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FREE QUASI-SYMMETRIC FUNCTIONS, PRODUCT ACTIONS AND QUANTUM FIELD THEORY OF PARTITIONS

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Abstract. We investigate two associative products over the ring of symmetric functions related to the intransitive and Cartesian products of permutation groups. As an application, we give an enumeration of some Feynman type diagrams arising in Bender's QFT (quantum field theory) of partitions. We end by exploring possibilities to construct noncommutative analogues.

Résumé. Nous étudions deux lois produits associatives sur les fonctions symétriques correspondant aux produits intransitif et cartésien des groupes de permutations. Nous donnons comme application l'énumération de certains diagrammes de Feynman apparaissant dans la QFT (théorie quantique des champs) des partitions de Bender. Enfin, nous donnons quelques pistes possibles pour construire des analogues non-commutatifs.

1. Introduction

In a relatively recent paper, Bender, Brody and Meister introduced a special Field Theory described by

$$G(z) = \left(e^{\left(\sum_{n\geq 1} L_n \frac{z^n}{n!}\frac{\partial}{\partial x}\right)}\right)\left(e^{\left(\sum_{m\geq 1} V_m \frac{x^m}{m!}\right)}\right)\Big|_{x=0} \tag{1}$$

in order to prove that any sequence of numbers $\{a_n\}$ can be generated by a suitable set of rules applied to some type of Feynman diagrams [1, 2]. These diagrams actually are 2-coloured multigraphs with no isolated vertex.

Expanding one factor of formula (1), one can observe surprising links between: the normal ordering problem (for bosons), the parametric Stieltjes moment problem and the convolution of kernels, substitution matrices (such as generalised Stirling matrices) and one-parameter groups of analytic substitutions [8, 9, 15]. Our aim in this paper is to make more explicit the connections between symmetric functions (either commutative or noncommutative) and the Feynman diagrams (either labelled or unlabelled) arising in Bender and al.'s Quantum Field Theory of Partitions, and used in combinatorial physics [15].

The paper is organized as follows. In Section 2, we define two binary operations on $\mathfrak{S} = \bigsqcup \mathfrak{S}_n$, respectively related to the intransitive and Cartesian products of permutation groups. We prove that both operations are associative, hence giving the graded vector space $\bigoplus_{n\geq 0} \mathbb{Q}[\mathfrak{S}_n]$ the structure of a 2-associative algebra. In Section 3, we show how the latter algebraic structure can be carried over to the commutative symmetric functions, and we further investigate the 2-associative algebra *Sym* with respect to distributivity. We also take advantage of the construction to recall, in its proper context, Pólya's cycle index theorem; as an application, we use it to establish an inductive formula for the generating functions of the Feynman diagrams associated with Bender's QFT of partitions. Section 4 is dedicated to noncommutative analogues of the constructions introduced in Section 3: we show how the Feynman diagrams obtained by expanding formula (1) are related to the algebras **FQSym** and **MQSym** [6].

2. Actions of a direct product of permutation groups

2.1. **Direct product actions.** The actions of the direct product of two permutation groups (in particular, on the structure of the cycles) give rise to interesting properties related to the enumeration of unlabelled objects [14]. We open this section with the definition of two actions (namely, intransitive and Cartesian). For greater detail about these constructions (or for constructions involving the wreath product) the reader is referred to [4].

Consider two pairs (G_1, X_1) and (G_2, X_2), where each G_i is a permutation group acting on the set X_i, either finite or infinite. The *intransitive action* of $G_1 \times G_2$ on $X_1 \sqcup X_2$ (here $\sqcup$ means disjoint union) is defined by the rule

$$(\sigma_1, \sigma_2)x = \begin{cases} \sigma_1 x & \text{if } x \in X_1, \\ \sigma_2 x & \text{if } x \in X_2. \end{cases} \tag{2}$$

This action will be denoted by $(G_1, X_1) \nrightarrow (G_2, X_2) := (G_1 \times G_2, X_1 \sqcup X_2)$.

The *Cartesian action* of $G_1 \times G_2$ on $X_1 \times X_2$ is defined by

$$(\sigma_1, \sigma_2)(x_1, x_2) = (\sigma_1 x_1, \sigma_2 x_2). \tag{3}$$

This action will be denoted by $(G_1, X_1) \mathbin{\times\!\!\!\!\searrow} (G_2, X_2) := (G_1 \times G_2, X_1 \times X_2)$. Note that neither of the two binary operations just defined is commutative. A natural question to ask is whether such a structure enjoys some algebraic properties. For example, is $\mathbin{\times\!\!\!\!\searrow}$ distributive over $\nrightarrow$?

In other words, what is the meaning of

$$(G_1, X_1) \mathbin{\times\!\!\!\!\searrow} ((G_2, X_2) \nrightarrow (G_3, X_3)) = (G_1 \times G_2 \times G_3, X_1 \times (X_2 \sqcup X_3))$$

and

$$\begin{aligned}((G_1, X_1) \mathbin{\times\!\!\!\!\searrow} (G_2, X_2)) \nrightarrow ((G_1, X_1) \mathbin{\times\!\!\!\!\searrow} (G_3, X_3)) \\ = (G_1 \times G_2 \times G_1 \times G_3, (X_1 \times X_2) \sqcup (X_1 \times X_3)).\end{aligned}$$

The groups $G_1 \times G_2 \times G_1 \times G_3$ and $G_1 \times G_2 \times G_3$ are not isomorphic, so distributivity does not hold, although the set-theoretical Cartesian product is distributive over disjoint union. However an examination of the structure of the cycles (see [4] for the general construction or Section 2.2 for a particular case) shows that the cycles are the same. More precisely, a cycle can appear with different multiplicities according to which group is acting, but if we focus on the set of the cycles, the two structures are similar.

We now exhibit a construction which accounts for such a phenomenon.

