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THE OCTAHEDRON RECURRENCE AND RSK-CORRESPONDENCE

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Abstract. We make the statement rigorous that the Robinson–Schensted–Knuth correspondence is a tropicalization of the Dodgson condensation rule.

1. Introduction

In [3] we proposed a new approach to the RSK (Robinson-Schensted-Knuth) correspondence based on bi-crystal structure of arrays. However, it was not easy to compare our construction with the classical one. The point is that the classical RSK-correspondence uses the bumping (or sliding) procedure (see, for example [5, 11]), while our array construction is done via condensation operations. Recent papers [1, 6, 8, 7] show connections of this issue with the Dodgson rule for calculation of determinants. Roughly speaking, the RSK-correspondence is a tropicalization of the Dodgson rule. To make this statement rigorous is one of the goals of this paper.

We start with an "algebraic" RSK-correspondence due to Noumi and Yamada [8]. Given a matrix X, we consider a pyramidal array of solid minors of X. It turns out that this array satisfies an algebraic variant of the octahedron recurrence. The main observation is that this array can also be constructed with the help of some square 'genetic' array. For example, if a 'genetic' array is positive then the corresponding matrix X is totally positive. Furthermore, any totally positive matrix can be obtained in this way (cf. [1, 2]).

Next we tropicalize this algebraic construction and consider *T-polarized* pyramidal arrays (that is, arrays satisfying octahedral relations). As a result we get several bijections, viz: a) a linear bijection between non-negative arrays and supermodular functions; b) a piecewise linear bijection between supermodular functions and the so called inframodular functions; c) a linear bijection between inframodular functions and plane partitions. A composition of these bijections yields a bijection between non-negative arrays and plane partitions coinciding with the modified RSK-correspondence defined in [3].

The paper is organized as follows. In Section 2 we present an algebraic version of the octahedron recurrence (OR). In Section 3 we consider tropically polarized functions and their relation to supermodular functions. In Section 4, a construction of tropically polarized functions via the tropical 'genetic' array is given. In Section 5, we show (see Theorem 3) that the tropical OR gives a bijection between the set of

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supermodular functions and that of inframodular functions. Section 6 contains a proof of Theorem 3.

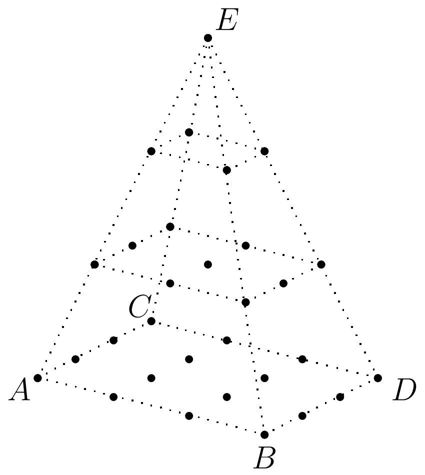
We would like to thank the referee for attracting our attention to the paper [8], and for proposing to consider our 'octahedron' RSK-correspondence from [4] as a tropicalization of the 'algebraic' RSK-correspondence due to Noumi and Yamada. We thank C. Krattenthaler and F. Zak for their numerous remarks and improvements.

2. Algebraic case

Although we are interested in two-dimensional arrays of numbers, relations between them will be constructed via three-dimensional arrays. Therefore, we shall operate also with functions on three-dimensional lattices.

Specifically, consider an $n \times n$-matrix $X = (X(r,c))$ with rows $r = 1, \dots, n$ and columns $c = 1, \dots, n$. (The general case of rectangular matrices can be reduced to this case without major difficulties. However, for simplicity, we confine ourselves to the case of square matrices.) We are interested in all solid minors of X. We arrange these minors in a three-dimensional pyramidal array.

Let $ABCDE$ be a pyramid with base $ABCD$, whose vertices $A = (0,0,0)$, $B = (2n,0,0)$, $C = (0,2n,0)$, $D = (2n,2n,0)$ are located at 'level' 0, and with top vertex $E = (n,n,n)$.



Denote by Π the subset of the pyramid consisting of the integer points (i,j,k), where i and j are equal to k modulo 2.

Given a matrix X we define a function $F = F_X$ on Π. On level zero, F is equal to 1. Elements of X are placed in the points on level one. More precisely, the number $X(r,c)$ is assigned to the point $(2r-1, 2c-1, 1)$. At level two we place the minors of X of size 2 and so on. Let $(i,j,k) \in \Pi$, so that $k \le i, j \le 2n-k$. Consider the sub-matrix of X formed by k consecutive rows $(i-k)/2+1, (i-k)/2+2, \dots, (i-k)/2+k$ and k consecutive columns $(j-k)/2+1, (j-k)/2+2, \dots, (j-k)/2+k$. We set $F(i,j,k)$ to be equal to the determinant of this sub-matrix. To visualize the construction, one has to consider a pyramidal cone (congruent to $ABCDE$) with apex at the point $(i,j,k) \in \Pi$. This cone meets level one along a sub-matrix of X.

The determinant of this sub-matrix is assigned to the point (i, j, k). For example, $F(n, n, n) = \det(X)$.

It is useful to consider the restriction of the function F to the faces of the pyramid. Suppose that a point $(i, j, k) \in \Pi$ lies on the face ABE, so that $i \geq j = k$. Then the corresponding minor is formed by the first k rows and the columns $(j-k)/2+1, (j-k)/2+2, \ldots, (j-k)/2+k$. Similarly, at the face ACE, the values of F consist of solid minors of sub-matrices whose consecutive columns start from 1.

