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ALGEBRA INVARIANTS FOR FINITE DIRECTED GRAPHS WITH RELATIONS

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Abstract. Finite directed graphs with relations define finite-dimensional algebras given as factor algebras of the corresponding path algebras. In joint work with T. Holm, we studied invariants of an associated generalized Cartan matrix for certain types of algebras where the Cartan invariants basically count nonzero paths; the main motivation for this came from the context of derived module categories and derived invariants. For the so-called (skewed) gentle algebras we computed explicit combinatorial formulae for the invariant factors of the Cartan matrices. Based on the talk at the FPSAC'05 conference in Taormina, some of this work is surveyed here, and some further results on algebras of a different type are presented in the final section.

1. Introduction

Finite directed graphs play an important rôle in the representation theory of finite-dimensional algebras, where they are called *quivers*. For a given quiver Q and a field K one obtains an algebra KQ by taking all paths in Q, including the trivial paths of length 0 at each vertex, as a K-basis, and defines multiplication as induced by concatenation of paths; this is called the *path algebra* to the quiver Q. The importance of the path algebras lies in a famous result by Gabriel that any finite-dimensional K-algebra over an algebraically closed field K is Morita equivalent to a factor algebra KQ/I, where I is an admissible ideal of KQ (i.e., contained in the square of the ideal of KQ generated by the arrows). Thus such factor algebras KQ/I are central objects of study in the representation theory of algebras; this situation is referred to as a *quiver with relations*. There are many interesting representation theoretic properties of such algebras for which one tries to find "combinatorial" methods for computing them. It is of particular importance to consider representation theoretic parameters of the algebra which are invariants for appropriate equivalence classes of algebras. In recent years a focus in the representation theory of algebras has been the investigation of derived equivalences of algebras; this is a homological notion: two algebras are derived equivalent if their derived module categories are equivalent. A lot of progress has recently been made in this very active area. It is a difficult problem to find invariants of algebras preserved by derived equivalences; only a few important representation theoretic parameters are known to be indeed invariants under derived equivalence, such as the number of simple modules, the dimension of the center of the algebra or the dimension of its Hochschild cohomology groups.

Here, we will discuss invariants of the *Cartan matrix* of a finite-dimensional algebra $A = KQ/I$. Cartan matrices contain crucial structural information on the algebra, as their entries are the multiplicities of simple A-modules as composition factors of projective indecomposable A-modules. It is in general difficult to compute the entries of the Cartan matrix and some famous conjectures are related to their properties. The main point to note here is that the unimodular equivalence class of the Cartan matrix of a finite dimensional algebra is invariant under derived equivalence.
From a combinatorial point of view it is important that for a finite-dimensional algebra $A = KQ/I$ given by a quiver with relations, the entries of the Cartan matrix can be computed by considering paths in the quiver Q which are non-zero in the algebra A.

The algebras we study here are the (skewed-) gentle algebras which are defined combinatorially by conditions on the quiver and relations. Gentle algebras occur naturally in many places in the representation theory of finite dimensional algebras, in particular in connection with derived categories. They made their first appearance in 1981 [1] when it was shown that the algebras which are derived equivalent to hereditary algebras of type $\mathbb{A}$ are precisely the gentle algebras whose underlying undirected graph is a tree. The algebras which are derived equivalent to hereditary algebras of type $\tilde{\mathbb{A}}$ are certain gentle algebras whose underlying graph has exactly one cycle [2]. Only recently it was proved that the class of gentle algebras has the remarkable property of being closed under derived equivalence [10]. For more background on the algebraic context the reader is referred to [5].

