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SYMMETRY AND UNIMODALITY IN THE q, x, y-HIT NUMBERS

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Abstract. We prove symmetry, and in some cases symmetry and unimodality, of polynomials related to the q, x, y-hit numbers introduced by Haglund. These results generalize theorems proven by Haglund for the q-hit numbers. We also apply one of these results to obtain a corollary concerning a generalization of the Eulerian numbers.

Résumé. Nous prouvons la symétrie et dans certains cas la symétrie et l'unimodalité des polynômes relatifs aux q, x, y nombres de contacts introduits par Haglund, généralisant ainsi certains théorèmes. Un de ces résultats nous permet d'obtenir un corollaire à propos d'une généralisation des nombres Eulériens.

1. Introduction

1.1. **Preliminaries.** We will use the notation SQ_n to denote the $n \times n$ square chess board. We will number the columns of SQ_n with 1 through n going from left to right across the bottom, and the rows of SQ_n with 1 through n going from bottom to top. We will label a square on SQ_n in column i row j with (i, j).

More generally, a *board* will be any subset of SQ_n for some $n \in \mathbb{N}$. A *Ferrers board* is a board with non-decreasing column heights from left to right, or more precisely a board of

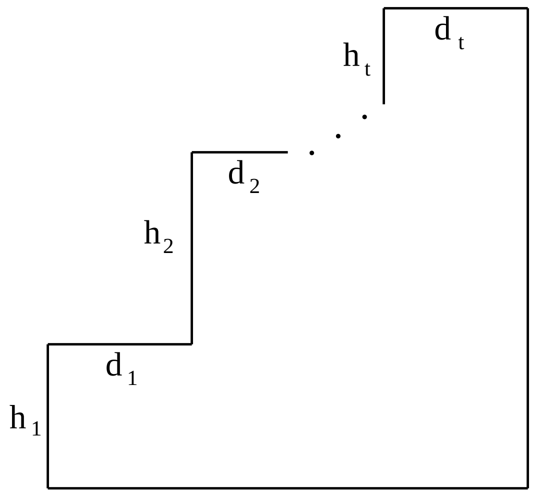

Figure 1. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$.

the form $\{(i,j) \in SQ_n | \ 1 \le j \le b_i, \ 1 \le i \le n\}$ where $b_1 \le b_2 \le \cdots \le b_n$. We will denote the Ferrers board with column heights $b_1, b_2, \ldots, b_n$ by $B(b_1, \ldots, b_n)$. We will also specify a Ferrers board by its step heights and depths. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$ is shown in Figure 1. We will call $B = B(b_1, \ldots, b_n) = B(h_1, d_1; \ldots; h_t, d_t)$ a *regular Ferrers board* if $b_i \ge i$ for $1 \le i \le n$, or equivalently if $h_1 + \cdots + h_i \ge d_1 + \cdots + d_i$ for $1 \le i \le t$ as was defined in [9]. In this paper we will focus on regular Ferrers boards.

A *rook placement* on a board $B \subseteq SQ_n$ is a subset of squares of B such that no two of these squares lie in the same row or the same column. As the name suggests, these squares represent positions on an $n \times n$ chess board where non-attacking rooks can be placed. Let $r_k(B)$ denote the number of k rook placements on B, and let $h_{n,k}(B)$ denote the number of n rook placements on SQ_n such that exactly k rooks lie on B. These are known as the *kth rook number* and the *kth hit number*, respectively, of the board B.

1.2. **Cycle-counting q-rook theory.** The cycle-counting q-rook numbers were first introduced in the unpublished work of Ehrenborg, Haglund, and Readdy [4], defined only for Ferrers boards. These rook numbers generalize both the q-rook numbers $R_k(q, B)$ of Garsia and Remmel [5], and the cycle-counting rook numbers $r_k(y, B)$ of Chung and Graham [2]. In order to describe them, we need to define the following three statistics.

The first statistic is denoted inv_B, a generalization of the number of inversions of a permutation. Given a placement P of rooks on a Ferrers board $B \subseteq SQ_n$, let each rook cancel all squares to the right in its row and below in its column. We can then define $\mathrm{inv}_B(P)$ to be the number of squares of B which neither contain a rook from P nor are cancelled.