If we put together values of F on the faces ABE and ACE, we obtain an $(n+1) \times (n+1)$ array. More precisely, consider the map

$$\alpha : \{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\} \to \Pi,$$

$$\alpha(i, j) = (2i - \min(i, j), 2j - \min(i, j), \min(i, j)).$$

Put $G = \alpha^*(F)$, that is, $G(i, j) := F(\alpha(i, j))$. For $j \leq i$, $G(i, j)$ is equal to the minor of X formed by the first j rows and columns $i - j + 1, \ldots, i$. For $i \geq j$, $G(i, j)$ is equal to the minor with the rows $j - i + 1, \ldots, j$ and the first i columns. Note, that $G(i, j) = 1$ if $ij = 0$, so actually we get an $n \times n$ array. (We denote this function by G in honor of Gauss since its construction is related to the Gauss method of elimination of unknowns.)

For a generic matrix X, the array G determines X. (By generic matrix we mean here a matrix with non-zero solid minors.)

Analogously, one can map the grid $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$ to the union of the faces DBE and DCE according to the rule

$$\beta(\tilde{i}, \tilde{j}) = (2n - 2\tilde{i} + \min(\tilde{i}, \tilde{j}), 2n - 2\tilde{j} + \min(\tilde{i}, \tilde{j}), \min(\tilde{i}, \tilde{j})).$$

The function $H = \beta^*(F)$ corresponds to the array consisting of solid minors of X adjacent to the south-east boundary of the matrix. For a generic matrix X, H also determines X.

In particular, the bijection between the G-data and H-data can be considered as a (birational) mapping from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$.

A more explicit transformation of G into H (and vice versa) is based on the following property of the function F, called the Dodgson condensation rule (see, for example, [9]). Namely, for $(i, j, k) \in \Pi$ and $k \geq 1$, we have

$$\begin{aligned} &F(i, j, k+1)F(i, j, k-1) \\ &\quad = F(i-1, j-1, k)F(i+1, j+1, k) - F(i-1, j+1, k)F(i+1, j-1, k). \quad (1) \end{aligned}$$

(The values of the function can be taken in any field or ring.)

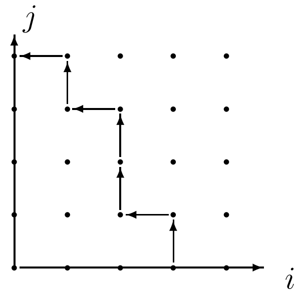
Definition. A function $F : \Pi \to \mathbb{R}$ is said to be *algebraically polarized* (or A-polarized) if $F(*, *, 0) = 1$ and the relations (1) are satisfied.

Thus, for any A-polarized function, its values at the vertices of each 'elementary' octahedron centered at (i, j, k), $i, j = k+1 \mod 2$, satisfy the relation (1). Because of this reason, one often speaks of the octahedron recurrence. If a function F is defined at any five vertices of an elementary octahedron, then its sixth value can be determined by (1). (Of course, for the generic case.) This gives a hint that an A-polarized function is determined by its values at some smaller subsets of Π. For example, an A-polarized function is determined by its values at the first level,

$F(2r-1, 2s-1, 1)$, $r, s = 1, \dots, n$ (that is by the matrix X). As well, F is determined by its values at any pair of adjacent faces (for example at faces ABE and ACE, where it is defined by the function G).

Now we discuss another construction of A-polarized functions via 'genetic' arrays.

Let us fix an array $W = (W(i,j),\ 1 \le i, j \le n)$. Pick a pair (i,j). A *path* from the column i to the row j is a sequence γ of integer pairs $(a_1, b_1), \dots, (a_s, b_s)$, such that $(a_1, b_1) = (i, 1)$, $(a_s, b_s) = (1, j)$, and for any $t = 2, \dots, s$, $(a_t, b_t) - (a_{t-1}, b_{t-1})$ equals either $(0, 1)$ or $(-1, 0)$.



A path γ from the 4-th column to the 5-th row.

For a path γ, we denote by W^γ the product $\prod_{t=1,\dots,s} W(a_t, b_t)$. Then we set

$$X(i,j) = F(2i-1, 2j-1, 1) := \sum_\gamma W^\gamma,$$

where γ runs over the set of all paths from i to j. More generally, let us consider a point $(i,j,k) \in \Pi$. It determines k consecutive rows $i_1 = (i-k)/2+1, \dots, i_k = (i-k)/2+k = (i+k)/2$ and k consecutive columns $j_1 = (j-k)/2+1, \dots, j_k = (j+k)/2$. We set

$$F(i,j,k) = \sum_{(\gamma_1,\dots,\gamma_k)} W^{\gamma_1} \cdots W^{\gamma_k},$$

where $(\gamma_1, \dots, \gamma_k)$ runs over all sets of k non-intersecting paths, the first path running from i_1 to j_1, the second running from i_2 to j_2, …, and the k-th running from i_k to j_k. As before, $F(i,j,0) = 1$.