In joint work with T. Holm [5] we have studied the Cartan matrices of (skewed-)gentle algebras and some refinements and generalization of this notion; the starting point of our investigation was a recent result by T. Holm [8] giving an explicit combinatorial formula for the Cartan determinants of gentle algebras. In [5], these results are improved to a determination of the invariant factors of the Cartan matrix C_A of a gentle algebra $A = KQ/I$, and the formulae are also extended to skewed-gentle algebras. Sections 2 and 3 survey some of these results, and (as in the talk at FPSAC'05) we take the opportunity to include an alternative proof for the fact that the number of oriented cycles with full zero relations in such a quiver is bounded by the number of vertices. The key to the refinement of the combinatorial analysis of the quiver is to put a weight on the paths according to their lengths instead of just counting them; in our context, this makes good sense as the relations on the quiver are homogeneous (in fact, they are even more special). Taking an indeterminate q corresponding to the weight of an arrow, this gives us a q-Cartan matrix $C_A(q)$ for the algebra; this may also be considered as a so-called filtered Cartan matrix, counting the multiplicities of the simple modules in the radical layers of the algebra. Setting $q = 1$ gives the ordinary Cartan matrix C_A. We have already pointed out that the unimodular equivalence class of the Cartan matrix of a finite-dimensional algebra is invariant under derived equivalence. But unfortunately, not even the determinant of the q-Cartan matrix is in general an invariant under derived equivalence.

There are further finite-dimensional algebras given by quivers and relations for which it is possible to determine the q-Cartan matrices and obtain nice formulae for their invariants or at least their determinant. A further family of algebras associated with cyclic quivers is considered in the final section 4; here, the q-Cartan matrices are special circulant matrices and we compute their determinants and invariants explicitly.

Finally, in more recent joint work with T. Holm [6] we have also dealt with a class of infinite-dimensional algebras which we termed *locally gentle algebras*; this class is again defined combinatorially by relaxing the conditions on the gentle quiver. Also, in this context it is then well motivated to study more general weights on the quiver. The combinatorial point of view adds further interest to these quivers, as for example secant configurations in regular $2n$-polygons may be viewed as examples of locally gentle quivers. For details the reader is referred to [6].

2. Gentle algebras

In this section, we want to describe an extension and refinement of the result on the determinant of the Cartan matrix of a gentle algebra from [8] obtained in [5] (see [5] for details).

First we have to give the definition of gentle algebras. They form an important subclass of the class of special biserial algebras which we now define.

Let K be an algebraically closed field. Let Q be a quiver, i.e., a finite directed graph, with set of vertices Q_0. Let I be an admissible ideal of the path algebra KQ, i.e., $I \subseteq J^2$, where J is the ideal of KQ generated by the arrows.
For a path p in Q we denote by $s(p)$ its start vertex and $t(p)$ its end vertex.
The pair (Q, I) is called *special biserial* if the following holds:
(i) For any vertex $v \in Q_0$ the set of lengths of the paths starting in v and not being in I is finite.
(ii) Each vertex $v \in Q$ is the end point of at most two arrows and the starting

point of at most two arrows.
(iii) For every arrow α there is at most one arrow β with $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$, and there is at most one arrow γ with $t(\gamma) = s(\alpha)$ and $\gamma\alpha \notin I$.

A special biserial pair (Q, I) is called *gentle*, if furthermore:
(iv) There is a generating set of I (as ideal) consisting of paths of lengths 2.
(v) For any arrow α there is at most one arrow β with $t(\alpha) = s(\beta)$ and $\alpha\beta \in I$, and there is at most one arrow γ with $t(\gamma) = s(\alpha)$ and $\gamma\alpha \in I$.

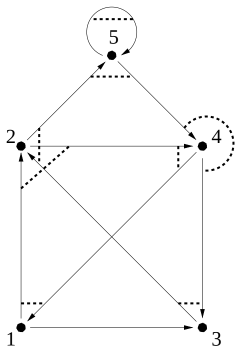
A K-algebra A is called *gentle* (resp. *special biserial*) if it is Morita equivalent to an algebra KQ/I, for (Q, I) gentle (resp. special biserial).
As we assume that the set of vertices Q_0 of Q is finite, condition (i) implies that the algebra KQ/I is finite-dimensional.
As mentioned in the introduction, in [6] the notion of gentle quivers is generalized to locally gentle quivers; for these quivers, condition (i) need not be satisfied, and thus the corresponding algebras can be infinite-dimensional.