The second statistic is denoted cyc, and is a generalization of the number of cycles of a permutation. Given a rook placement P on a board $B \subseteq SQ_n$, it is possible to associate to P a simple directed graph G_P on n vertices. This fact was first noted in [6] (see also [2] and [3]). There is an edge from i to j in G_P if and only if there is a rook from P on the square (i, j). We can then define $\mathrm{cyc}(P)$ to be the number of cycles in G_P.

The third statistic, denoted E, depends on the following fact. Given any placement P of j non-attacking rooks in columns 1 through $i-1$ of a Ferrers board B (where $j \le i-1$), it is an easy exercise to see that if $b_i \ge i$ then there is exactly one square in column i where placement of a rook will complete a new cycle in the digraph G_P. If $b_i < i$ then there is no square where placing a rook will complete a new cycle. Note that a regular Ferrers board will have such a square in each of its columns (since $b_i \ge i$ for all $1 \le i \le n$). Now for i with $b_i \ge i$ we can define $s_i(P)$ to be the unique square which, considering only rooks from P in columns 1 through $i-1$ of P, completes a new cycle. Then let $E(P)$ be the number of i such that $b_i \ge i$ and there is no rook from P in column i on or above square $s_i(P)$. For the rook placement P pictured in Figure 2, we see that $\mathrm{inv}_B(P) = 4$, $\mathrm{cyc}(P) = 2$, and $E(P) = 2$ (corresponding to $i = 4$ and $i = 5$).

We will use the common notation $[x] = (1 - q^x)/(1 - q)$ to denote the *q-analog* of the real number x, and $[n]!$ to denote the product $[n][n-1]\cdots[2][1]$, the q-analog of $n!$. For $n, k \in \mathbb{N}$ we denote by $\begin{bmatrix} n \\ k \end{bmatrix}$ the q-analog of the binomial coefficient $\binom{n}{k}$, equal to

$$\frac{[n]!}{[k]![n-k]!} = \frac{[n][n-1]\cdots[n-k+1]}{[k]!}$$

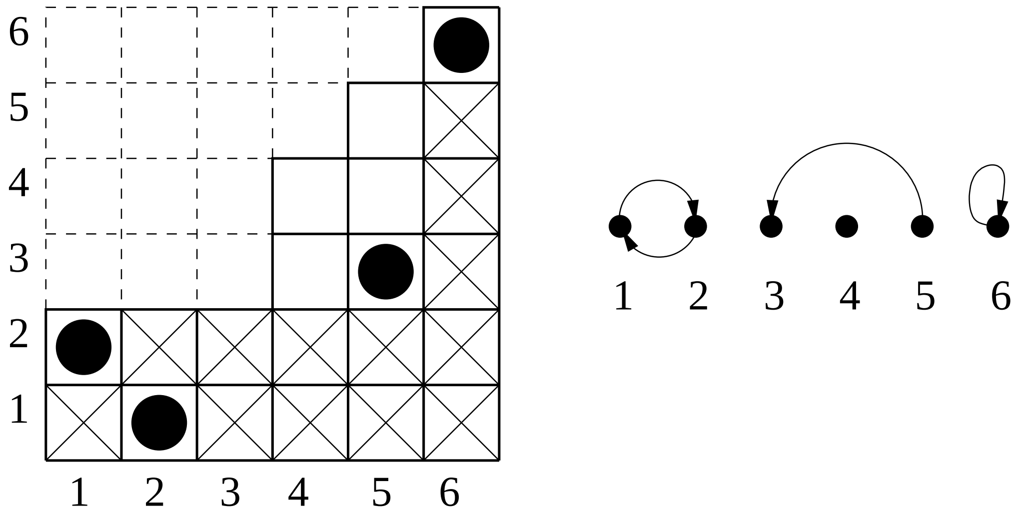

FIGURE 2. The placement P on B and the associated digraph G_P.