Theorem 1. *The function F is A-polarized.*

The proof is a combination of the Dodgson rule and the Lindström-Gessel-Viennot theorem (see [10, Theorem 2.7.1]). The theorem shows that $F(i,j,k)$ is indeed equal to the minor of the matrix $X = (X(i,j))$ 'illuminated' from the point (i,j,k). More generally, the Lindström theorem asserts that, for an increasing tuple of rows $i_1, \dots, i_k$ and an increasing tuple of columns $j_1, \dots, j_k$, the corresponding minor of the matrix X is equal to

$$\sum_{(\gamma_1,\dots,\gamma_k)} W^{\gamma_1} \cdots W^{\gamma_k},$$

where the tuple $(\gamma_1, \dots, \gamma_k)$ runs over all sets of k non-intersecting paths, the first path running from i_1 to j_1, the second running from i_2 to j_2, …, and the k-th running from i_k to j_k.

Let F be the A-polarized function constructed via a 'genetic' array W. We are interested in its values at the faces ABE and ACE. Specifically, we are interested in relations between the function $G = \alpha^*(F)$ and the array W.

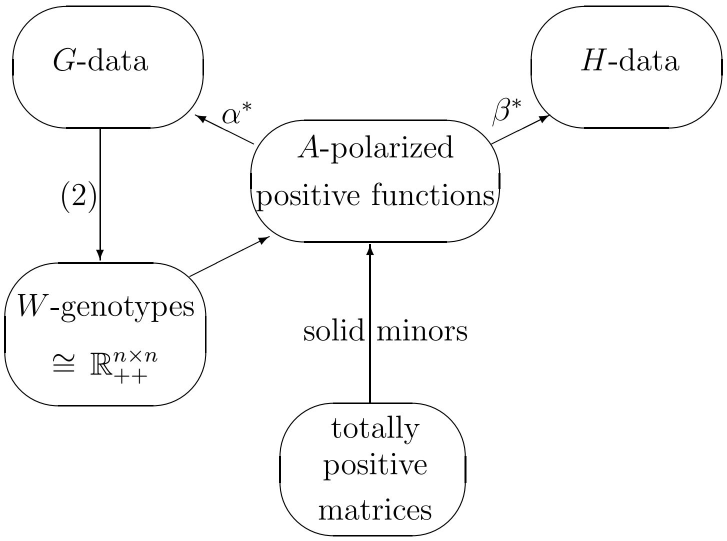
Consider a pair (i, j) with $i \geq j$. Then $G(i, j) := F(2i-j, j, j)$ is equal to the minor of X formed by the columns $i_1 = i-j+1, \ldots, i_j = i$ and rows $1, \ldots, j$. But there is only one collection of j non-intersecting paths from columns $i_1 = i-j+1, \ldots, i_j = i$ to rows $1, \ldots, j$, and these paths cover the whole rectangle $\{1, \ldots, i\} \times \{1, \ldots, j\}$. Therefore, the corresponding minor of the matrix X is equal to the product of all $W(a, b)$'s, $1 \leq a \leq i$, $1 \leq b \leq j$.

From this, we immediately obtain that F, and even G, determines W. Namely, we have

$$W(i,j) = \frac{G(i,j)G(i-1,j-1)}{G(i-1,j)G(i,j-1)}. \tag{2}$$

(Here we assume that all values $G(*,*)$ are invertible.)

For totally positive initial matrices (more precisely, for matrices with strictly positive minors) we get (after [1, 2]) the following diagram of bijections:



If we replace totally positive matrices by arbitrary matrices, we obtain birational isomorphisms between the corresponding parts of this diagrams. In [6, 8], the bijection between the set of W-genotypes ($\cong \mathbb{R}^{n\times n}_{++}$) and the set of Gauss data of type H was considered as an algebraic[1] RSK-correspondence.

In the next sections we tropicalize these constructions.

3. Tropically polarized functions

We again consider functions on the subset Π of the pyramid $ABCDE$. To distinguish from the algebraic case, we will use small letters to denote functions in the tropical (or combinatorial) case.

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[1]Such a bijection was called in [8] tropical RSK-correspondence, but it seems that using this term in the algebraic framework is a bit misleading.

Definition. A function $f : \Pi \to \mathbb{R}$ is said to be *tropically polarized* (shortly, T-polarized) if $f(*, *, 0) = 0$ and

$$f(i-1, j-1, k) + f(i+1, j+1, k) =$$
$$\max(f(i, j, k-1) + f(i, j, k+1), f(i-1, j+1, k) + f(i+1, j-1, k)) \qquad (3)$$

for any i and j compatible with $k+1$ modulo 2.

Example 1. Consider the function $q : \Pi \to \mathbb{R}$ given by $q(i, j, k) = k$. Obviously this function is T-polarized.

Example 2. Consider the following function p on Π:

$$p(i, j, k) = (i + j - 1)k.$$

It is T-polarized. In fact, for an elementary octahedron around (i, j, k) (where $i - k$ and $j - k$ are odd numbers) the sum of the values of p at end points of any diagonal of this octahedron is equal to $2(i + j - 1)k$.

Note that, for a T-polarized function f, the function $f + \alpha p$ is T-polarized for any real α.

The next example generalizes the preceding ones.

Example 3. Let ϕ and ψ be functions defined on $\{0, 2, \dots, 2n\}$. Consider the following function on Π:

$$f(i, j, k) = \phi(i - k) - \phi(i + k) + \psi(j - k) - \psi(j + k).$$

We claim that f is tropically polarized function. In fact, $f(i-1, j-1, k) + f(i+1, j+1, k)$ is equal to

$$\begin{aligned}\phi(i-k-1) - \phi(i+k-1) + \psi(j-k-1) - \psi(j+k-1)\\ + \phi(i-k+1) - \phi(i+k+1) + \psi(j-k+1) - \psi(j+k+1). \quad (4)\end{aligned}$$

One can check that $f(i, j, k-1) + f(i, j, k+1)$ is also equal to (4), as well as $f(i-1, j+1, k) + f(i+1, j-1, k)$. Thus, for an elementary octahedron, the values of f at end points of any of its diagonal are the same and the relations (3) are fulfilled. Since $f(i, j, 0) = 0$, f is T-polarized.

