{r+1}-X)\,,$$

and this shows that the contribution of T has been multiplied by a factor independent of T.

Summation over all tableaux T of weight $[n^{k-1},\zeta]$ gives the equality

$$Q'_{[n^k,\zeta]}(t^r-X) = R(t^r,X) Q'_{[n^{k-1},\zeta]}(t^{r+1}-X)$$

and finishes the proof. $\square$

For example, for $n=2$, let us illustrate, on a single tableau, the inductive step from $Q'_{22211}(t-X)$ to $Q'_{222211}(1-X)$. The tableau $\begin{array}{|c|c|c|c|}\hline 6 \\ \cline{1-1} 5 \\ \cline{1-1} 3 & 4 \\ \cline{1-2} 2 & 2 & 3 & 4 \\ \hline\end{array}$ has charge 5 and gives the contribution

$$t^5 S_{4211}(t-X) = t^5 S_{211}(\text{-}X)\, S_4(t-X) = t^5 S_{211}(\text{-}X)\, R(t,X)\, t^2\,.$$

Its different factorizations, the issuing new tableaux and their contributions are

[6]Indeed, when y a single letter, $S_\rho(y-X)$ is equal to the sum $\sum_\xi y^{|\rho/\xi|} S_\xi(-X)$.

$$\begin{array}{cc} 2 & 3 \end{array}\ \begin{array}{ll} 6 & \\ 5 & \\ 4 & 4 \\ 2 & 3 \end{array} \quad\rightarrow\quad \begin{array}{llll} 6 & & & \\ 5 & & & \\ 4 & 4 & & \\ 2 & 3 & & \\ 1 & 1 & 2 & 3 \end{array} \qquad t^7 S_{2211}(-X)\, R(1,X)$$

$$\begin{array}{ccc} 2 & 3 & 6 \end{array}\ \begin{array}{ll} 5 & \\ 4 & 4 \\ 2 & 3 \end{array} \quad\rightarrow\quad \begin{array}{lllll} 5 & & & & \\ 4 & 4 & & & \\ 2 & 3 & & & \\ 1 & 1 & 2 & 3 & 6 \end{array} \qquad t^8 S_{221}(-X)\, R(1,X)$$

$$\begin{array}{ccc} 2 & 3 & 4 \end{array}\ \begin{array}{ll} 6 & \\ 5 & \\ 4 & \\ 2 & 3 \end{array} \quad\rightarrow\quad \begin{array}{lllll} 6 & & & & \\ 5 & & & & \\ 4 & & & & \\ 2 & 3 & & & \\ 1 & 1 & 2 & 3 & 4 \end{array} \qquad t^8 S_{2111}(-X)\, R(1,X)$$

$$\begin{array}{cccc} 2 & 3 & 4 & 6 \end{array}\ \begin{array}{ll} 5 & \\ 4 & \\ 2 & 3 \end{array} \quad\rightarrow\quad \begin{array}{llllll} 5 & & & & & \\ 4 & & & & & \\ 2 & 3 & & & & \\ 1 & 1 & 2 & 3 & 4 & 6 \end{array} \qquad t^9 S_{211}(-X)\, R(1,X)\,.$$

The sum of these contributions is

$$\begin{aligned} &t^7 R(1,X)\left(S_{2211}(-X) + t\, S_{221}(-X) + t\, S_{2111}(-X) + t^2\, S_{211}(-X)\right) \\ &\qquad = t^7 R(1,X)\, S_{2211}(t-X) = t^7 R(1,X) R(t,X) S_{211}(t-X)\,. \end{aligned}$$

In the special case $X=\{x\}$ of cardinality 1, and $r=0$, one recovers that [8, p.226]

$$Q'_\lambda(1-x) = Q_\lambda\left(\frac{1-x}{1-t}\right) = t^{\mathbf{n}(\lambda)}(1-x)(1-xt^{-1})\cdots(1-xt^{1-\ell(\lambda)})\,, \quad (23)$$

an identity which allows to describe the *principal specializations* $Q'_\lambda(1-t^N) = Q_\lambda((1-t^N)/(1-t))$ when taking $y=t^N$.