for $k \le n$ and equal to 0 for $k > n$. It is a well known fact that $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in q. More generally for $z \in \mathbb{C}$ we will write $\begin{bmatrix} z \\ k \end{bmatrix}$ for

$$\frac{[z][z-1]\cdots[z-k+1]}{[k]!}.$$

As in [4], we now define the *kth cycle-counting q-rook number* of a Ferrers board B by the equation

$$R_k(y,q,B) = \sum_{P \ k \text{ rooks on } B} [y]^{\mathrm{cyc}(P)} q^{\mathrm{inv}_B(P)+(y-1)E(P)}, \tag{1}$$

where the sum is taken over all placements P of k non-attacking rooks on B. Letting $y = 1$ in (1) yields the q-rook numbers of [5], and letting $q \to 1$ gives the cycle-counting rook numbers of [2]. The $R_k(y,q,B)$ satisfy the useful equation

$$\sum_{k=0}^{n} R_{n-k}(y,q,B)[z][z-1]\cdots[z-k+1] = \prod_{i \text{ with } b_i \ge i} [z + b_i - i + y] \prod_{i \text{ with } b_i < i} [z + b_i - i + 1], \tag{2}$$

a version of the well-known factorization theorems proven for the $r_k(B)$ [7], $R_k(q,B)$ [5], and $r_k(y,B)$ [2].

Haglund [9, p. 449] further extended this model by defining the *q, x, y-hit numbers* algebraically by the equation

$$\sum_{k=0}^{n} A_{n,k}(x,y,q,B) z^k = \sum_{k=0}^{n} R_{n-k}(y,q,B)[x][x+1]\cdots[x+k-1] z^k \prod_{i=k+1}^{n} (1 - zq^{x+i-1}). \tag{3}$$

The $A_{n,k}(x,y,q,B)$ generalize the $a_{n,k}(x,y,B)$ also discussed in [9] (obtained by letting $q \to 1$ in (3)), along with the q-hit numbers of Garsia and Remmel [5] (letting $x = y = 1$) and the cycle-counting hit numbers in the model of Chung and Graham [2] (when $x = 1$ and $q \to 1$).

The case $x = y$ is studied in [1], where for a regular Ferrers board B the combinatorial interpretation

$$A_{n,k}(y,y,q,B) = \sum_{\substack{P\ n \text{ rooks on } SQ_n,\\ n-k \text{ rooks on } B}} [y]^{\text{cyc}(P)} q^{(n-\text{cyc}(P))(y-1)+b_{n,B}(P)+E(P)}$$

is given. Here the sum is taken over all placements of n non-attacking rooks on SQ_n such that exactly $n-k$ of the rooks lie on B. The statistic E is as defined above, and $b_{n,B}(P)$ is the number of squares on SQ_n which neither contain a rook from P nor are cancelled, after applying the following cancellation scheme:

(1) each rook cancels all squares to the right in its row;
(2) each rook on B cancels all squares above it in its column (squares both on B and strictly above B);
(3) each rook on B which is also on a square which completes a cycle cancels all squares below it in its column as well;
(4) each rook off B cancels all squares below it but above B.

While no combinatorial interpretation is known for the $A_{n,k}(x,y,q,B)$ when $x \neq y$, the author suspects that one exists similar to that given for the $a_{n,k}(x,y,B)$ in [9, p. 418]. Such an interpretation would enhance the results that follow.

In Section 2 we sketch an easy proof of the symmetry and unimodality of $A_{n,k}(a,b,q,B)$ for $a, b \in \mathbb{N}$. Our proof for regular Ferrers boards is a simplified version of that given in [10], for an analogous result concerning the q-hit numbers. We also deduce two corollaries in this section. In Section 3, we prove symmetry of the polynomial $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ for any regular Ferrers board $B = B(h_1,d_1;\ldots;h_t,d_t)$. Finally in Section 4, we prove unimodality of $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ for a certain class of regular Ferrers boards.

2. Symmetry and Unimodality of $A_{n,k}(a,b,q,B)$

If $B = B(h_1,d_1;\ldots;h_t,d_t) \subseteq SQ_n$ is a Ferrers board, let us denote by $B - h_p - d_p$ the Ferrers board $B(h_1,d_1;\ldots;h_p-1,d_p-1;\ldots h_t,d_t) \subseteq SQ_{n-1}$, obtained from B by decreasing the pth step by 1. We will write Area(B) for the number of squares in the board B.