Denote by Pol the set of T-polarized functions on Π. The set Pol is a polyhedral complex of cones. A cone of the complex is specified by choosing a cutting of each elementary octahedron into two half-octahedron such that both halves contain the *propagation vector* $(2, 2, 0)$. To wit, each elementary octahedron is determined by its center (i, j, k), where $0 < k < n$, $k \leq i, j \leq 2n-k$ and, modulo 2, i and j differ from k. Denote by Π' the set of centers of elementary octahedra. The end points of the diagonal parallel to the propagation vector are the octahedron vertices $(i-1, j-1, k)$ and $(i+1, j+1, k) \in \Pi$. Through this diagonal we can make either a vertical cut ($\uparrow$) or a horizontal cut ($\to$) in order to split the octahedron into halves. To each function $\sigma : \Pi \to \{\uparrow, \to\}$ there corresponds a cone $C(\sigma)$ of functions for which the maximum in the left-hand side of (3) is attained at $f(i, j, k-1) + f(i, j, k+1)$ if the corresponding entry in σ is $\uparrow$, and at $f(i-1, j+1, k) + f(i+1, j-1, k)$ if the corresponding entry in σ is $\to$. Then we have $Pol = \bigcup_\sigma C(\sigma)$. The intersection of all these cones consists of functions which are affine[2] on every elementary octahedron (as in Example 3).

For a function f on Π, we denote by $f_1 = res_1(f)$ the restriction of f to the points of Π of the form $(*, *, 1)$ (that is on the level 1). Specifically, f_1 is defined on the grid $\{1, 3, \ldots, 2n-1\} \times \{1, 3, \ldots, 2n-1\}$. In contrast to the algebraic case, for a T-polarized function f, the function f_1 is not arbitrary, it is supermodular. Let us recall that a function $g : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$ is called *supermodular* if

$$g(i,j) - g(i-1,j) - g(i,j-1) + g(i-1,j-1) \geq 0 \tag{5}$$

for all i and j.

Proposition 1. *If f is a T-polarized function then f_1 is supermodular.*

Indeed, by the definition of a polarized function, we have

$$f(i-1,j-1,1) + f(i+1,j+1,1) \geq f(i-1,j+1,1) + f(i+1,j-1,1).$$

(It is clear that the restriction of f to each level of Π is a supermodular function as well.) Denote by *Supmod* the set of supermodular functions on the grid $\{1, 3, \ldots, 2n-1\} \times \{1, 3, \ldots, 2n-1\}$.

Proposition 2. *The mapping $res_1 : Pol \to Supmod$ is surjective.*

Proof. Let b be a supermodular function on $\{1, 3, \ldots, 2n-1\} \times \{1, 3, \ldots, 2n-1\}$. Define the function f on Π by the rule

$$f(i,j,k) = b(i-k+1, j-k+1) + b(i-k+3, j-k+3) + \cdots + b(i+k-1, j+k-1).$$

For example, for $k = 1$, this sum consists of a single summand $b(i,j)$, so that $f_1 = b$. It remains to verify that f is T-polarized. We do this by proving two claims.

Claim 1.

$$f(i-1,j-1,k) + f(i+1,j+1,k) \geq f(i-1,j+1,k) + f(i+1,j-1,k).$$

The left hand side is (by definition) the sum

$$\begin{aligned} &b(i-1-k+1, j-1-k+1) + \cdots + b(i-1+k-1, j-1+k-1) \\ &\qquad + b(i+1-k+1, j+1-k+1) + \cdots + b(i+1+k-1, j+1+k-1). \end{aligned}$$

Since b is supermodular, we have

$$\begin{gathered} b(i-k, j-k) + b(i-k+2, j-k+2) \geq b(i-k, j-k+2) + b(i-k+2, j-k), \\ \cdots\cdots \\ b(i+k-2, j+k-2) + b(i+k, j+k) \geq b(i+k-2, j+k) + b(i+k, j+k-2). \end{gathered}$$

Summing up these inequalities, on the left hand side we get the above sum, and on the right hand side we get the sum

$$\begin{aligned} &b(i-k, j-k+2) + \cdots + b(i+k-2, j+k) \\ &\qquad + b(i-k+2, j-k) + \cdots + b(i+k, j+k-2). \end{aligned}$$

[2] That is, $f(i-1,j-1,k) + f(i+1,j+1,k) = f(i,j,k-1) + f(i,j,k+1) = f(i-1,j+1,k) + f(i+1,j-1,k)$ for all $(i,j,k) \in \Pi$.

The first of these sums is equal to $f(i-1,j+1,k)$ and the second one is equal to $f(i+1,j-1,k)$. So, this claim is proven.

Claim 2.

$$f(i-1,j-1,k)+f(i+1,j+1,k)=f(i,j,k-1)+f(i,j,k+1).$$

In fact, both sides are equal to

$$b(i-k,j-k)+2b(i-k+2,j-k+2) \\ +\cdots+2b(i+k-2,j+k-2)+b(i+k,j+k).$$

□

We see from the proof that the mapping res_1 provides a (linear) bijection between the cone $C(\uparrow,\uparrow,\ldots,\uparrow)$ and the cone *Supmod*. For other cones in *Pol* this is not the case. Thus, in general, the 'matrix' f_1 does not determine a T-polarized function f.