In the case of an alphabet of cardinality 2, $X=\{x_1,x_2\}$, the remaining partition ζ is such that $\zeta = 1^\beta$ for some $\beta\in\mathbb{N}$, and therefore, the factor $Q'_\zeta(t^j - X)$ is equal to

$$Q_{1^\beta}\left(\frac{t^j - X}{1-t}\right) = b_{1^\beta}\, S_{1^\beta}\left(\frac{t^j - x_1 - x_2}{1-t}\right)\,.$$

In total,

$$\begin{aligned} &Q'_{2^k+\nu,\,1^\beta}(1-x_1-x_2) \\ &\qquad = t^{\mathbf{n}(\nu)}\,(1\text{-}t)\cdots(1\text{-}t^\beta)\,\prod_{i=0}^{k-1}(t^i\text{-}x_1)(t^i\text{-}x_2)\, S_{1^\beta}\left(\frac{t^k\text{-}x_1\text{-}x_2}{1-t}\right)\,. \quad (24) \end{aligned}$$

5. Generating Functions

The results of the preceding sections may be used to compute generating series of Hall–Littlewood polynomials. For example, given the series $\sum_{\mu\in\mathfrak{Part}} c_\mu P_\mu(X)$, with some arbitrary coefficients c_μ, suppose that one wants to evaluate the product

$$\sigma_1(-X)\sum c_\mu P_\mu X = \prod_{x\in X}(1-x)\sum c_\mu P_\mu X\ .$$

To do so, one first extends the family c_μ to a family $c_v : v \in \mathbb{Z}^n$ by imposing the relations (3). It amounts introducing a second alphabet Y, and putting $c_\mu = Q_\mu Y$. Since $\sigma_1(XY(1\text{–}t)) = \sum Q_\mu Y P_\mu X$, then

$$\begin{aligned}
\sigma_1(-X)\sum_\mu c_\mu P_\mu(X) &= \sigma_1\big(-X + XY(1\text{–}t)\big)\\
&= \prod_{x\in X}(1\text{–}x)\prod_{x\in X, y\in Y}\frac{1-txy}{1-xy}\\
&= \sigma_1\left(X(Y-\frac{1}{1-t})(1-t)\right)\\
&= \sum_{\lambda\in\mathfrak{Part}} P_\lambda X\, Q_\lambda\left(Y-\frac{1}{1\text{–}t}\right)\ .
\end{aligned}$$

Knowing the expansion of $Q_\lambda(Y - 1/(1\text{–}t))$, or equivalently, of $Q'_\lambda(Y'-1)$, with $Y' = Y(1\text{–}t)$, given in (13), one concludes

$$\sigma_1(-X)\sum_\mu c_\mu P_\mu X = \sum_\lambda\sum_{v\in\{0,1\}^n}(\text{-}1)^{|v|}c_{\lambda-v}P_\lambda X\ . \tag{25}$$

Let us give another similar example.

Proposition 5.1. *Given two alphabets X, Y, we have*

$$\begin{aligned}
\sigma_1\left(X + XY(1\text{–}t)\right) &= \prod_{x\in X}\frac{1}{1-x}\prod_{y\in Y}\frac{1-txy}{1-xy}\\
&= \sum_{\lambda\in\mathfrak{Part}}\sum_{\mu\subseteq\lambda} P_\lambda X P_\mu Y\, b_\mu Q'_{\lambda/\mu}1\,,
\end{aligned} \tag{26}$$

the value of $b_\mu Q'_{\lambda/\mu}1$ being given in (18).

Proof. One writes $X + XY(1\text{–}t) = X(Y + (1\text{–}t)^{-1})(1\text{–}t)$, and therefore one obtains

$$\begin{aligned}
\sigma_1\left(X + XY(1\text{–}t)\right) &= \sum_\lambda P_\lambda X Q_\lambda(Y + (1\text{–}t)^{-1})\\
&= \sum_{\lambda,\mu} P_\lambda X P_\mu b_\mu Y Q_{\lambda/\mu}((1\text{–}t)^{-1})\,,
\end{aligned}$$

which is the required formula. □

For example, the coefficient of $P_{42}X$, in terms of the $P_\mu = P_\mu Y$, is

$$t^2P_0 + t(1-t^2)P_1 + (1-t^2)P_2 + t(1-t)(1-t^2)P_{11} \\ + (1-t)P_3 + (1-t)(1-t^2)P_{21} + (1-t)P_4 + (1-t)^2P_{31} + (1-t)(1-t^2)P_{22} \\ + (1-t)^2P_{41} + (1-t)^2P_{32} + (1-t)^2P_{42}\,.$$