By condition (iv), the relations for a gentle quiver are homogeneous, so the length of a non-zero path in KQ/I is well-defined. Then we define the q-Cartan matrix of $A = KQ/I$ as $C_A(q) = (c_{ij}(q))_{i,j\in Q_0}$, where for $i, j \in Q_0$

$$c_{ij}(q) = \sum_{n\geq 0} a_n(i,j)q^n$$

with $a_n(i, j)$ the number of paths from i to j which are non-zero in A (and different in A).

Example. In the following picture, the dotted lines (or arcs) between two arrows or in a loop correspond to the generating relations for the quiver, i.e., they indicate that composing the corresponding arrows is zero in the algebra $A = KQ/I$.



Here is the q-Cartan matrix for this gentle quiver with relations:

$$C_A(q) = \begin{pmatrix} 1+q^5 & q+q^2+q^7 & q+q^4+q^6+q^9 & q^3+q^4+q^8 & q^2+q^3 \\ q^4 & 1+q^6 & q^2+q^5+q^8 & q+q^3+q^7 & q+q^2 \\ 0 & q & 1+q^3 & q^2 & 0 \\ q & q^3 & q+q^2+q^5 & 1+q^4 & 0 \\ q^2+q^3 & q^4+q^5 & q^3+q^4+q^6+q^7 & q+q^2+q^5+q^6 & 1+q \end{pmatrix}$$

The following property does not only hold for gentle algebras but for those where we have dropped the final condition (v) in the definition of gentle pairs; in [5], it is a useful reduction tool in the proof of the main result.

Lemma 2.1. *Let $A = KQ/I$ be a special biserial algebra, where I is generated by paths of length 2. Let α be an arrow in Q, not a loop, such that there is no arrow β with $s(\alpha) = t(\beta)$ and $\beta\alpha \in I$, or there is no arrow γ with $t(\alpha) = s(\gamma)$ and $\alpha\gamma \in I$. Let Q' be the quiver obtained from Q by removing the arrow α, I' the corresponding relation ideal and $A' = KQ'/I'$. Then the q-Cartan matrices $C_A(q)$ and $C_{A'}(q)$ are unimodularly equivalent (over $\mathbb{Z}[q]$).*

For the combinatorially defined gentle algebras we can provide an explicit combinatorial description for a very nice normal form of its q-Cartan matrix; below, we say that a matrix C is unimodularly equivalent (over $\mathbb{Z}[q]$) to a matrix D, if there are matrices S, T with entries in $\mathbb{Z}[q]$ and determinant 1 such that $SCT = D$.

Theorem 2.2. [5] *Let $A = KQ/I$ be a gentle algebra, defined by a gentle pair (Q, I). Denote by c_k the number of oriented k-cycles in Q with full zero relations. Then the q-Cartan matrix $C_A(q)$ is unimodularly equivalent (over $\mathbb{Z}[q]$) to a diagonal matrix with entries $(1 - (-q)^k)$, with multiplicity c_k, $k \geq 1$, and all further diagonal entries being* 1.

This result has some immediate nice consequences.

Corollary 2.3. *Let $A = KQ/I$ be a gentle algebra, and denote by c_k the number of oriented k-cycles in Q with full zero relations. Then the q-Cartan matrix $C_A(q)$ has determinant*

$$\det C_A(q) = \prod_{k\geq 1}(1 - (-q)^k)^{c_k} .$$

Example. In the example given before, the q-Cartan matrix has determinant

$$\det C_A(q) = 1 + q + q^3 - q^5 - q^7 - q^8 = (1 + q)(1 + q^3)(1 - q^4) .$$

Indeed, the quiver has a loop at vertex 5 (which is a 1-cycle with a zero relation), a 3-cycle with full zero relations from vertex 1 to 2 to 4 and back to 1, and a 4-cycle with full zero relations running over 2,5,4,3 and back to 2.

For an algebra $A = KQ/I$ as above, let $ec(A)$ and $oc(A)$ be the number of oriented cycles in Q with full zero relations of even and odd length, respectively. Setting $q = 1$, the corollary above immediately implies the main result from [8] which was in fact the starting point of our investigations in [5].