Suppose

$$f(q) = \sum_{i=M}^{N} a_i q^i,$$

is a polynomial in q with $a_M, a_N \neq 0$. We call $M+N$ the *virtual degree* of f. We will say the polynomial $f(q)$ is zsu(d) if either

(1) $f(q)$ is identically zero, or
(2) $f(q)$ is in $\mathbb{N}[q]$, symmetric, and unimodal with virtual degree d.

Note that for $s \in \mathbb{N}$, q^s is zsu$(2s)$ and $[s]$ is zsu$(s-1)$. It is also easy to see that if f and g are polynomials which are both zsu(d), then $f+g$ is also zsu(d). We will use the following lemmas to prove the main proposition of this section. A proof of Lemma 2.1 can be found in [11].

Lemma 2.1. *If f is* $\mathrm{zsu}(d)$ *and g is* $\mathrm{zsu}(e)$*, then fg is* $\mathrm{zsu}(d+e)$*.*

Lemma 2.2. *Let $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $B - h_t - d_t \subseteq SQ_{n-1}$ as described earlier. Then*

$$\begin{aligned} A_{n,k}(x,y,q,B) &= [k+y+d_t-1]A_{n-1,k}(x,y,q,B-h_t-d_t) \\ &\quad + q^{k+y+d_t-2}[n+x-y-d_t-k+1]A_{n-1,k-1}(x,y,q,B-h_t-d_t) \end{aligned}$$

for any $1 \leq k \leq n$.

Proof. Let $p = t$ in Lemma 5.7 of [9]. □

The following is now a simple corollary of the above lemmas. We offer a brief sketch of the proof.

Proposition 2.3. *Let $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $a, b \in \mathbb{N}$. If $n + a + 1 \geq b + d_t + k$, then $A_{n,k}(a, b, q, B)$ is* $\mathrm{zsu}(\mathrm{Area}(B) + n(b+k-1) + k(a-1) - \binom{n+1}{2})$ *for $0 \leq k \leq n$.*

Proof. The proof is by induction on $\mathrm{Area}(B)$. When $k = 0$, we use Lemma 2.1 with (2) and (3) to prove that

$$A_{n,0}(a,b,q,B) = \prod_{i=1}^{n} [b_i - i + b],$$

which is $\mathrm{zsu}(\mathrm{Area}(B) + n(b-1) - \binom{n+1}{2})$. We then use Lemma 2.2, Lemma 2.1, and the fact that two polynomials which are $\mathrm{zsu}(d)$ sum to another polynomial which is $\mathrm{zsu}(d)$ for the case when $k > 0$. Note the assumption $n + a + 1 \geq b + d_t + k$ is necessary to ensure that the factor $[n + a - b - d_t - k + 1]$ in the recurrence is a polynomial in q. □

An immediate corollary is the following.

Corollary 2.4. *For any regular Ferrers board $B \subseteq SQ_n$ and $m \in \mathbb{N}$, the polynomial*

$$\sum_{\substack{P \text{ } n \text{ rooks on } SQ_n, \\ n-k \text{ rooks on } B}} [m]^{\mathrm{cyc}(P)} q^{(m-\mathrm{cyc}(P))(y-1)+b_{n,B}(P)+E(P)}$$

is $\mathrm{zsu}(\mathrm{Area}(B) + n(m+k-1) + k(m-1) - \binom{n+1}{2})$*.*

Proof. Let $x = y = m$ in Proposition 2.3, and use the combinatorial interpretation for $A_{n,k}(y, y, q, B)$ given in Section 1.2. Note that assuming B is a regular Ferrers board, we always have $n + m + 1 \geq m + d_t + k$. This is because the last d_t columns of B have height n, so in a placement of n non-attacking rooks on SQ_n with k rooks off B, we must have $n - d_t \geq k$. □

A cycle-counting version of the Eulerian numbers is given in [1], defined by the equation

$$\tilde{E}_{n,k}(y,q) = \sum_{\sigma \in S_n,\ \mathrm{des}(\sigma)=k-1} [y]^{\mathrm{rlmin}(\sigma)} q^{(n-\mathrm{rlmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)}. \tag{4}$$