Using the explicit values (24), one obtains as a corollary for $X = (t - x_1 - x_2)(1 - t)^{-1}$, $Y = (1 - y)(1 - t)^{-1}$, the expansion of

$$\sigma_1\left(\frac{(t - x_1 - x_2)(2 - y)}{1 - t}\right) \\ = \sigma_1\left(2\frac{t}{1-t} + \frac{y(x_1 + x_2)}{1 - t} - \frac{yt}{1-t} - 2\frac{x_1 + x_2}{1 - t}\right) \\ = \prod_{i=0}^{\infty} \frac{1}{(1 - t^{i+1})^2} \frac{1}{(1 - t^i yx_1)(1 - t^i yx_2)} \\ \cdot (1 - yt^{i+1})\big((1 - t^i x_1)(1 - t^i x_2)\big)^2\,.$$

Warnaar evaluates a similar generating series, with simpler coefficients. Let, for any pair of partitions,

$$\theta(\lambda, \mu) = t^{\mathbf{n}(\lambda/\mu)-|\mu|}\,.$$

Then Warnaar's identity [10, Th. 1.1] is the following.

Theorem 5.2. *For any pair of alphabets X, Y, one has*

$$\sigma_1\left(X + Y + (\frac{1}{t} - 1)XY\right) = \sum_{\lambda,\mu\in\mathfrak{Part}} \theta(\lambda, \mu)P_\lambda X P_\mu Y\,. \tag{27}$$

Proof. Writing the left-hand side $\sigma_1 X\, \sigma_1\left(Y(1 + (t^{-1} - 1)X)\right)$, and using that $\sigma_1 XY((1-t) = \sum Q_\mu X P_\mu Y$, one rewrites the identity to prove as

$$Q_\lambda\left(\frac{1}{1-t} + t^{-1}X\right) \stackrel{?}{=} \frac{1}{\sigma_1 X} \sum_\mu \theta(\lambda, \mu)P_\mu X\,,$$

or

$$\sum_\mu t^{-|\mu|}Q_\mu X\, Q_{\lambda/\mu}\left(\frac{1}{1-t}\right) = \sum_\mu t^{-|\mu|}Q_\mu X\, Q'_{\lambda/\mu}1 \\ \stackrel{?}{=} \sigma_1(-X) \sum_\mu \theta(\lambda, \mu)P_\mu X\,. \tag{28}$$

Thanks to (25), the right-hand side can be written

$$\sum_{v\in\{0,1\}^n} \theta(\lambda, \mu - v)P_\mu X = \sum_{v\in\{0,1\}^n} \theta(\lambda, \mu - v)b_\mu^{-1}Q_\mu X\,.$$

Therefore, (28) is a consequence of (18) and is, in fact, equivalent to (26). □

Let us mention that Warnaar used another expression of $\theta(\lambda,\mu)$, putting

$$\theta(\lambda,\mu) = t^{\mathbf{n}(\lambda)+\mathbf{n}(\mu)-(\lambda^\sim,\mu^\sim)}\,, \tag{29}$$

where $(\lambda^\sim,\mu^\sim) = \sum \lambda_i^\sim \mu_i^\sim$.

6. Scalar Product

The combinatorics of Hall–Littlewood polynomials fundamentally reduces to the fact that $\{P_\lambda\}$ is an orthogonal basis of the ring of polynomials, and that it can be extended to a family satisfying the *straightening relations* (3). Given these two ingredients, we can forget Hall and Littlewood altogether.