Corollary 2.4. *Let $A = KQ/I$ be a gentle algebra. Then for the determinant of its ordinary Cartan matrix C_A the following holds.*

$$\det C_A = \begin{cases} 0 & \textit{if } ec(A) > 0 \\ 2^{oc(A)} & \textit{else} \end{cases}$$

The Theorem also gives the following corollary.

Corollary 2.5. *Let $A = KQ/I$ be a gentle algebra. Then there are at most $|Q_0|$ oriented cycles with full zero relations in the quiver Q.*

Instead of deriving this from the main result (as done in [5]), this may be proved directly. As an illustration, we give this alternative proof here.
Clearly the result holds when $|Q_0| = 1$. So we assume now that $|Q_0| > 1$, and we prove the result by induction.
We may remove all arrows going in or out of a vertex and not having a zero relation at this vertex, without losing the property of the quiver with relations being gentle and without changing the number of vertices and the number of oriented cycles with full relations. After this removal, all vertices are of degree 0, 2 or 4, and there is a zero relation at all vertices of degree 2. By induction, we may assume that there is no vertex of degree 0 and that the quiver is connected. If all vertices are of degree 4, then there are paths of arbitrary lengths, contradicting the property of being gentle. Hence there is a vertex of degree 2, which then belongs to a unique oriented cycle with full zero relations. Removing this vertex and the arrows incident to it reduces the number of vertices as well as the number of oriented cycles with full zero relations by 1 (note that any arrow in a gentle quiver belongs to at most one oriented cycle with full zero relations). Hence the result follows by induction.

Remark 2.6. Note that in our context $|Q_0|$ is the number $l(A)$ of simple A-modules, which is also invariant under derived equivalence. Hence this implies that the Cartan determinant of a gentle algebra A is at most $2^{l(A)}$.

Recall that the property of an algebra being gentle is invariant under derived equivalence [10]. Also, we have pointed out earlier that the invariant factors of the ordinary Cartan matrix $C_A = C_A(1)$ are invariants of the derived equivalence class of the algebra $A = KQ/I$. Thus we now have some easily computable invariants for gentle algebras to distinguish the derived equivalence classes.

Corollary 2.7. *Let $A = KQ/I$ and $A' = KQ'/I'$ be gentle algebras as above. If A is derived equivalent to A', then $ec(A) = ec(A')$ and $oc(A) = oc(A')$.*

Our new invariants are a quite powerful tool for distinguishing gentle algebras up to derived equivalence which cannot be separated by the more classical invariants. An illustration on how to use the invariants to tell non-equivalent gentle algebras apart is given in [5], where the 9 gentle algebras with two simples and the 18 gentle algebras with three simples and vanishing Cartan determinant are discussed in detail.

3. Skewed-gentle algebras

Also skewed-gentle algebras are defined combinatorially. They were introduced in [7]; for the notation and definition we follow here mostly [4], but we try to explain how the construction works rather than stating the technical definitions.

We start with a gentle pair (Q, I). A set Sp of vertices of the quiver Q is an admissible set of *special* vertices if the quiver with relations obtained from Q by adding loops with square zero at these vertices is again gentle; we denote this gentle pair by $(Q^{\mathrm{sp}}, I^{\mathrm{sp}})$. The triple (Q, Sp, I) is then called *skewed-gentle.*

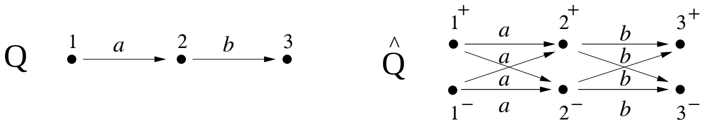
We want to point out that the admissibility of the set Sp of special vertices is both a local as well as a global condition. Let v be a vertex in the gentle quiver (Q, I); then we can only add a loop at v if v is of degree 1 or 0 or if it is of degree 2 with a non-loop zero relation. Hence only vertices of this type are potential special vertices. But for the choice of an admissible set of special vertices we also have to take care of the global condition that after adding all loops, the pair $(Q^{\mathrm{sp}}, I^{\mathrm{sp}})$ still does not have paths of arbitrary lengths.