2.2. **Explicit realization.** We will denote by $\circ_N$ the natural action of $\mathfrak{S}_n$ on $\{0,\dots,n-1\}$. Let $\mathfrak{S}_n$ and $\mathfrak{S}_m$ be two symmetric groups, we note by $\circ_I$ the intransitive action of $\mathfrak{S}_n\times\mathfrak{S}_m$ on $\{0,\cdots,n+m-1\}$ and by $\circ_C$ the *Cartesian action* of $\mathfrak{S}_n\times\mathfrak{S}_m$ on $\{0,\dots,nm-1\}$. More precisely, for $\sigma_1\in\mathfrak{S}_n$ and $\sigma_2\in\mathfrak{S}_m$,

$$(\sigma_1,\sigma_2)\circ_I i=\begin{cases}\sigma_1\circ_N i & \text{if } 0\le i\le n-1,\\ (\sigma_2\circ_N(i-n))+n & \text{if } n\le i\le n+m-1,\end{cases}\tag{4}$$

for $0\le i\le n+m-1$, and

$$(\sigma_1,\sigma_2)\circ_C(j+nk)=(\sigma_1\circ_N j)+n(\sigma_2\circ_N k)\tag{5}$$

for $0\le j\le n-1$ and $0\le k\le m-1$.

The *intransitive product* is the map $\nrightarrow\ :\mathfrak{S}_n\times\mathfrak{S}_m\to\mathfrak{S}_{n+m}$ defined by

$$\sigma_1\nrightarrow\sigma_2=\sigma_1\sigma_2[n],\tag{6}$$

where $\sigma_2[n]$ denotes σ_2 composed with the shifted substitution $i\mapsto i+n$ (here permutations are considered as words and $\nrightarrow$ is nothing else but shifted concatenation).

Example 2.1. Let $\sigma_1=1320\in\mathfrak{S}_4$ and $\sigma_2=534120\in\mathfrak{S}_6$. Here, we denote a permutation of $\mathfrak{S}_n$ by the word whose ith letter is the image of i under the natural action on $\{0,\dots,n-1\}$). With this notation, we obtain

$$\sigma_1\nrightarrow\sigma_2=1320978564$$

and

$$\sigma_2\nrightarrow\sigma_1=5341207986.$$

Clearly, it turns out that $\nrightarrow$ is not commutative.

The following proposition shows that the natural action of (the image under $\nrightarrow$ of $\mathfrak{S}_n\times\mathfrak{S}_m$ in) $\mathfrak{S}_{n+m}$ coincides with the intransitive action of $\mathfrak{S}_n\times\mathfrak{S}_m$.

Proposition 2.2. *We have* $(\sigma_1\nrightarrow\sigma_2)\circ_N i=(\sigma_1,\sigma_2)\circ_I i$.

Let us introduce a similar construction for the Cartesian action: we define a map $\times\!\!\!\searrow:\mathfrak{S}_n\times\mathfrak{S}_m\to\mathfrak{S}_{nm}$ by

$$\sigma_1\times\!\!\!\searrow\sigma_2=\prod_{i,j}c_i\times\!\!\!\searrow c'_j,\tag{7}$$

where $\sigma_1=c_1\cdots c_k$ (respectively $\sigma_2=c'_1\cdots c'_{k'}$) is the decomposition of σ_1 (respectively σ_2) into a product of cycles and, for two cycles $c=(i_0,\cdots,i_{l-1})$, $c'=(j_0,\cdots,j_{l'-1})$,

$$c\times\!\!\!\searrow c'=\prod_{s=0}^{\gcd(l,l')-1}(\phi(s,0),\phi(s+1,1)\cdots,\phi(\mathrm{lcm}(l,l')-1,\mathrm{lcm}(l,l')-1)),\tag{8}$$

where $\phi(k,k')=i_{k \bmod l}+nj_{k' \bmod l'}$. Just like the intransitive action, the Cartesian action coincides with the natural action of (the image under $\times\!\!\!\searrow$ of of $\mathfrak{S}_n\times\mathfrak{S}_m$ in) $\mathfrak{S}_{nm}$.

Proposition 2.3. *We have* $(\sigma_1\times\!\!\!\searrow\sigma_2)\circ_N i=(\sigma_1,\sigma_2)\circ_C i$.

Proof. From (7), it suffices to prove the property when $\sigma_1=c$ and $\sigma_2=c'$ are two cycles. But (8) is equivalent to

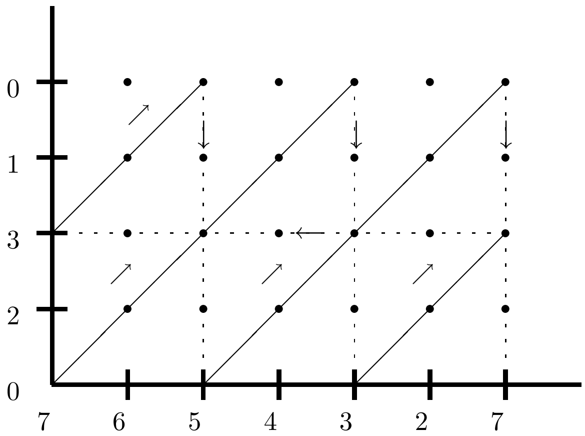
$$\begin{aligned}c\times\!\!\!\searrow c'&=\prod_{s=0}^{\gcd(l,l')-1}(i_s+nj_0,(c,c')\circ_C(i_s+nj_0),\dots,(c^{\mathrm{lcm}(l,l')-1},c'^{\mathrm{lcm}(l,l')-1})\circ_C(i_s+nj_0))\\&=\prod_{s=0}^{\gcd(l,l')-1}(i_s+nj_0,c\circ_C i_s+nc'\circ_N j_0,\dots,c^{\mathrm{lcm}(l,l')-1}\circ_N i_s+nc'^{\mathrm{lcm}(l,l')-1}\circ_N j_0)),\end{aligned}$$

which completes the proof. $\square$

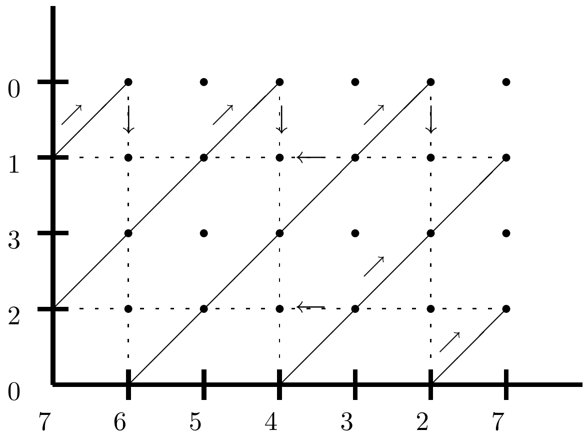
Example 2.4. Consider the two permutations $\sigma_1 = 2031 \in \mathfrak{S}_4$ and $\sigma_2 = 01723456 \in \mathfrak{S}_8$. The permutation σ_1 consists of a unique cycle $c_1 = (0, 2, 3, 1)$ and $\sigma_2 = c'_1 c'_2 c'_3$ is the product of the three cycles $c'_1 = (0)$, $c'_2 = (1)$ and $c'_3 = (7, 6, 5, 4, 3, 2)$. Hence, the permutation $\sigma_1 \nearrow\!\!\!\!\searrow \sigma_2$ is the product of four cycles given by