Thus, let us consider the ring of Laurent series in $x_1,\ldots,x_n$, with a finite number of terms with exponent $\geq [0,\ldots,0]$, modulo the ideal generated by the relations

$$(x^v + x^{vs_i})(x_{i+1} - tx_i) \simeq 0\,,\ i=1,\ldots,\ n{-}1,\ v\in\mathbb{Z}^n,\ x^v \simeq 0 \text{ if } v_n < 0\,. \tag{30}$$

Any element of this ring can be written uniquely as a linear combination of dominant monomials $x^\lambda : \lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

We define a scalar product by its restriction to dominant monomials:

$$((x^\lambda, x^\mu)) = b_\lambda\, \delta_{\lambda,\mu}\,,\ \lambda,\mu\in\mathfrak{Part}\,. \tag{31}$$

We may observe that the scalar product on Laurent polynomials used in [2],

$$(f,g)_t := CT\left(f(x_1,\ldots,x_n)\, g\left(\frac{1}{x_n},\ldots,\frac{1}{x_1}\right) \prod_{1\leq i<j\leq n} \frac{1-x_i/x_j}{1-tx_i/x_j}\right)\,, \tag{32}$$

where CT means "constant term", is such that

$$(Q_\lambda, x^\mu) = b_\lambda\, \delta_{\lambda,\mu}\,.$$

This indicates that the present study is related to more general constructions about non-symmetric Hall–Littlewood polynomials.

In the present set-up, to recover Warnaar's generating function, we essentially need to interpret the function $\theta(\lambda,\mu)$, $\lambda,\mu\in\mathfrak{Part}$, as a scalar product. To do so, let us also write $\theta(x^\lambda,x^\mu)$ for $\theta(\lambda,\mu)$, and extend the definition of θ to all monomials by linearity using relations (30).

Proposition 6.1. *For any pair $\lambda,\mu\in\mathfrak{Part}$, we have*

$$\left(\left(\frac{x^\lambda t^{-|\lambda|}}{\prod_{i=1}^n (1-t/x_i)}\,,\ \frac{x^\mu}{\prod_{i=1}^n (1-1/x_i)} \right)\right) = \theta(\lambda,\mu)\,. \tag{33}$$

Proof. Multiplying[7] x^μ by $\prod_{i=1}^n (1-1/x_i)$, let us prove the equivalent statement that

$$\Big(\Big(\frac{x^\lambda t^{-|\lambda|}}{\prod_{i=1}^n (1-t/x_i)}\,,\, x^\mu\Big)\Big) = \sum_{v\in\{0,1\}^n} (\text{-}1)^{|v|}\theta(\lambda,\mu-v)\,. \qquad (34)$$

Let a,k be such that $\mu_1 = a = \cdots = \mu_k > \mu_{k+1}$, let $r=\lambda^\sim_a$. Then, for any $j : 0\le j\le k$, one has

$$\theta(\lambda,\mu-[1^j,0^{n-j}]) = \theta(\lambda,\mu)t^{-0-1-\cdots-(j-1)}t^{jr}\,.$$

Summing over all vectors $u\in\{0,1\}^n$, $u_i=0$ for $i>k$, and taking into account reordering, one gets

$$\begin{aligned}\sum_u (\text{-}1)^{|u|}\theta(\lambda,\mu-u) &= \theta(\lambda,\mu)\sum_{j=0}^k (\text{-}1)^j \begin{bmatrix} k\\ j\end{bmatrix} t^{-\binom{j}{2}}t^{jr}\\ &= \theta(\lambda,\mu)(1-t^{r-0})(1-t^{r-1})\cdots(1-t^{r-k+1})\,.\end{aligned}$$

By induction $[\mu_{k+1},\mu_{k+2},\ldots]\to\mu=[a^k,\mu_{k+1},\ldots]$, this proves that

$$\sum_{v\in\{0,1\}^n} (-1)^{|v|}\theta(\lambda,\mu-v) = \theta(\lambda,\mu)(1-t^{\nu_1})(1-t^{\nu_2-1})(1-t^{\nu_3-2})\cdots\,, \qquad (35)$$

$\nu_1,\nu_2,\nu_3,\ldots$ being the parts of $\lambda^\sim$ of index $\mu_1,\mu_2,\mu_3,\ldots$. Notice that the product is null if $\mu\not\subseteq\lambda$.

