Séminaire Lotharingien de Combinatoire **54A** (2006), Article B54Af

A COMBINATORIAL INTERPRETATION FOR THE COEFFICIENTS IN THE KRONECKER PRODUCT $s_{(n-p,p)} * s_\lambda$

Cristina M. Ballantine[1*] and Rosa C. Orellana[2**]

[1]College of the Holy Cross, Worcester, MA 01610, USA
Email: cballant@holycross.edu

[2]Dartmouth College, Hanover, NH 03755, USA
Email: Rosa.C.Orellana@Dartmouth.edu

Abstract. In this paper we give a combinatorial interpretation for the coefficient of s_ν in the Kronecker product $s_{(n-p,p)} * s_\lambda$, where $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)}) \vdash n$, if $\ell(\lambda) \geq 2p-1$ or $\lambda_1 \geq 2p-1$; that is, if λ is not a partition inside the $2(p-1) \times 2(p-1)$ square. For λ inside the square our combinatorial interpretation provides an upper bound for the coefficients. In general, we are able to combinatorially compute these coefficients for all λ when $n > (2p-2)^2$. We use this combinatorial interpretation to give characterizations for multiplicity free Kronecker products. We have also obtained some formulas for special cases.

Introduction

Let χ^λ and χ^μ be the irreducible characters of S_n (the symmetric group on n letters) indexed by the partitions λ and μ of n. The *Kronecker product* $\chi^\lambda\chi^\mu$ is defined by $(\chi^\lambda\chi^\mu)(w) = \chi^\lambda(w)\chi^\mu(w)$ for $w \in S_n$. Then $\chi^\lambda\chi^\mu$ is the character that corresponds to the diagonal action of S_n on the tensor product of the irreducible representations indexed by λ and μ. We have

$$\chi^\lambda\chi^\mu = \sum_{\nu\vdash n} g_{\lambda,\mu,\nu}\chi^\nu,$$

where $g_{\lambda,\mu,\nu}$ is the multiplicity of χ^ν in $\chi^\lambda\chi^\mu$. Hence the coefficients $g_{\lambda,\mu,\nu}$ are non-negative integers.

By means of the Frobenius map one defines the Kronecker (internal) product on the Schur symmetric functions by

$$s_\lambda * s_\mu = \sum_{\nu\vdash n} g_{\lambda,\mu,\nu} s_\nu.$$

A formula for decomposing the Kronecker product is unavailable, although the problem has been studied for nearly one hundred years. In recent years Lascoux [5], Remmel [7, 8], Remmel and Whitehead [9] and Rosas [11] derived closed formulas for Kronecker products of Schur functions indexed by two row shapes or hook shapes. Gessel [3]

*Partially supported by the Fulbright Commission.
**Partially supported by the Wilson Foundation.

obtained a combinatorial interpretation for zigzag partitions. However, a combinatorial interpretation is still lacking even in the case when both λ and μ are two row partitions.

The objective of this paper is to provide a combinatorial interpretation for the Kronecker coefficients comparable to the Littlewood-Richardson rule which is defined in terms of the so-called Littlewood-Richardson tableaux (see Section 1 for the definition). In this paper we give a combinatorial interpretation for the coefficient of s_ν in $s_{(n-p,p)} * s_\lambda$, if $\lambda_1 \geq 2p-1$ or $\ell(\lambda) \geq 2p-1$, in terms of what we call *Kronecker Tableaux*. In particular, our combinatorial interpretation holds for all λ if $n > (2p-2)^2$. For a general λ, the number of Kronecker tableaux always gives an upper bound for the Kronecker coefficients. Furthermore, using our combinatorial rule we obtain that $g_{(n-p,p),\lambda,\nu} = 0$ whenever the intersection of λ and ν has less than p boxes.

The techniques we use to obtain our main theorem are purely combinatorial and rely both on the Jacobi-Trudi identity and on the Garsia–Remmel rule [4] for decomposing the Kronecker product of a homogeneous symmetric function and a Schur symmetric function.