(1) $c_1 \nearrow\!\!\!\!\searrow c'_1 = (0, 2, 3, 1)$,
(2) $c_1 \nearrow\!\!\!\!\searrow c'_2 = (4, 6, 7, 5)$,
(3) $c_1 \nearrow\!\!\!\!\searrow c'_3 = (28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9)$
$(30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8)$.

To illustrate Proposition 2.3, it suffices to draw the cycles in the Cartesian product $\{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ whose elements are re-labelled through $(i, j) \mapsto i + nj$. For example, the two cycles appearing in $c_1 \nearrow\!\!\!\!\searrow c'_3$ give the following partition of $\{0, 1, 2, 3\} \times \{2, 3, 4, 5, 6, 7\}$.



$(28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9)$



$(30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8)$

On the other hand, the permutation $\sigma_2 \nearrow\!\!\!\!\searrow \sigma_1$ is the product of the four cycles

(1) $c'_1 \nearrow\!\!\!\!\searrow c_1 = (0, 16, 24, 8)$,
(2) $c'_2 \nearrow\!\!\!\!\searrow c_1 = (1, 17, 25, 9)$,
(3) $c'_3 \nearrow\!\!\!\!\searrow c_1 = (7, 22, 29, 12, 3, 18, 31, 14, 5, 20, 27, 10)$
$(6, 21, 28, 11, 2, 23, 30, 13, 4, 19, 26, 15)$.

Clearly, $\sigma_1 \nearrow\!\!\!\!\searrow \sigma_2 \neq \sigma_2 \nearrow\!\!\!\!\searrow \sigma_1$: the binary operation $\nearrow\!\!\!\!\searrow$ is not commutative.

2.3. **Algebraic structure.** The advantage of the new structures over the ones defined in Section 2.1 consists in the omission of the operations over the groups. Hence, algebraic properties come into light quite naturally.

First, the two operations are associative.

Proposition 2.5. Associativity.
Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be 3 permutations

(1) $\sigma_1 \nrightarrow (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \nrightarrow \sigma_2) \nrightarrow \sigma_3$
(2) $\sigma_1 \nearrow\!\!\!\!\searrow (\sigma_2 \nearrow\!\!\!\!\searrow \sigma_3) = (\sigma_1 \nearrow\!\!\!\!\searrow \sigma_2) \nearrow\!\!\!\!\searrow \sigma_3$

Proof. 1) Set $\eta_1 = \sigma_1 \nrightarrow (\sigma_2 \nrightarrow \sigma_3)$ and $\eta_2 = (\sigma_1 \nrightarrow \sigma_2) \nrightarrow \sigma_3$. One has

$$\eta_1 \circ_N i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \leq i \leq n-1, \\ \sigma_2 \circ_N (i-n) + n & \text{if } n \leq i \leq m+n-1, \\ \sigma_3 \circ_N (i-n-m) + n + m & \text{if } n+m \leq i \leq n+m+p-1, \end{cases}$$

for each $0 \leq i \leq n + m - 1$, and the same holds for $\eta_2 \circ_N i$. It follows that $\eta_1 = \eta_2$.

2) The strategy is the same. First, we set $\eta_1 = \sigma_1 \nearrow\!\!\!\!\searrow (\sigma_2 \nearrow\!\!\!\!\searrow \sigma_3)$ and $\eta_2 = (\sigma_1 \nearrow\!\!\!\!\searrow \sigma_2) \nearrow\!\!\!\!\searrow \sigma_3$. The action of η_1 can be computed as follows

$$\eta_1 \circ_N (i + ni') = \sigma_1 \circ_N i + n(\sigma_2 \nearrow\!\!\!\!\searrow \sigma_3) \circ_N i' = \sigma_1 \circ_N i + n\sigma_2 \circ_N j + nm\sigma_3 \circ_N k,$$

where $0 \le i \le n-1$, $0 \le i' \le mp-1$, $0 \le j \le m-1$ and $0 \le k \le p-1$.

On the other hand, the action of η_2 is

$$\eta_2 \circ_N (k' + nmk) = (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \circ_N k' + nm\sigma_3 \circ_N k = \sigma_1 \circ_N i + n\sigma_2 \circ_N j + nm\sigma_3 \circ_N k,$$

where $0 \le i \le n-1$, $0 \le j \le m-1$, $0 \le k \le p-1$ and $0 \le k' \le nm-1$. Hence, $\eta_1 \circ_N i = \eta_2 \circ_N i$ for $0 \le i \le nmp-1$ and $\eta_1 = \eta_2$. □

From Examples 2.1 and 2.4, neither $\mathbin{+\!\!\!\!\rightarrow}$ nor $\mathbin{\times\!\!\!\!\searrow}$ is commutative. But, one has the property of left distributivity.