Using (18), one rewrites the right hand side of (35) as

$$\theta(\lambda,\mu)t^{-\mathbf{n}(\lambda/\mu)}\,b_\mu\,Q'_{\lambda/\mu}\,1\,. \qquad (36)$$

On the other hand,

$$Q'_\lambda(1+t^{-1}X) = \sum_{u\in\mathbb{N}^n} Q'_{\lambda-u}(t^{-1}X) = \sum t^{-|\mu|}Q'_\mu X\,Q'_{\lambda/\mu}1\,.$$

This implies that

$$t^{-|\lambda|}\prod_{i=1}^n (1-t/x_i)^{-1} \simeq \sum_{u\in\mathbb{N}^n} x^{\lambda-u}t^{|u|-|\lambda|}$$

is congruent to $\sum_\mu t^{-|\mu|}x^\mu Q'_{\lambda/\mu}1$. Therefore,

$$((\frac{x^\lambda t^{-|\lambda|}}{\prod_{i=1}^n (1-t/x_i)}\,,\,x^\mu)) = t^{-|\mu|}b_\mu Q'_{\lambda/\mu}1\,,$$

and this is the required identity (34), using that $\theta(\lambda,\mu)=t^{\mathbf{n}(\lambda/\mu)-|\mu|}$. □

[7]Multiplying by x^{-u}, $u\in\mathbb{N}^n$, is compatible with relation (30): $x^v=0$ if $v_n<0$.

7. Note

The functions Q'_λ have an interpretation in terms of the cohomology of flag manifolds [1], [8, III.7]. They describe graded multiplicities of representations of the symmetric group. The functions Q_λ also have an interpretation, as Euler–Poincaré characteristic of line bundles over the flag manifold [4]. In combinatorial terms, this gives the following definition.

Let π_ω be the symmetrizing operator defined before. Given a partition $\lambda \in \mathbb{N}^n$, let m_0 be the multiplicity of the part 0 (=$n - \ell(\lambda)$). Then [8, III.2]

$$Q_\lambda(X) = \frac{(1-t)^n}{(1-t)\cdots(1-t^{m_0})}\, x^\lambda \prod_{i<j}(1 - tx_j/x_i)\ \pi_\omega\,. \qquad (37)$$

However, the fact that the normalization factor depends on λ prevents us from using (37), which is equivalent to formula (2.14) of [8], as a definition for Q_v when v is not a partition (contrary to what is stated p. 214 of [8]).

Indeed, let $n = 2$. Then

$$\begin{aligned} x^{02}(1 - tx_2/x_1)\pi_\omega &= \frac{x^{02}(1 - tx_2/x_1)}{1 - x_2/x_1} + \frac{x^{20}(1 - tx_1/x_2)}{1 - x_1/x_2} \\ &= t(x^{20} + x^{11} + x^{02}) - x^{11}\,. \end{aligned}$$

On the other hand, the relations (3) impose[8]

$$Q'_{02} = t\,Q'_{20} + (t-1)Q'_{11}\,,$$

and the same relation must be valid for Q_{02}, resulting from the transformation $X \to X(1-t)$. But

$$tQ_{20}(X) + (t-1)Q_{11}(X) = (t - t^2)\,(x^{20} + x^{11} + x^{02}) + (t-1)(1 - t + t^3)x^{11}$$

is not proportional to the image of x^{02}.

The confusing fact is that

$$(x^{02} - tx^{20} + (1-t)x^{11})\,(1 - tx_2/x_1)\,\pi_\omega = 0\ ,$$

that is, the images of monomials under the operator $(1 - tx_2/x_1)\pi_\omega$ satisfy the same relations as the functions Q_v. However, because of the different normalization factors, this does not imply that the image of x^{02} be proportional to Q_{02}.

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[8] Macdonald [8, p. 214] writes $Q_{16} = tQ_{61} + (t^2-1)Q_{52} + (t^3-t)Q_{43}$, $Q_{15} = tQ_{51} + (t^2-1)Q_{42} + (t^2-t)Q_{33}$. Read $t^m - t^{m-1}$ instead of $t^{m+1} - t^m$.

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