Here $\mathrm{des}(\sigma)$ denotes the number of descents, and $\mathrm{rlmin}(\sigma)$ the number of right-to-left minima, in the permutation σ. A *right-to-left minimum* of $\sigma_1\sigma_2\cdots\sigma_n$ is an entry σ_i which is smaller than σ_j for all $j > i$ (so for example $\mathrm{rlmin}(51243) = 3$, corresponding to the right-to-left minima 3, 2, and 1). Note that right-to-left minima have the same overall distribution as cycles in S_n, justifying the term "cycle-counting." It was proven in [1] that

$$\tilde{E}_{n,k}(y,q) = A_{n,k-1}(y,y,q,\mathbb{T}_n), \tag{5}$$

where $\mathbb{T}_n = B(1,2,\ldots,n)$ denotes the triangular Ferrers board. In light of (5) and Proposition 2.3, the following can be easily proven.

Corollary 2.5. *For $m \in \mathbb{N}$, the polynomial*

$$\sum_{\sigma\in S_n,\ \mathrm{des}(\sigma)=k-1} [m]^{\mathrm{rlmin}(\sigma)} q^{(n-\mathrm{rlmin}(\sigma))(m-1)+\mathrm{maj}(\sigma)}$$

is $\mathrm{zsu}(n(m+k-2)+(k-1)(m-1))$.

3. Symmetry of $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$

In this section we prove a more general symmetry result for all regular Ferrers boards, namely the symmetry of the polynomial

$$\frac{A_{n,k}(a,b,q,B)}{\prod_{i=1}^{t}[d_i]!},$$

where $B = B(h_1,d_1;\ldots;h_t,d_t)$. Throughout the rest of the paper we will use the notation H_i for the partial sum $h_1+\cdots+h_i$, and D_i for $d_1+\cdots+d_i$. We have the following lemmas.

Lemma 3.1. *Let $B = B(b_1,\ldots,b_n) = B(h_1,d_1;\ldots;h_t,d_t)$ be a regular Ferrers board, $j \in \mathbb{N}$. Then*

$$\frac{\prod_{i=1}^{n}[j+b_i-i+y]}{\prod_{i=1}^{t}[d_i]!} = \prod_{i=1}^{t}\begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}.$$

Proof. We see that

$$\prod_{i=1}^{n}[j+b_i-i+y] = \prod_{i=1}^{t}\Big([j+H_i-D_{i-1}+y-1][(j+H_i-D_{i-1}+y-1)-1] \\ \cdots[(j+H_i-D_{i-1}+y-1)-d_i+1]\Big).$$

Thus

$$\frac{\prod_{i=1}^{n}[j+b_i-i+y]}{\prod_{i=1}^{t}[d_i]!} \\ = \prod_{i=1}^{t}\frac{[j+H_i-D_{i-1}+y-1]\cdots[(j+H_i-D_{i-1}+y-1)-d_i+1]}{[d_i]!},$$

which is

$$\prod_{i=1}^{t}\begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}$$

by definition. $\square$

Lemma 3.2. *Let $B = B(b_1, \ldots, b_n) = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board. Then*

$$\frac{A_{n,k}(x, y, q, B)}{\prod_{i=1}^{t}[d_i]!} = \sum_{j=0}^{k} \begin{bmatrix} n+x \\ k-j \end{bmatrix} \begin{bmatrix} x+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \prod_{i=1}^{t} \begin{bmatrix} j + H_i - D_{i-1} + y - 1 \\ d_i \end{bmatrix}.$$

Proof. By Lemma 5.1 of [9], we have

$$A_{n,k}(x, y, q, B) = \sum_{j=0}^{k} \begin{bmatrix} n+x \\ k-j \end{bmatrix} \begin{bmatrix} x+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \prod_{i=1}^{n} [j + b_i - i + y].$$

The lemma now follows trivially from Lemma 3.1. $\square$

We can now prove the following.