Proposition 2.6. Semi-distributivity.
Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be three permutations

$$\sigma_1 \mathbin{\times\!\!\!\!\searrow} (\sigma_2 \mathbin{+\!\!\!\!\rightarrow} \sigma_3) = (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \mathbin{+\!\!\!\!\rightarrow} (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_3).$$

Proof. We use the same method as in the proof of Proposition 2.5. First, let us define $\eta_1 = \sigma_1 \mathbin{\times\!\!\!\!\searrow} (\sigma_2 \mathbin{+\!\!\!\!\rightarrow} \sigma_2)$ and $\eta_2 = (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \mathbin{+\!\!\!\!\rightarrow} (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_3)$. The action of η_1 is

$$\begin{aligned}\eta_1 \circ_N (i + nj) &= \eta_1 \circ_N i + n(\sigma_2 \mathbin{+\!\!\!\!\rightarrow} \sigma_3) \circ_N j \\ &= \begin{cases} \sigma_1 \circ_N i + n\sigma_2 \circ_N j & \text{if } 0 \le j \le m-1, \\ \sigma_1 \circ_N i + n\sigma_3 \circ_N (j-m) + m & \text{if } m \le j \le p+m-1, \end{cases}\end{aligned} \tag{9}$$

where $0 \le i \le n-1$ and $0 \le j \le m+p-1$.

On the other hand, one has

$$\eta_2 \circ_N k = \begin{cases} (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \circ_N k & \text{if } 0 \le k \le nm-1, \\ (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_3) \circ_N (k-nm) + nm & \text{if } nm \le k \le n(m+p)-1. \end{cases} \tag{10}$$

If $0 \le k \le mn-1$, we set $k = i + nj$ where $0 \le i \le n-1$ and $0 \le j \le m-1$. Hence,

$$(\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \circ_N k = \sigma_1 \circ_N i + n\sigma_2 \circ_N j. \tag{11}$$

Similarly, if $nm \le k \le n(m+p)-1$, we set $(k-nm) = i + nj$ where $0 \le i \le n-1$ and $0 \le j \le p-1$. Hence,

$$(\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_3) \circ_N (k-nm) + nm = \sigma_1 \circ_N i + n(\sigma_3 \circ_N (j-m) + m). \tag{12}$$

Substituting (11) and (12) in (10), one recovers the right hand side of (9). It follows immediately that $\eta_1 = \eta_2$. □

The two binary operations can be extended by linearity to the graded vector space $\bigoplus_{n\ge 0} \mathbb{Q}[\mathfrak{S}_n]$ and endow this space with the structure of a 2-associative algebra, i.e., a vector space equipped with 2 associative products [11]). In the next section, we construct a product $\star$ in *Sym* (the algebra of symmetric functions) defined on the power sums and appearing when one examines the cycle index of a Cartesian product. This product is the image of $\mathbin{\times\!\!\!\!\searrow}$ under a particular homomorphism of 2-associative algebras. We will prove that this last property implies the associativity of $\star$ and the distributivity of $\star$ over $\times$ (the natural product in *Sym*) and $+$.

3. Cycle index algebra

3.1. **Cartesian product in *Sym*.** We first construct a homomorphism of 2-associative algebras $\bigoplus_{n\ge 0} \mathbb{Q}[\mathfrak{S}_n] \to Sym$.

The arrow maps a permutation $\sigma \in \mathfrak{S}_n$ to a product of power sums. For $j \ge 1$, let $c_j(\sigma)$ be the number of cycles of length j in σ and set

$$\mathfrak{Z}(\sigma) = \prod_{j=0}^{\infty} p_j^{c_j(\sigma)}, \tag{13}$$

where p_i denotes the ith power sum symmetric function. We claim that $\mathfrak{Z}$ is a homomorphism of algebras mapping $\mathbin{+\!\!\!\!\rightarrow}$ to $\times$ (the usual product in *Sym*) and such that $\mathbin{\times\!\!\!\!\searrow}$

is compatible with $\mathfrak{Z}$ to the extent that there exists an associative product on Sym such that $\mathfrak{Z}$ is also a homomorphism mapping $\times\!\!\!\searrow$ to it. This second law is given on the power sums basis by

$$\prod_{1\le i\le\infty} p_i^{\alpha_i} \star \prod_{1\le j\le\infty} p_j^{\beta_j} = \prod_{1\le i,j\le\infty} p_{\mathrm{lcm}(i,j)}^{\alpha_i\beta_j \gcd(i,j)}. \tag{14}$$

(The sequences $(\alpha_i)_{i\ge1}$, $(\beta_j)_{j\ge1}$ have finite support.) It is straightforward to check the following facts.

Proposition 3.1. i) *The map* $\mathfrak{Z} : \bigoplus_{n\ge0}\mathbb{Q}[\mathfrak{S}_n] \to Sym$ *is a homomorphism of 2-associative algebras mapping the two products* $\dotplus$; $\times\!\!\!\searrow$ *respectively to* $\times$; $\star$. *(Recall that* $\times$ *denotes the usual product of Sym.) More precisely, for* $\sigma,\ \tau \in \sqcup_{n\ge0}\mathfrak{S}_n = \mathfrak{S}$ *one has*

$$\mathfrak{Z}(\sigma \dotplus \tau) = \mathfrak{Z}(\sigma)\mathfrak{Z}(\tau)\ ;\ \mathfrak{Z}(\sigma \times\!\!\!\searrow \tau) = \mathfrak{Z}(\sigma)\star\mathfrak{Z}(\tau). \tag{15}$$

ii) *The product* $\star$ *is associative, commutative and distributive over* $\times$.

Proof. i) For the first relation of (15), one just notices that $c_j(\sigma \dotplus \tau) = c_j(\sigma) + c_j(\tau)$. For the second relation, one observes that the Cartesian product of a i-cycle and a j-cycle produces $\gcd(i,j)$ cycles of length $\mathrm{lcm}(i,j)$. Thus, one has $c_r(\sigma \times\!\!\!\searrow \tau) = \sum_{\mathrm{lcm}(p,q)=r}\gcd(p,q)c_p(\sigma)c_q(\tau)$, whence (15).

ii) When $\sigma\in\mathfrak{S}_n$ is a cycle of maximum length, one has $\mathfrak{Z}(\sigma) = p_n$, hence the image of $\mathfrak{Z}$ contains also all the products of power sums and we get $\mathrm{Im}(\mathfrak{Z}) = Sym$. Then, by Proposition 3.1(i), $\star$ is distributive on the left over $\times$. Complete distributivity is a consequence of the commutativity of $\star$, which straightforwardly follows from the definition. □

The following structural result goes into particulars of the distributivity of $\star$ over $\times$.