Given a skewed-gentle triple (Q, Sp, I), we now construct a new quiver with relations $(\hat{Q}, \hat{I})$ by doubling the special vertices, introducing arrows to and from these vertices corresponding to the previous such arrows and replacing a previous zero relation at the vertices by a mesh relation.
More precisely, we proceed as follows. The non-special vertices in Q are also vertices in the new quiver; any arrow between non-special vertices as well as corresponding relations are also kept. Any special vertex $v \in Sp$ is replaced by two vertices v^+ and v^- in the new quiver. An arrow a in Q from a non-special vertex w to v (or from v to w) will be doubled to arrows $a^{\pm} : w \to v^{\pm}$ (or $a^{\pm} : v^{\pm} \to w$, resp.) in the new quiver; an arrow between two special vertices v, w will correspondingly give four arrows between the pairs $v^{\pm}$ and $w^{\pm}$. We say that these new arrows lie over the arrow a. Any relation $ab = 0$ where $t(a) = s(b)$ is non-special gives a corresponding zero relation for paths of length 2 with the same start and end points lying over ab. If v is a special vertex of degree 2 in Q, then the corresponding zero relation at v, say $ab = 0$ with $t(a) = v = s(b)$, is replaced by mesh commutation relations saying that any two paths of length 2 lying over ab, having the same start and end points but running over v^+ and v^-, respectively, coincide in the factor algebra to the new quiver with relations $(\hat{Q}, \hat{I})$. We will speak of $(\hat{Q}, \hat{I})$ as a skewed-gentle quiver *covering* the gentle pair (Q, I). Note that also here the generating relations are homogeneous.

A K-algebra is then called *skewed-gentle* if it is Morita equivalent to a factor algebra $K\hat{Q}/\hat{I}$, where $(\hat{Q}, \hat{I})$ comes from a skewed-gentle triple (Q, Sp, I) as described above.

Examples. (1) We take the gentle quiver Q as shown below, with relation ideal I generated by ab and ba. Then we can take $Sp = \{2\}$, i.e., only the vertex 2 is specified as a special vertex. This gives the skewed-gentle quiver $\hat{Q}$ shown below, with relation ideal $\hat{I}$ generated by $a^+b^+ - a^-b^-$, $b^\pm a^\pm$, $b^\pm a^\mp$.

(2) We take the gentle quiver Q as shown below, with relation ideal generated by ab. This time we take $Sp = Q_0$, i.e., all vertices are chosen to be special. This gives the skewed-gentle quiver $\hat{Q}$ shown below, where for simplicity all arrows lying over a or b, respectively, are also marked a or b, respectively, but for writing down the relations generating the relation ideal $\hat{I}$ we will put signs on, so that e.g. a_+^- denotes the arrow going from 1^+ to 2^-. In this notation the generating relations are given by $a_+^+b_+^+ - a_+^-b_-^+$, $a_+^+b_+^- - a_+^-b_-^-$, $a_-^+b_+^+ - a_-^-b_-^+$, $a_-^+b_+^- - a_-^-b_-^-$.



Our result on gentle algebras generalizes nicely to skewed-gentle algebras (see [5] for details) as follows.

Theorem 3.1. [5] *Let (Q, I) be a gentle quiver, $(\hat{Q}, \hat{I})$ a covering skewed-gentle quiver. Let $\hat{A} = K\hat{Q}/\hat{I}$ be the corresponding skewed-gentle algebra. Denote by c_k the number of oriented k-cycles in (Q, I) with full zero relations.*
Then the q-Cartan matrix $C_{\hat{A}}(q)$ is unimodularly equivalent (over $\mathbb{Z}[q]$) to a diagonal matrix with entries $1 - (-q)^k$, with multiplicity c_k, $k \geq 1$, and all further diagonal entries being 1.