Theorem 3.3. *Let $B = B(h_1, d_1; \ldots; h_t, d_t)$ be a regular Ferrers board (so $H_i \geq D_i$ for $1 \leq i \leq t$). Let $a, b \in \mathbb{N}$ with $a \geq b \geq 1$, and set*

$$L_k^{a,b}(B) = \text{Area}(B) + n(b-1) + k(n+a-1) - \sum_{i=1}^{t} d_i D_i.$$

Then $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$ is either zero or symmetric with virtual degree $L_k^{a,b}(B)$.

Proof. By Lemma 3.2, $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]! =$

$$\sum_{j=0}^{k} \begin{bmatrix} n+a \\ k-j \end{bmatrix} \begin{bmatrix} a+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \prod_{i=1}^{t} \begin{bmatrix} j + H_i - D_{i-1} + b - 1 \\ d_i \end{bmatrix},$$

which is a polynomial in q (the first two q-binomial coefficients in each summand are clearly polynomials, and the third is since $H_i \geq D_i \geq D_{i-1}$ and $b \geq 1$). Using the fact that $\begin{bmatrix} r \\ s \end{bmatrix}$ is symmetric with virtual degree $s(r-s)$ and Lemma 2.1, we see that each term on the right side above has virtual degree $(k-j)(n+a-k+j) + j(a-1) + (k-j)(k-j-1) + \sum_{i=1}^{t} d_i(j + H_i - D_i + b - 1)$ (which is exactly $L_k^{a,b}(B)$). We then conclude that if $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$ is non-zero, then it is symmetric with virtual degree $L_k^{a,b}(B)$. $\square$

4. Unimodality of $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$

In this section we give some sufficient conditions on the regular Ferrers board B for the polynomial of the previous section to also be unimodal. Let us first define some more notation.

Suppose we have integers $h_1, \ldots, h_t$, $d_1, \ldots, d_t$, and $e_1, \ldots, e_t$ with $d_i \in \mathbb{P}$, $h_i \in \mathbb{N}$, and $0 \leq e_i \leq d_i$. We will denote the vector $(e_1, e_2, \ldots, e_t)$ by $\vec{e}$. We will continue to denote the partial sum $h_1 + \cdots + h_i$ by H_i, $d_1 + \cdots + d_i$ by D_i, and we will also let $E_i = e_1 + \cdots + e_i$. We make the convention that $H_0 = D_0 = E_0 = 0$. For fixed $h_1, \ldots, h_t$ and $d_1, \ldots, d_t$ we can define

$$P(\vec{e}, x, y) = \prod_{i=1}^{t} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} + y - 1 \\ d_i - e_i \end{bmatrix} \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} + x - y \\ e_i \end{bmatrix}$$

and prove the following lemmas.

Lemma 4.1. *Let $B = B(h_1, d_1; \dots h_{t-1}, d_{t-1}; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $B' = B(h_1, d_1; \dots; h_{t-1}, d_{t-1}) \subseteq SQ_{H_{t-1}}$. Then*

$$A_{n,k}(x,y,q,B) = [d_t]! \sum_{s=k-d_t}^{k} A_{H_{t-1},s}(x,y,q,B') \begin{bmatrix} y+d_t+s-1 \\ d_t-k+s \end{bmatrix} \times \begin{bmatrix} n-y-d_t+x-s \\ k-s \end{bmatrix} q^{(k-s)(y+k-1)}.$$

Proof. Let $p = t$ in Corollary 5.10 of [9] and note that because B is a regular Ferrers board, $H_t = D_t = n$. □

Lemma 4.2. *Let $B = B(h_1, d_1; \dots; h_t, d_t)$ be a regular Ferrers board. Then*

$$A_{n,k}(x,y,q,B) = \prod_{i=1}^{t} [d_i]! \sum_{e_1+\cdots+e_t=k,\ 0 \le e_i \le d_i} P(\vec{e}, x, y) \prod_{i=1}^{t} q^{e_i(H_i - D_i + E_i + y - 1)}. \tag{6}$$

Proof. By induction on t. When $t = 1$ we have that $d_1 = n$, and Lemma 4.1 gives us

$$A_{n,k}(x,y,q,B) = [d_1]! \sum_{s=k-n}^{k} A_{0,s}(x,y,q,\emptyset) \begin{bmatrix} y+n+s-1 \\ d_1-k+s \end{bmatrix} \times \begin{bmatrix} n-y-n+x-s \\ k-s \end{bmatrix} \times q^{(k-s)(y+k-1)}. \tag{7}$$