In the next section we show that a tropical variant of the 'ontogenesis' proves to be more relevant.

4. Tropical ontogenesis

We consider a tropical version of the construction of functions F via genetic arrays W (see Section 2). Let $s=(s(i,j),\,1\le i,j\le n)$ be a (genetic) array of real numbers (we allow negative entries as well). For a path $\gamma=((a_1,b_1),\ldots,(a_s,b_s))$, we denote $s(\gamma)=s(a_1,b_1)+\cdots+s(a_s,b_s)$. We associate to s an (ontogenetic) function $f=\Phi(s)$ on Π by the rule

$$f(i,j,k)=\max_{(\gamma_1,\ldots,\gamma_k)}[s(\gamma_1)+\cdots+s(\gamma_k)],$$

where $(\gamma_1,\ldots,\gamma_k)$ runs over all sets of k-tuples of non-intersecting paths, the first path running from $(i-k)/2+1$ to $(j-k)/2+1$, …, and the k-th path running from $(i-k)/2+k$ to $(j-k)/2+k$.

For example, $f(2,2,2)=\max\{s(1,2)+s(1,1)+s(2,1),s(1,2)+s(2,2)+s(2,1)\}$.

Example 1′. Consider the diagonal array $s(i,j)=\delta_{ij}$. Then $\Phi(\delta)=q$, where q is the function from Example 1.

Example 2′. Consider the array $s\equiv 1$. Then $\Phi(s)=p$, where p is the function from Example 2. In fact, any path from the i-th column to the j-th row contains $i+j-1$ nodes. Therefore, for such a path γ, $s(\gamma)=i+j-1$.

Theorem 2. *The function $f=\Phi(s)$ is T-polarized.*

Proof. This is a tropicalization of the proof of Theorem 1. □

To demonstrate the machinery behind this theorem, we consider in detail one simple particular case.

Example 4. Consider the elementary octahedron centered at the point $(3,3,2)$. We would like to check the corresponding octahedron equality

$$f(2,2,2)+f(4,4,2)=\max(f(3,3,1)+f(3,3,3),f(2,4,2)+f(4,2,2)).$$

To start with, we write the values of f as sums of relevant $s(i,j)$. For the sake of brevity, we will write ij instead of $s(i,j)$, and we set $S = 11+12+13+21+22+23+31+32+33$. We have

$$\begin{aligned}
f(2,2,2) &= 11+12+21+22,\\
f(4,4,2) &= \max(S-33, S-22, S-11),\\
f(3,3,1) &= \max(12+11+21, 12+22+21),\\
f(3,3,3) &= S,\\
f(2,4,2) &= S-31-32-33,\\
f(4,2,2) &= S-13-23-33.
\end{aligned}$$

Thus we have to verify the equality

$$\begin{aligned}
&11+12+21+22+\max(S-33, S-22, S-11)\\
&= \max(\max(12+11+21, 12+22+21)+S, S-31-32-33+S-13-23-33).
\end{aligned}$$

Subtracting S, we see that the proof of the octahedron equality reduces to the verification the relation

$$\begin{aligned}
&11+12+21+22+\max(-33,-22,-11)\\
&\qquad = \max(\max(12+11+21, 12+22+21), S-31-32-33-13-23-33).
\end{aligned}$$

Subtracting $11+12+21+22$, we arrive at the obvious equality

$$\max(-33,-22,-11) = \max(\max(-22,-11),-33).$$

Denote by $Arr = \mathbb{R}\otimes(\{1,\dots,n\}\times\{1,\dots,n\}) \cong \mathbb{R}^{n\times n}$ the set of $n\times n$ arrays. Due to Theorem 2, we have a piecewise linear mapping

$$\Phi : Arr \to Pol.$$

Proposition 3. *The mapping Φ is a bijection.*

Proof. We construct the inverse mapping. Let f be a T-polarized function, and let $g = \alpha^*(f)$. Since g determines f (see Lemma 1 below), the proposition will follow from a bijection between s and g, where the array s is defined by the rule

$$s(i,j) = g(i,j) - g(i-1,j) - g(i,i-1) + g(i-1,j-1).$$

□

Note that s and g are related by an invertible linear transformation.

Thus we see that any T-polarized function f is determined by its genetic array s. As we have seen, the genotype $s \equiv 1$ determines the function p from Example 2.

One can consider a more general case. Let MD be the set of arrays s such that $s(i,j) \le s(i+1,j+1)$ for all i, j (when both sides are defined). Then, for any $(i,j,k)\in\Pi$, we have

$$\Phi(s)(i,j,k) = \sum_{2a\le i+k,\,2b\le j+k} s(a,b) - \sum_{2a'\le i-k,\,2b'\le j-k} s(a',b').$$

Indeed, for an array $s\in MD$, one can take the maximal path from column i to row j in the shape of $((i,1),\dots,(i,j),(i-1,j),\dots,(1,j))$.

5. RSK-correspondence

In Section 2 we considered two maps α and β from the square grid $\{0, 1, \dots, n\} \times \{0, 1, \dots, n\}$ to the pyramid Π. As in Section 2, one can use these maps to restrict T-polarized functions on this grid. Note that the restricted functions vanish on the south-west boundary of this grid. So, we actually deal with the functions on the grid of size $n \times n$.