Proposition 3.2. *Let* $\mathbb{N}^*$ *and* $\mathfrak{p}$ *respectively stand for the set of positive natural numbers and the set of prime numbers. Let* $\mathbb{N}^{(\mathbb{N}^*)}$ *(respectively* $\mathbb{N}^{(\mathfrak{p})}$*) denote the set of sequences of natural numbers indexed by* $\mathbb{N}^*$ *(respectively indexed by* $\mathfrak{p}$*) with finitely many non-zero elements. Let* P *be the set of products of power sums, i.e.,* $P = \{\prod_{i=1}^{\infty} p_i^{\alpha_i}\}_{(\alpha_i)_{i\ge1}\in\mathbb{N}^{(\mathbb{N}^*)}}$. *Then* P *is closed under* $\times$ *and* $\star$*: more precisely* $(P,\times,\star)$ *is isomorphic to a subsemiring of the* $\mathbb{Z}$*-algebra* $\mathbb{Z}[\mathbb{N}^{(\mathfrak{p})}]$ *of the monoid* $(\mathbb{N}^{(\mathfrak{p})},\sup)$ *(where* sup *stands for the componentwise supremum).*

Proof. The fact that P is closed under $\times$ and $\star$ follows from the definition and (14). Now P contains the two units (1 and p_1) of the 2-associative algebra Sym, therefore (as a consequence of the properties established for the products $\times$, $\star$) it is a semiring. For every $q\in\mathfrak{p}$ and $n\in\mathbb{N}^*$, let $\nu_q(n)$ denote the exponent of q in the decomposition of n in prime factors ($n = \prod_{q\in\mathfrak{p}} q^{\nu_q(n)}$); for a fixed n, we naturally identify $q\mapsto\nu_q(n)$ with the sequence $(\nu_2(n),\nu_3(n),\ldots)\in\mathbb{N}^{(\mathfrak{p})}$. Define an arrow $\phi : P\to\mathbb{Z}[\mathbb{N}^{(\mathfrak{p})}]$ by

$$\phi(\prod_{1\le i\le\infty} p_i^{\alpha_i}) = \sum_{1\le i\le\infty} i\alpha_i(q\mapsto\nu_q(i)). \tag{16}$$

As $\phi(m_1m_2) = \phi(m_1)+\phi(m_2)$ by construction (16), it suffices to prove that

$$\phi(p_i\star p_j) = \phi(p_i)\times_{\sup}\phi(p_j),$$

where $\times_{\sup}$ stands for the product in $\mathbb{Z}[(\mathbb{N}^{(\mathfrak{p})}, \sup)]$. But

$$\begin{aligned}\phi(p_i \star p_j) &= \phi(p_{\mathrm{lcm}(i,j)}^{\gcd(i,j)})\\ &= \gcd(i,j)\phi(p_{\mathrm{lcm}(i,j)})\\ &= \gcd(i,j)\,\mathrm{lcm}(i,j)(q \mapsto \nu_q(\mathrm{lcm}(i,j))\\ &= \gcd(i,j)\,\mathrm{lcm}(i,j)(q \mapsto \sup(\nu_q(i), \nu_q(j)))\\ &= ij(q \mapsto \sup(\nu_q(i), \nu_q(j)))\\ &= \phi(p_i) \times_{\sup} \phi(p_j).\end{aligned}$$

Since the arrow ϕ is clearly into, the claim is proved. □

3.2. **Cycle index.** Let $\mathfrak{S} = \bigsqcup_{n\geq 0} \mathfrak{S}_n$ be the disjoint union of all the symmetric groups and $\mathfrak{S}_{sg} = \bigcup_{n\geq 0} (\mathfrak{S}_n)_{sg}$ be the set of all the subgroups of all symmetric groups. For the sake of simplicity, we identify a permutation group $G \in (\mathfrak{S}_n)_{sg}$ with its action $(G, \{0, \ldots, n-1\})$ (see Section 2.1). The laws $\nrightarrow$ and $\times\!\!\!\searrow$ can be defined over $\mathfrak{S}_{sg}$; for $G_1 \in (\mathfrak{S}_n)_{sg}$ and $G_2 \in (\mathfrak{S}_m)_{sg}$, set

$$G_1 \nrightarrow G_2 := (G_1 \times G_2, \{0, \ldots, n+m-1\}), \tag{17}$$

where G_1 acts on $\{0, \ldots, n-1\}$ and G_2 acts on $\{n, \ldots, n+m-1\}$, and

$$G_1 \times\!\!\!\searrow G_2 := (G_1 \times G_2, \{0, \ldots, nm-1\}), \tag{18}$$

where the action on $\{0, \ldots, nm-1\}$ is given by $(\sigma_1, \sigma_2)k = \psi^{-1}((\sigma_1, \sigma_2)\psi(k))$, the map ψ being the bijection $\psi : \{0, \ldots, nm-1\} \to \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ defined by $\psi(i + nj) = (i, j)$ if $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$ and $(\sigma_1, \sigma_2)(i, j) = (\sigma_1 i, \sigma_2 j)$. Note that both $\nrightarrow$ and $\times\!\!\!\searrow$ are associative but $\times\!\!\!\searrow$ is not distributive over $\nrightarrow$.

Let $Z : \mathfrak{S}_{sg} \to Sym$ be defined by

$$Z(G) = \mathfrak{Z}\left(\frac{1}{|G|}\sum_{\sigma\in G} \sigma\right), \tag{19}$$

where the map $\mathfrak{Z}$ is defined by Equation 13 above.

$Z(G)$ is called *Pólya's cycle index* (or *Pólya's cycle indicator polynomial*) of G [14].

Example 3.3. (1) The cycle index of the symmetric group $\mathfrak{S}_n$ is $Z(\mathfrak{S}_n) = h_n$.

(2) The cycle index of the alternating group A_n is $Z(A_n) = h_n + e_n$.

Here h_n (respectively e_n) denotes the nth complete (respectively the nth elementary) symmetric function. These examples appear as exercises in [12, p. 29, Ex. 9].