In this case we have that $H_1 = D_1 = d_1 = n$ and $D_0 = H_0 = 0$, so we get that the $s = 0$ term in (7) is equal to

$$[d_1]! \begin{bmatrix} H_1 - D_0 + y - 1 \\ d_1 - k \end{bmatrix} \begin{bmatrix} D_1 + D_0 - H_1 + x - y \\ k \end{bmatrix} \times q^{k(H_1 - D_1 + k + y - 1)}. \tag{8}$$

Note that by definition

$$A_{0,s}(x,y,q,\emptyset) = \delta_{s,0},$$

so the only nonzero summand in (7) occurs when $s = 0$ and hence (8) is actually equal to (7). Finally if we recall that $E_1 = e_1$ and $E_0 = 0$, we can rewrite (8) as

$$[d_1]! \sum_{e_1=k,\ 0 \le e_1 \le d_1} \begin{bmatrix} H_1 - D_0 + E_0 + y - 1 \\ d_1 - e_1 \end{bmatrix} \times \begin{bmatrix} D_1 + D_0 - H_1 - E_0 + x - y \\ e_1 \end{bmatrix} \times q^{e_1(H_1 - D_1 + E_1 + y - 1)},$$

which is exactly of the form of (6).

For $t > 1$, Lemma 4.1 gives that

$$A_{n,k}(x,y,q,B) = [d_t]! \sum_{E_{t-1}=E_t-d_t}^{E_t} A_{H_{t-1},E_{t-1}}(x,y,q,B') \begin{bmatrix} y+d_t+E_{t-1}-1 \\ d_t-e_t \end{bmatrix} \times \begin{bmatrix} n-y-d_t+x-E_{t-1} \\ e_t \end{bmatrix} \times q^{e_t(y+E_t-1)}. \quad (9)$$

Here we are letting $E_{t-1} = s$ and defining $e_t = k - s$ and $E_t = E_{t-1} + e_t = k$. Since B is regular $H_t = D_t = n$, so $H_t - D_{t-1} = D_t - D_{t-1} = d_t$ and (9) can be rewritten as

$$A_{n,k}(x,y,q,B) = [d_t]! \sum_{e_t=0}^{d_t} A_{H_{t-1},E_{t-1}}(x,y,q,B') \begin{bmatrix} H_t - D_{t-1} + E_{t-1} + y - 1 \\ d_t - e_t \end{bmatrix} \times \begin{bmatrix} D_t + D_{t-1} - H_t - E_{t-1} + x - y \\ e_t \end{bmatrix} \times q^{e_t(H_t - D_t + E_t + y - 1)}.$$

By the inductive hypothesis, the above is equal to

$$[d_t]! \sum_{e_t=0}^{d_t} \left\{ \prod_{i=1}^{t-1} [d_i]! \sum_{e_1+\cdots+e_{t-1}=E_{t-1},\ 0\le e_i\le d_i} \prod_{i=1}^{t-1} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} + y - 1 \\ d_i - e_i \end{bmatrix} \times \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} + x - y \\ e_i \end{bmatrix} q^{e_i(H_i - D_i + E_i + y - 1)} \right\} \times \begin{bmatrix} H_t - D_{t-1} + E_{t-1} + y - 1 \\ d_t - e_t \end{bmatrix} \begin{bmatrix} D_t + D_{t-1} - H_t - E_{t-1} + x - y \\ e_t \end{bmatrix} q^{e_t(H_t - D_t + E_t + y - 1)},$$

which is

$$\prod_{i=1}^{t} [d_i]! \sum_{e_1+\cdots+e_t=k,\ 0\le e_i\le d_t} P(\vec{e},x,y) \prod_{i+1}^{t} q^{e_i(H_i - D_i + E_i + y - 1)},$$