Lemma 1. *The maps $\alpha^* : Pol \to \mathbb{R}^{n\times n}$ and $\beta^* : Pol \to \mathbb{R}^{n\times n}$ are bijections.*

Proof. This follows from the octahedron recurrence (3). Let us check that α^* is a surjection. Using (3), we propagate a function g from faces ABE and ACE to points of Π. That is, for every elementary octahedron such that the function is defined at five of its vertices except the end point of the diagonal parallel to the propagation vector $(2, 2, 0)$, we set the value at the remaining vertex by using (3). Suppose that as a result of this procedure some of the points of Π remain non-filled. Let p be such a point with minimal value of $i + j$. This point can not lie in the base of the pyramid or in the faces ABE and ACE since all these have values. Therefore, we can move from p in Π along the vectors $(-2, -2, 0)$, $(0, -2, 0)$, $(-2, 0, 0)$, $(-1, -1, 1)$, and $(-1, -1, 1)$. On the other hand, the function f is already defined at these nodes; hence, by (3), we can define $f(p)$.

The injectivity of α^* is proven similarly.

The case of the map β^* is dealt with in a similar way. □

Thus we have three bijections

$$\begin{array}{ccc} \mathbb{R}^{n\times n} \xleftarrow{\alpha^*} & Pol & \xrightarrow{\beta^*} \mathbb{R}^{n\times n} \\ & \Phi\big\uparrow & \\ & Arr & \end{array}$$

Now we characterize the sets $\alpha^*(\Phi(Arr_+))$ and $\beta^*(\Phi(Arr_+))$, where $Arr_+ \cong \mathbb{R}^{n\times n}_+$ is the set of non-negative arrays. The OR-map $\beta^* \circ (\alpha^*)^{-1}$ gives a bijection between these sets.

The image $\alpha^*(\Phi(Arr_+))$ admits a rather simple description. Namely, it consists of supermodular functions which vanish on the south and west boundaries ($i = 0$ or $j = 0$) of the $(n + 1) \times (n + 1)$-grid. Indeed, if $g = \alpha^*(\Phi(s))$ then $s(i, j) = g(i, j) + g(i - 1, j - 1) - g(i, j - 1) - g(i - 1, j)$ by Proposition 3. Therefore g is supermodular if and only if s is non-negative.

To describe $\beta^*(\Phi(Arr_+))$, we need a notion complementary to that of supermodularity. A function h on a square (or rectangular) grid is called *inframodular* if

$$h(i, j) + h(i + 1, j) \geq h(i, j - 1) + h(i + 1, j + 1)$$

and

$$h(i, j) + h(i, j + 1) \geq h(i - 1, j) + h(i + 1, j + 1)$$

for all i and j (when all the terms are defined). The inframodularity means that the "diagonal partial difference" $\partial_d h(i, j) := h(i, j) - h(i - 1, j - 1)$ is decreasing as a function of i and as a function of j.

A function is called *discretely concave* if it is supermodular and inframodular.

We claim that the image $\beta^*(\Phi(Arr_+))$ consists of inframodular functions. More precisely, we have the following theorem.

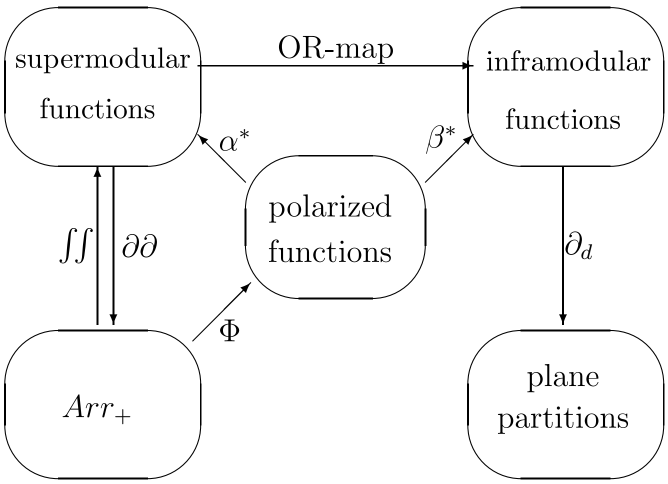
Theorem 3. *Let g be a supermodular function on $\{0,1,\ldots,n\}\times\{0,1,\ldots,n\}$ such that $g(*,0)=g(0,*)=0$. Then the function $h=\beta^*(\alpha^{*-1}(g))$ is inframodular, $h(*,0)=h(0,*)=0$, and $h(n-1,n-1)\le h(n,n)$. The converse is also true.*

We prove Theorem 3 in the next section.

While supermodular functions are related to non-negative arrays, inframodular functions are related to plane partitions. Here, by *plane partition*, we mean an arbitrary function $p:\{1,\ldots,n\}\times\{1,\ldots,n\}\to\mathbb{R}_+$ which is weakly decreasing along rows and along columns. If we replace $\mathbb{R}_+$ on $\mathbb{Z}_+$, we obtain classical plane partitions.

Now, if we have an inframodular function h on $\{0,1,\ldots,n\}\times\{0,1,\ldots,n\}$ then $p=\partial_d(h)$ is a weakly decreasing function on $\{1,\ldots,n\}\times\{1,\ldots,n\}$. If, in addition, $h(n,n)\ge h(n-1,n-1)$ then $p(n,n)\ge 0$, from which it follows that the same holds for all $p(i,j)$. Finally, the function p uniquely determines the boundary function h provided that h vanishes on the south-west boundary. To wit, $h(i,j)=p(i,j)+p(i-1,j-1)+p(i-2,j-2)+\cdots$.