Since $\mathfrak{Z}$ is a homomorphism of 2-associative algebras, one recovers the classical relations (see [4])

$$Z(G_1 \nrightarrow G_2) = Z(G_1)Z(G_2) \tag{20}$$

$$Z(G_1 \times\!\!\!\searrow G_2) = Z(G_1) \star Z(G_2) \tag{21}$$

Example 3.4. (1) The cycle index of the intransitive product of two symmetric groups $\mathfrak{S}_n$ and $\mathfrak{S}_m$ is

$$Z(\mathfrak{S}_n \nrightarrow \mathfrak{S}_m) = h_n h_m.$$

(2) The cycle index of the Cartesian product of two symmetric groups $\mathfrak{S}_n$ and $\mathfrak{S}_m$ is

$$Z(\mathfrak{S}_n \times\!\!\!\searrow \mathfrak{S}_m) = h_n \star h_m = \sum_{\substack{|\lambda|=n\\|\rho|=m}} m_\lambda \star m_\rho = \sum_{\substack{|\lambda|=n\\|\rho|=m}} \frac{1}{z_\lambda z_\rho} \prod_{i,j} p_{\mathrm{lcm}(\lambda_i,\rho_j)}^{\gcd(\lambda_i,\rho_j)},$$

where the sum runs over all (integer) partitions $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$ of n and all (integer) partitions $\rho = \rho_1 \geq \rho_2 \geq \cdots$ of m; m_λ denotes the monomial symmetric function indexed by λ and $z_\lambda = \prod i^{n_i} n_i!$, where n_i is the number of parts of λ equal to i.

3.3. **Enumeration of a type of Feynman diagrams related to the Quantum Field Theory of partitions.** The cycle indexes are classical tools used in combination with Pólya's theorem, for the enumeration of unlabelled objects [10, 14]. Let us review Pólya's general method.

Consider a permutation group G acting on a finite set $X = \{x_1, \cdots, x_n\}$. Let $L = \{l_0, \ldots, l_p, \ldots\}$ be another (possibly infinite) set, and $f : X \to L$. The *type* $t(f)$ of f is the vector $(i_0, \ldots, i_p, \ldots)$, where i_k is the number of elements of X whose image by f is l_k. The *shape* $s(f)$ of f is the integer partition obtained by sorting in the decreasing order $t(f)$ and erasing the zeroes. For example, a function f having the type $t(f) = (0, 1, 0, 9, 1, 2, 0, \ldots, 0, \ldots)$ has the shape $s(f) = (9, 2, 1, 1)$. The group G naturally acts on L^X by $(\sigma \circ_N f)(x) = f(\sigma \circ x)$, where $\circ$ denotes the action of G on X, and $\circ_N$ preserves the shape. Besides, Pólya's cycle index of G, $Z(G)$, is a symmetric polynomial and can expanded in the basis $\{m_\lambda\}_{\lambda \vdash n}$ of monomial symmetric polynomials. Pólya's cycle index theorem asserts that the coefficient of m_λ in this expansion is the number $d^s_\lambda(G, L)$ of G-classes on L^X with given shape λ:

$$Z(G) = \sum_\lambda d^s_\lambda(G, L) m_\lambda. \tag{22}$$

Now, let us apply this method to enumerate the Feynman diagrams arising in the expansion of formula (1). These diagrams are unlabelled 2-coloured multigraphs (or 2-coloured graphs with edges weighted by positive integers) with no isolated vertex. By a 2-coloured multigraph, we mean an undirected multigraph whose vertex set is partitioned into a set of white vertices and a set of black vertices, such that every undirected multiedge joins a white vertex with a black vertex.

First, we enumerate all unlabelled 2-coloured multigraphs. Such a computation can be found in [10], so we will only sketch the general case[1]. Henceforth, and until the end of the present section, 2-coloured multigraphs will be simply referred to as 'multigraphs'. Let n and m be the numbers of white vertices and the number of black vertices of the multigraph, respectively. We represent the multiedge set of the multigraph as a function e from $\{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ to $\mathbb{N}$. The type (respectively the shape) of a such a multigraph is the type (respectively the shape) of its multiedge set, i.e., $t(e)$ (respectively $s(e)$). The ith component of the type vector gives the number of (multi)edges with weight i.

As we consider unlabelled multigraphs, we identify multigraphs that can be obtained from one another by independently permuting the white vertices and the blacks vertices, *i.e* by a Cartesian action of an ordered pair $(\sigma_1, \sigma_2) \in \mathfrak{S}_n \times \mathfrak{S}_m$. Therefore, the number $d_I(n, m)$ of multigraphs with type I is equal to the number of orbits with type I, for the action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\mathbb{N}^{\{0,\ldots,n-1\}\times\{0,\ldots,m-1\}}$. Hence, the generating function of the shape is given by

$$g(n, m) := \sum_\lambda d^s_\lambda(n, m) m_\lambda = Z(\mathfrak{S}_n) \star Z(\mathfrak{S}_m). \tag{23}$$

Specializing the symmetric function appearing in 23 to the alphabet $\{y_0, \ldots, y_k, \ldots, \}$, the coefficient $d^t_I(n, m)$ of $\prod y_k^{i_k}$ in the expansion of $g(n, m)$ is equal to the number of multigraphs with type $I = (i_0, \ldots, i_k, \ldots)$,

$$g(n, m) = \sum_{I=(i_0,\ldots,i_p,\ldots)} d^t_I(n, m) \prod_{k=0}^{\infty} y_k^{i_k}. \tag{24}$$

[1] Of course, the following computations (and more general ones) could be carried out within the framework of the theory of species [3].

Note that one can enumerate multigraphs having (multi)edges with weights less than or equal to p by specializing to the finite alphabet $\{y_0, \dots, y_p\}$.

Let us define the generating functions of the type of our Feynman diagrams

$$F(n,m) := \sum_{I=(i_0,\dots,i_p,\dots)} f_I^t(n,m) \prod_{k=0}^{\infty} y_k^{i_k}, \tag{25}$$

where $f_I^t(n,m)$ denotes the number of Feynman diagrams of type I. Observe that $F(n,m)$ is a symmetric function over the alphabet $\{y_1, \dots, y_p, \dots\}$ but not over $\{y_0, \dots, y_p, \dots\}$.

Example 3.5. Let us give the first examples of generating functions, for weights in $\{0, 1, 2\}$.