as desired. □

Lemma 4.3. *Let $B = B(h_1, d_1; \ldots; h_t, d_t)$ be a regular Ferrers board, $a, b \in \mathbb{N}$ with $a \ge b \ge 1$. Let e_i, d_i, h_i, E_i, D_i, and H_i be as in the definition of $P(\vec{e}, x, y)$. Assume that B is such that $d_{i-1} + d_i \ge h_i$ for $1 \le i \le t$ (where $d_0 := 0$). If any of the numerators of the q-binomial coefficients in*

$$P(\vec{e},a,b) = \prod_{i=1}^{t} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} + b - 1 \\ d_i - e_i \end{bmatrix} \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} + a - b \\ e_i \end{bmatrix}$$

are negative, then $P(\vec{e}, a, b) = 0$.

Proof. First note that $H_i - D_{i-1} + E_{i-1} + b - 1 \ge 0$ for $1 \le i \le t$, since $H_i \ge D_i \ge D_{i-1}$ and $b \ge 1$, so none of the numerators in the first q-binomial coefficient of the product are ever negative.

Now suppose that $D_k + D_{k-1} - H_k - E_{k-1} + a - b < 0$ for some k with $0 \le k \le t$. Note $D_1 + D_0 - H_1 - E_0 + a - b = d_1 - h_1 + a - b$, and since we assumed $d_{i-1} + d_i \ge h_i$ (and in particular $d_1 \ge h_1$) and $a \ge b$, we have that $d_1 - h_1 + a - b \ge 0$. Thus we see that such a k must be greater than 2.

Now choose j such that $D_i + D_{i-1} - H_i - E_{i-1} + a - b \geq 0$ for $1 \leq i < j$, but $D_j + D_{j-1} - H_j - E_{j-1} + a - b < 0$ (such a j exists because of the remarks in the previous paragraph). Then $D_j + D_{j-1} - H_j - E_{j-1} + a - b < 0$ implies $D_j + D_{j-1} - H_j - E_{j-2} + a - b < e_{j-1}$, which is equivalent to $d_j + d_{j-1} - h_j + D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b < e_{j-1}$, which implies $D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b < e_{j-1}$ (since $d_j + d_{j-1} \geq h_j$). Hence

$$\begin{bmatrix} D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b \\ e_{j-1} \end{bmatrix} = 0,$$

since the numerator is non-negative by definition of j but less than the denominator, so the product $P(\vec{e}, a, b)$ is 0 as well. □

We are now ready to prove the main theorem of this section, the unimodality of the $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$.

Theorem 4.4. *Let $B = B(h_1, d_1; \ldots; h_t, d_t)$ be a regular Ferrers board such that $d_{i-1} + d_i \geq h_i$ for $1 \leq i \leq t$. Let $a, b \in \mathbb{N}$ with $a \geq b \geq 1$, and set*

$$L_k^{a,b}(B) = \mathrm{Area}(B) + n(b-1) + k(n + a - 1) - \sum_{i=1}^{t} d_i D_i$$

as before. Then $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$ is $\mathrm{zsu}(L_k^{a,b}(B))$.

Proof. We apply Lemma 4.2, which says that

$$\frac{A_{n,k}(a, b, q, B)}{\prod_{i=1}^{t}[d_i]!} = \sum_{e_1 + \cdots + e_t = k,\ 0 \leq e_i \leq d_i} P(\vec{e}, a, b) \prod_{i=1}^{t} q^{e_i(H_i - D_i + E_i + b - 1)},$$

and all of the terms on the right hand side above are in $\mathbb{N}[q]$ by Lemma 4.3. Using the fact that $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ is $\mathrm{zsu}(s(r-s))$ (for a proof of the unimodality see [8]) along with Lemma 2.1, each term above is $\mathrm{zsu}(\sum_{i=1}^{t}\{(d_i - e_i)(H_i - D_i + E_i + b - 1) + e_i(D_i + D_{i-1} - H_i - E_i + a - b) + 2e_i(H_i - D_i + E_i + b - 1)\})$. A simple calculation shows this is the same $\mathrm{zsu}(L_k^{a,b}(B))$. Thus $A_{n,k}(a, b, q, B)/\prod_{i=1}^{t}[d_i]!$ is $\mathrm{zsu}(L_k^{a,b}(B))$ as well. □

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