Thus, due to Theorem 3, we get the following commutative diagram of bijections:



In particular, the composition $\partial_d\circ\beta^*\circ\Phi$ gives a natural bijection between the set of non-negative arrays and that of plane partitions, which can be considered as a kind of RSK-correspondence (see [3], 14.6–14.8). Since the octahedron recurrence (3) propagates integer-valued data to integer-valued data, our bijection sends non-negative integer arrays to ordinary integer-valued plane partitions.

We conclude this section by a comparison of the bijection $\partial_d\circ\beta^*\circ\Phi$ with the classical RSK-correspondence. This relationship is based on a simple natural bijection between the set of plane partitions and the set of pairs of semi-standard Young tableaus (with n rows) of the same shape. Let us recall the definition of this bijection (see the details in [11] or [3]).

Let p be an (integer-valued) plane partition. Then the following weakly decreasing n-tuple $\lambda = (p(1,1), p(2,2), \ldots, p(n,n))$ is a partition (of the number $f(n,n,n) = \sum_{a,b} s(a,b)$ when p originates from the genetic array s; the partition λ is the shape of the array s in the sense of [3]).

Now consider the $(n-1)$-tuple $\lambda' = (p(2,1), p(3,2), \ldots, p(n,n-1))$. ($\lambda'$ is the shape of the array s' obtained from s by forgetting the last column.) Because of monotonicity, the sequences λ and λ' interlace, that is

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda'_{n-1} \geq \lambda_n.$$

The same holds for the other diagonals. Thus, the restriction of p to the triangle below the principal diagonal yields a sequence of n interlacing partitions ($\lambda^{(0)} = \lambda, \lambda^{(1)} = \lambda', \lambda^{(2)}, \ldots, \lambda^{(n)} = \emptyset$), that is, a Gel'fand-Tsetlin pattern or, equivalently, a semi-standard Young tableau. Specifically, to obtain the tableau one needs to fill every horizontal strip $\lambda^{(i)} - \lambda^{(i+1)}$ with the letter $n - i$.

Analogously, the upper half of p gives another semi-standard Young tableau. This tableau has the same shape λ. Conversely, a pair of tableaux of the same shape gives a plane partition.

This permits to reformulate Theorem 3 as follows.

Theorem 3′. *The map $\partial_d \circ \beta^* \circ \Phi$ gives a bijection between non-negative integer $(n \times n)$-arrays and pairs of semi-standard Young tableaux of the same shape (filled with the alphabet $\{1, \ldots, n\}$).*

It is easy to see (using Appendix B from [3]) that the bijection coincides with the (modified) RSK-correspondence from [3]. In order to get the original RSK, we should replace one of the tableaux, viz 'Q-symbol', which corresponds to the half of the function h living on the face BED, by its Schützenberger transform.

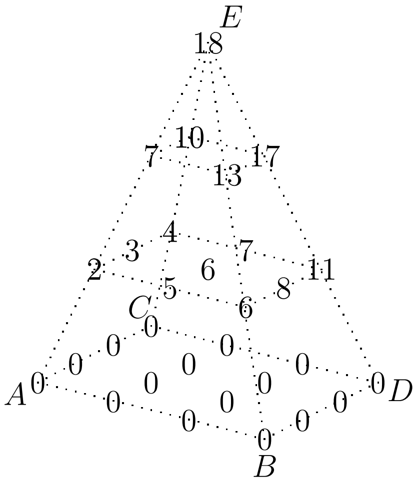
Example 5. We illustrate this theorem by an example. Consider the array

$$s = \begin{matrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 3 & 1 \end{matrix}\ .$$

The mapping $\iint$ takes s to the function

$$g = \begin{matrix} 0 & 4 & 10 & 18 \\ 0 & 3 & 7 & 13 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{matrix}\ .$$

Using the octahedron recurrence, we propagate g to Π (see the picture below).



On the faces $BED \cup CED$ we get the function

$$h = \begin{array}{cccc} 0 & 6 & 13 & 18 \\ 0 & 8 & 17 & 10 \\ 0 & 11 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{array}.$$

Applying ∂_d gives the plane partition

$$p = \begin{array}{ccc} 6 & 5 & 1 \\ 8 & 6 & 3 \\ 11 & 7 & 4 \end{array}.$$

The lower half of p gives the semi-standard Young tableau

$$\begin{array}{l} 3 \\ 2\ 2\ 2\ 3\ 3\ 3 \\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ 3 \end{array}$$

of the shape $(11, 6, 1)$. The upper half of p gives the semi-standard Young tableau

$$\begin{array}{l} 3 \\ 2\ 2\ 2\ 2\ 2\ 3 \\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 3\ 3\ 3 \end{array}$$

of the same shape $(11, 6, 1)$.

6. Proof of Theorem 3

Before proving Theorem 3, we would like to illustrate it by a simple example. Consider the function g given by

$$\begin{array}{ccc} 0 & c & e \\ 0 & a & b \\ 0 & 0 & 0 \end{array}.$$

Then its OR-image h is

$$\begin{array}{ccc} 0 & b & e \\ 0 & a' & c \\ 0 & 0 & 0 \end{array},$$

where (according to (3)) $a + a' = \max(e + 0, b + c)$.