(1) $F(1,1) = y_1 + y_2$,
(2) $F(2,1) = F(1,2) = y_1^2 + y_1y_2 + y_2^2$,
(3) $F(2,2) = y_0^2y_1^2 + y_0^2y_2^2 + y_0^2y_1y_2 + y_0y_1^3 + 3y_0y_1^2y_2 + 3y_0y_1y_2^2 + y_0y_2^3 + y_1^4 + y_1^3y_2 + 3y_1^2y_2^2 + y_1y_2^3 + y_2^4$.

One can remark that under this specialization,

$$F(2,2) + F(2,1)y_0^2 + F(1,2)y_0^2 + F(1,1)y_0^3 + y_0^4 = 3m_{22} + m_4 + 3m_{211} + m_{31} = g(2,2).$$

The latter equality can be formulated in a more general setting.

Theorem 3.6. *One has the following decomposition of the cycle index:*

$$Z(\mathfrak{S}_n \times\!\!\!\searrow \mathfrak{S}_m) = y_0^{nm} + \sum_{(1,1)\leq_{lex}(k,p)\leq_{lex}(n,m)} F(k,p)y_0^{nm-kp}. \tag{26}$$

Proof. It suffices to notice that a 2-coloured multigraph is either a 2-coloured multigraph with no isolated vertex (i.e., a Feynman diagram) or the union of some isolated vertex and a smaller 2-coloured multigraph. $\square$

This yields a nice induction formula for the $F(n,m)$'s.

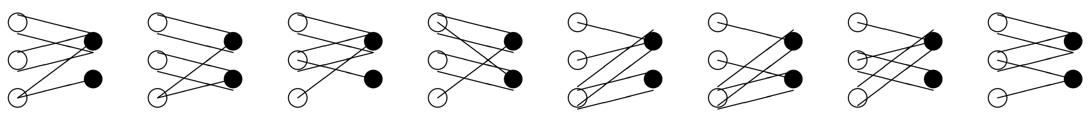
Example 3.7. From Theorem 3.6, one has

$$F(3,2) = Z(\mathfrak{S}_3 \times\!\!\!\searrow \mathfrak{S}_2) - F(3,1)y_0^3 - F(2,2)y_0^2 - F(2,1)y_0^4 - F(1,2)y_0^4 - F(1,1)y_0^5 - y_0^6.$$

From Example 3.5, it suffices to compute $F(3,1) = y_1^3 + y_2^3$ to enumerate Feynman diagrams whose edges are weighted by 0, 1 or 2. After simplification, one obtains

$$\begin{aligned} F(3,2) = {} & y_2^6 + y_2^5y_1 + 3y_2^4y_1 + 3y_2^4y_1y_0 + 2y_2^4y_0^2 + 3y_2^3y_1^3 + 6y_2^3y_1^2y_0 + 5y_2^3y_1y_0^2 \\ & + y_2^3y_0^3 + 3y_2^2y_1^4 + 3y_2^2y_1^3y_0 + 8y_2^2y_1^2y_0^2 + 3y_2^2y_1y_0^3 + y_2y_1^5 + 3y_2y_1^4y_0 + 5y_2y_1^3y_0^2 \\ & + 3y_2y_1^2y_0^3 + y_1^6 + y_1^5y_0 + y_1^3y_0^3 + 2y_1^2y_0^4. \end{aligned}$$

For example, there are eight $(2,2,2)$-Feynman diagrams:



4. Noncommutative realizations

4.1. **Free quasi-symmetric cycle index algebra.** Let $(A, <)$ be an ordered alphabet and $w \in A^*$ a word of length n. One denotes by $Std(w)$ the standardization of w, i.e., the permutation $\sigma \in \mathfrak{S}_n$ defined by

$$\sigma(i) = (\text{Number of letters} = w[i] \text{ in } w[1..i] + \text{number of letters} < w[i] \text{ in } w). \tag{27}$$

Recall that the algebra **FQSym** of free quasi-symmetric functions is defined by one of its bases, indexed by $\mathfrak{S}$ and defined as follows:

$$\mathbf{F}_\sigma = \sum_{Std(w)=\sigma^{-1}} w \in \mathbb{Z}\langle\langle A\rangle\rangle. \tag{28}$$

In [6], it is shown that **FQSym** is freely generated by the $\mathbf{F}_\sigma$'s, where σ runs over the connected permutations (see [5]) (i.e., permutations such that $\sigma([1,k]) \neq [1,k]$ for each k). The algebra **FQSym** is spanned by a linear basis, $\{\mathbf{F}^\sigma\}_{\sigma\in\mathfrak{S}}$, whose product implements the intransitive action $\nrightarrow$:

$$\mathbf{F}^\sigma = \mathbf{F}_{\sigma_1}\cdots\mathbf{F}_{\sigma_n}, \tag{29}$$

where $\sigma = \sigma_1 \nrightarrow \cdots \nrightarrow \sigma_n$ is the maximal factorisation of σ as a product of connected permutations. As a consequence of this definition, one has

$$\mathbf{F}^\sigma\mathbf{F}^\tau = \mathbf{F}^{\sigma\nrightarrow\tau}. \tag{30}$$

This naturally induces an isomorphism of algebras

$$\begin{aligned} \underline{\mathfrak{Z}} : \left(\bigoplus_{n\geq 0}\mathbb{Q}[\mathfrak{S}_n], \nrightarrow, +\right) &\rightarrow (\mathbf{FQSym}, \cdot, +) \\ \sigma &\mapsto \mathbf{F}^\sigma. \end{aligned} \tag{31}$$

One defines the product $\star$ on **FQSym** by $\mathbf{F}^\sigma \star \mathbf{F}^\tau := \mathbf{F}^{\sigma \times\!\!\!\searrow \tau}$, so $\underline{\mathfrak{Z}}$ becomes a morphism of 2-associative algebras. Furthermore, $\star$ is associative, distributive over the sum and semi-distributive over the shifted concatenation.

4.2. **Free quasi-symmetric analogue of Pólya's cycle index.** Recall that $\mathfrak{S}_{sg}$ denotes the set of all permutation groups. Following the same pattern as in the commutative setting (see Sections 3.1 and 3.2 above), one defines a map $\underline{Z} : \mathfrak{S}_{sg} \to \mathbf{FQSym}$ by

$$\underline{Z}(G) := \underline{\mathfrak{Z}}\left(\frac{1}{|G|}\sum_{\sigma\in G}\sigma\right) = \frac{1}{|G|}\sum_{\sigma\in G}\mathbf{F}^\sigma. \tag{32}$$

$\underline{Z}(G)$ will be called *Pólya's free quasi symmetric cycle index* of G.