Remark 3.2. Thus, the q-Cartan matrix $C_A(q)$ for the gentle algebra A to (Q, I), and the q-Cartan matrix $C_{\hat{A}}(q)$ for a skewed-gentle cover $\hat{A}$ are unimodularly equivalent to diagonal matrices which only differ by adding as many further 1's on the diagonal as there are special vertices chosen in Q; in particular, with notation as above,

$$\det C_{\hat{A}}(q) = \det C_A(q) = \prod_{k \geq 1} (1 - (-q)^k)^{c_k}$$

and thus also for the ordinary Cartan matrices

$$\det C_A = \det C_{\hat{A}} \, .$$

4. Cycles and circulants

There are some more types of algebras for which one can determine the unimodular equivalence class of their q-Cartan matrices. As an example, we consider q-Cartan matrices to cyclic quivers with relations which occur in other interesting contexts.
In this situation, circulants make an appearance, and we first define the relevant notation.

For $x = (x_1, \dots, x_n)$ the circulant matrix to x is

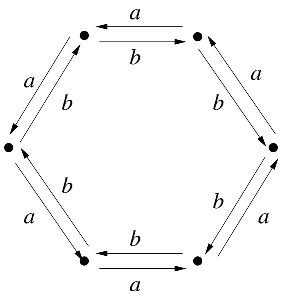
$$C = \begin{pmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ x_n & x_1 & \dots & x_{n-2} & x_{n-1} \\ \vdots & & & & \\ x_2 & x_3 & \dots & x_n & x_1 \end{pmatrix} .$$

Let ω be a primitive n-th root of unity in $\mathbb{C}$. Then C has the eigenvalues

$$\sum_{k=1}^{n} x_k (\omega^j)^{k-1} \; , \; j = 1, \dots, n.$$

Special circulants appear as q-Cartan matrices for cyclic quivers with monomial relations and commutation relations which are homogeneous of degree 2.

Example. Here is the corresponding cyclic quiver for the case of 6 vertices:



where we take as generating relations: a^2, b^2, $ab - ba$ (at each vertex of the quiver).

The q-Cartan matrix for such a cyclic quiver is just the circulant to $v = (1 + q^2, q, 0 \dots, 0, q)$. We will describe a more precise result on these q-Cartan matrices below, but as it is easy to do we compute here the determinant. Note that as in the case of gentle quivers we have a contribution $1 - (-q)^n$ for each of the two oriented cycles with full zero relations in the quiver.

Theorem 4.1. *Let q be an indeterminate, $v = (1 + q^2, q, 0 \dots, 0, q)$ (of length n), and $C(q)$ the circulant to v. Then we have*

$$\det C(q) = (1 - (-q)^n)^2 .$$

Proof. By the above, we have

$$\begin{aligned}\det C(q) &= \prod_{j=1}^{n}(1+q^2+q\omega^j+q(\omega^j)^{n-1}) = \prod_{j=1}^{n}(1+q^2+q(\omega^j+\overline{\omega^j})) \\ &= \prod_{j=1}^{n}(q+\omega^j)(q+\overline{\omega^j}) = (\prod_{j=1}^{n}(q+\omega^j))^2\end{aligned}.$$

Now $-\omega^j$ is a zero of q^n-1, if n is even and a zero of q^n+1, if n is odd, for all j, hence we have the assertion. ☐

In fact, it is not hard to transform the circulant $C(q)$ to $v=(1+q^2,q,0\dots,0,q)$ into a better form:

Theorem 4.2. *Assume $n\geq 3$, and let $C(q)$ be as above.*
(i) *Over $\mathbb{Z}[q]$, we can transform $C(q)$ unimodularly into the form*

$$\begin{pmatrix} E_{n-2} & 0 & 0 \\ 0 & 1-(-q)^n & \left(\sum_{j=1}^{n-1} q^{2j-1}\right)-(-q)^{n-1} \\ 0 & 0 & 1-(-q)^n \end{pmatrix}$$

where E_{n-2} is the identity matrix of type $n-2$.
(ii) *Over $\mathbb{Q}[q]$, we have the following unimodular equivalences:*

For n odd, $\quad C(q)\sim \operatorname{diag}(1^{n-2},\sum_{j=0}^{n-1}(-q)^j, q^{n+1}+q^n+q+1)\,.$

For n even, $\quad C(q)\sim \operatorname{diag}(1^{n-2},\sum_{j=0}^{(n-2)/2} q^{2j}, q^{n+2}-q^n-q^2+1)\,.$

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