If g is supermodular, then the following four inequalities hold:

$$b, c \geq a \geq 0 \text{ and } a + e \geq b + c.$$

In order to verify the inframodularity of h, we have to check the following five inequalities:

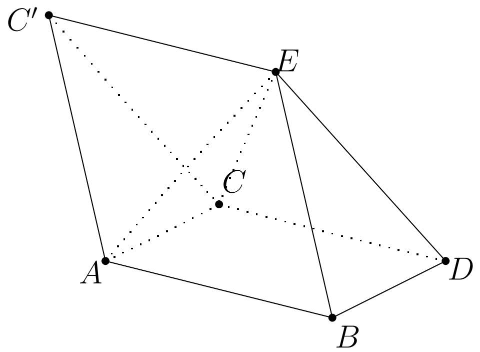
$$a' \geq b, c;\ a' + b \geq e;\ a' + c \geq e;\ e \geq a'.$$

Since $b + c \geq b + a$, we have $a' + a = \max(e, b + c) \geq b + a$ and, consequently, $a' \geq b$. Since $b \geq a$, we have $a' + a + b = \max(e, b + c) + b \geq e + b \geq e + a$; cancellation of a gives $a' + b \geq e$. Finally, since $e + a \geq e$ and $e + a \geq b + c$, we have $e + a \geq \max(e + 0, b + c) = a' + a$ and $e \geq a'$.

Vice versa, let h satisfy these five inequalities. Then adding a to both sides of the inequality $e \geq a'$, we get $e + a \geq \max(e, b + c)$. This implies $e + a \geq e$, and hence $a \geq 0$ and $e + a \geq b + c$. Since $a' + b \geq e$ and $a' + b \geq c + b$, we have $a' + b \geq \max(e, b + c) = a' + a$, that is $b \geq a$. Thus g is supermodular.

Proof of Theorem 3. We will exploit a result from [4]. For this we need to transform our data in order to satisfy the conditions of [4, Corollary 5]. To wit, we have to transform data from the pyramid to a prism.

Let us turn the face ACE around the edge AE in order to get a prism $AC'CBED$. We locate the function g on the front face $AC'EB$, and we assign zero values to the points of the triangle face $AC'C$. In accordance with (3), we propagate these data to the prism. In view of the non-negativity of g, the values of the propagated function on ACE coincide with the old values. Therefore the octahedron recurrence on the prism gives the same polarized function f on the pyramid $ABCDE$.



Now, we can make positive all negative H-breaks, $g(i,j)+g(i+1,j)-g(i,j-1)-g(i+1,j+1)$ and all negative V-breaks, $g(i,j)+g(i,j+1)-g(i-1,j)-g(i+1,j+1)$, of the function g by adding to g two appropriate functions of the variables $i+k$ and $j+k$, respectively. As a result we get that the modified function $\tilde{g}$ on $ABEC'$ (as well as the corresponding function on $AC'C$) is a discrete concave function (that is, supermodular and inframodular). By Corollary 5 of Theorem 1 in [4], the corresponding polarized function $\tilde{f}$ is a discrete concave function on Π. In particular,

$\tilde{f}$ is discretely concave on the faces DBE and DCE. Now we subtract the functions which we added in order to make positive the negative H-, V-breaks of g. We obtain the 'old' function h. One can see that this subtraction does not change the H- and V-breaks on the boundary $BED \cup CED$. Thus, the function h is inframodular.

It remains to verify $h(n-1, n-1) \leq h(n, n)$, that is $a' = f(n+1, n+1, n-1) \leq f(n, n, n) = e$. By induction we can suppose that

$$e' = f(n, n, n-2) \leq f(n-1, n-1, n-1) = a.$$

Set $b = f(n+1, n-1, n-1)$ and $c = f(n-1, n+1, n-1)$. Because of the octahedron recurrence, we get

$$a' + a = \max(e + e', b + c).$$

Supermodularity of g implies $a + e \geq b + c$. Therefore,

$$a' + a \leq \max(e + e', a + e) = \max(e', a) + e \leq \max(a, a) + e = a + e,$$

and we get $a' \leq e$.

The converse is proved by a similar construction. First we turn the face BED around the edge DE and get a square $CEB'D$. We locate the function h on this square. We set it to be identically zero on the new face $BB'D$. These data are propagated by the octahedron recurrence (3) in the opposite direction $(-2, -2, 0)$. On the pyramid, we obtain the function f. Again, we can add to h an appropriate function of $i + k$ in order to improve negative supermodular breaks of h. We get a discrete concave function $\tilde{h}$. Therefore (by the same Corollary 5 from [4]) the corresponding $\tilde{f}$ is a discrete concave function on the prism. Subtracting from this function the addendum to h, we come back to f. This subtraction does not change supermodular breaks of g except, possibly, the supermodular breaks along the main diagonal.

Thus, to complete the proof, we have to show that g has non-negative breaks along the main diagonal. That is, (in the above notations) we need to show that $a + e \geq b + c$. We have $a' \leq e$ and

$$a' + a = \max(e + e', b + c).$$

Hence

$$a + e = a + a' + e - a' \geq a + a' = \max(e + e', b + c) \geq b + c.$$

Moreover, $a + e \geq e + e'$, which proves the inequality $a \geq e'$ and yields a basis for an induction argument. This completes the proof of supermodularity of g and that of the theorem. $\square$

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