Note 4.1. *There is another basis of* **FQSym** *indexed by permutations, namely* $\{\mathbf{G}^\sigma\}_{\sigma\in\mathfrak{S}}$. *It is obtained by setting* $\mathbf{G}_\sigma = \mathbf{F}_{\sigma^{-1}}$, *and applying the same construction as above to get a linear basis multiplicative with respect to* $\nrightarrow$ *(see Equation (30)), yields*

$$\mathbf{G}^\sigma = \mathbf{G}_{\sigma_1}\cdots\mathbf{G}_{\sigma_n}, \tag{33}$$

where $\sigma = \sigma_1 \nrightarrow \cdots \nrightarrow \sigma_n$ *is the maximal factorisation of* σ *as a product of connected permutations. In this case,* σ^{-1} *splits maximally into* $\sigma_1^{-1} \nrightarrow \cdots \nrightarrow \sigma_n^{-1}$, *so one also has* $\mathbf{G}^\sigma = \mathbf{F}^{\sigma^{-1}}$ *and formula (32) can be rewritten*

$$\underline{Z}(G) = \frac{1}{|G|}\sum_{\sigma\in G}\mathbf{G}^\sigma. \tag{34}$$

The polynomial $\underline{Z}(G)$ has properties similar to that of $Z(G)$, in particular with respect to the laws $\nrightarrow$ and $\times\!\!\!\searrow$.

Proposition 4.2. *Let* $G_1, G_2 \in \mathfrak{S}_{sg}$ *be two permutation groups, one has*

(1) $\underline{Z}(G_1 \nrightarrow G_2) = \underline{Z}(G_1)\underline{Z}(G_2)$.
(2) $\underline{Z}(G_1 \times\!\!\!\searrow G_2) = \underline{Z}(G_1) \star \underline{Z}(G_2)$.

Consider the homomorphism $z : \mathbf{FQSym} \to Sym$ defined by $z(\mathbf{F}^\sigma) = \mathfrak{Z}(\sigma)$. Note that it is not a Hopf homomorphism because $z(\mathbf{F}^{231}) = p_3$.

The following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{S}_{sg} & \xrightarrow{\underline{Z}} & \mathbf{FQSym} \\ Z\downarrow & {}^{z}\swarrow & \uparrow \underline{\mathfrak{Z}} \\ Sym & \xleftarrow[\mathfrak{Z}]{} & \bigoplus_{n\geq 0}\mathbb{Q}[\mathfrak{S}_n] \end{array} \tag{35}$$

Example 4.3. (1) The free quasi-symmetric cycle index of $\mathfrak{S}_n$ is

$$\mathbf{H}_n := \underline{Z}(\mathfrak{S}_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\mathbf{F}^{\sigma}.$$

One can regard $\mathbf{H}_n$ as a free quasi-symmetric analogue of the complete symmetric function h_n: indeed $z(\mathbf{H}_n) = Z(\mathfrak{S}_n) = h_n$.

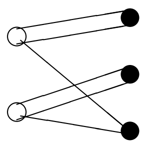
(2) One can define free quasi-symmetric analogues of elementary symmetric functions, by considering the free quasi-symmetric cycle index of the alternative groups:

$$\mathbf{E}_n := \underline{Z}(A_n) - \underline{Z}(\mathfrak{S}_n).$$

We get $z(\mathbf{E}_n) = Z(A_n) - Z(\mathfrak{S}_n) = e_n$.

(3) The knowledge of analogues of other symmetric functions should be useful to understand the combinatorics of the free quasi-symmetric cycle indexes. In particular, it should be interesting to find free quasi-symmetric functions whose images by z are the monomial symmetric functions.

4.3. **Realizations in MQSym.** We will call *labelled diagrams* the Feynman diagrams as defined in Section 3.3, but with p white (respectively q black) vertices labelled bijectively by integers in $[1..p]$ (respectively in $[1..q]$). When one draws such a diagram, one implicitly assumes that the labelling goes from top to bottom:



Labelled diagram of the matrix $\left(\begin{smallmatrix}2&0&1\\0&2&1\end{smallmatrix}\right)$

Now, to such a $p\times q$ *labelled diagram*, we can associate a matrix in $\mathbb{N}^{p\times q}$ and this correspondence is one-to-one. The condition that no vertex be isolated is equivalent to the condition that there be no complete line or column of zeroes, i.e., the representative matrix is *packed* [6]. In the same way, the diagrams are in one-to-one correspondence with the classes of packed matrices under the permutations of lines and columns as shown below (the vertical arrows are then one-to-one):

$$\begin{array}{ccc} \text{Packed matrices} & \xrightarrow{Class} & \text{Classes of packed matrices} \\ \downarrow & & \downarrow \\ \text{Labelled diagrams} & \longrightarrow & \text{Diagrams} \end{array} \tag{36}$$

There is an interesting structure of Hopf algebra (in fact an enveloping algebra) over the diagrams [7] which can be pulled back in a natural way to labelled diagrams.

The correspondence described above allows to construct a new Hopf algebra structure on **MQSym** and a Hopf algebra structure on the space spanned by the classes.

5. Conclusion

Other realizations in Hopf algebras seem feasible. For example, let us consider the Hopf algebras of graphs $GQSym^{110}$ and $GTSym^{110}$ defined in [13]. An interesting mapping from $\bigoplus_{n\geq 0}\mathbb{Q}[\mathfrak{S}_N]$ to $GQSym^{110}$ or $GTSym^{110}$ can be constructed, sending each cycle to an equivalent loop.

More precisely, J.-Y. Thibon (personal communication) showed to us how to construct a non-commutative Hopf algebra which is the dual of a quotient of a subalgebra of $GTSym^{110}$. This algebra is Hopf homomorphic to Sym and has two bases indexed by permutations, whose commutative images are proportional to power sums and monomial symmetric functions, respectively. This construction gives natural noncommutative analogues of Pólya's cycle index.

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