One can easily deduce from existing formulas that $s_{(n-1,1)} * s_\lambda$ is multiplicity free if and only if $\lambda = (a^k, b^l)$ or $\lambda = (a^k)$, where a, k, b, l are non-negative integers. If $p \geq 2$, we have used our combinatorial rule to determine the partitions λ for which the Kronecker product $s_{(n-p,p)} * s_\lambda$ is multiplicity free. We have determined that if $n \geq 6$, $s_{(n-2,2)} * s_\lambda$ is multiplicity free if and only if $\lambda = (n), (1^n), (n-1,1), (2,1^{n-2})$ or $\lambda = (a^k)$, where a, k are non-negative integers. If $n \geq 16$, $s_{(n-3,3)} * s_\lambda$ is multiplicity free if and only if $\lambda = (n), (1^n), (n-1,1), (2,1^{n-2})$ and if n is even also $\lambda = (n/2, n/2)$ or $\lambda = (2^{n/2})$. If $p \geq 4$ and $n > (2p-2)^2$, then $s_{(n-p,p)} * s_\lambda$ is multiplicity free if and only if $\lambda = (n), (1^n), (n-1,1), (2,1^{n-2})$.

Other applications of our combinatorial interpretation for $g_{(n-p,p),\lambda,\nu}$ include formulas for some special partitions λ and ν. The formulas obtained do not have cancellations and are easy to program.

The paper is organized as follows. In Section 1 we give preliminary definitions and set the notation used throughout the paper. In Section 2 we give a variation of the Remmel–Whitney [10] algorithm for expanding the skew Schur function $s_{\lambda/\mu}$. We then use this algorithm to prove that the symmetric function $s_{\lambda/\alpha}s_\alpha - s_{\lambda/\beta}s_\beta$, where $\beta = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$, is Schur positive if and only if $\lambda_1 \geq 2\alpha_1 - 1$. In Section 3 we define the Kronecker tableaux and give the combinatorial interpretation for $g_{(n-p,p),\lambda,\nu}$. In the last section we apply our combinatorial rule to give characterizations for multiplicity free $s_{(n-p,p)} * s_\lambda$. We also give closed formulas for several special cases. For instance, we give a general formula for the coefficient of $s_{(n-t,t)}$ in the product $s_{(n-p,p)} * s_\lambda$ and show that these coefficients are unimodal for some special cases of λ.

Acknowledgement. The authors are grateful to Christine Bessenrodt for useful suggestions and comments.

1. Preliminaries and Notation

Details and proofs for the contents of this section can be found in [12, Chap. 7]. A *partition* of a non-negative integer n is a weakly decreasing sequence of non-negative integers, $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$, such that $|\lambda| = \sum \lambda_i = n$. We write $\lambda \vdash n$ to mean λ is a partition of n. The nonzero integers λ_i are called the *parts* of λ. We identify a partition with its *Young diagram*, i.e., the array of left-justified squares (boxes) with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. The rows are arranged in matrix form from top to bottom. By the box in position (i, j) we mean the box in the i-th row and j-th column of λ. The *length* of λ, denoted $\ell(\lambda)$, is the number of rows in the Young diagram.

Given two partitions λ and μ, we write $\mu \subseteq \lambda$ if and only if $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for $1 \leq i \leq \ell(\mu)$. If $\mu \subseteq \lambda$, we denote by λ/μ the skew shape obtained by removing the boxes corresponding to μ from λ. The length and parts of a skew diagram are defined in the same way as for Young diagrams.

Let $D = \lambda/\mu$ be a skew shape and let $a = (a_1, a_2, \cdots, a_k)$ be a sequence of positive integers such that $\sum a_i = |D| = |\lambda| - |\mu|$. A *decomposition* of D of type a, denoted $D_1 + \cdots + D_k = D$, is given by a sequence of shapes

$$\mu = \lambda^{(0)} \subseteq \lambda^{(1)} \ldots \subseteq \lambda^{(k)} = \lambda,$$

where $D_i = \lambda^{(i)}/\lambda^{(i-1)}$ and $|D_i| = a_i$.

A *semi-standard Young tableau* (SSYT) *of shape* λ/μ is a filling of the boxes of the skew shape λ/μ with positive integers so that the numbers weakly increase in each row from left to right and strictly increase in each column from top to bottom. The *type* of a SSYT T is the sequence of non-negative integers $(t_1, t_2, \ldots)$, where t_i is the number of i's in T.

$$\begin{array}{ccccccccc}
 & & & 2 & 2 & 5 & 7 & 8 & 9 \\
 & 4 & 5 & 7 & 9 & & & & \\
2 & 8 & & & & & & & \\
8 & & & & & & & &
\end{array}$$

A SSYT of shape $\lambda = (9, 5, 2, 1)/(3, 1)$ and type $(0, 3, 0, 1, 2, 0, 2, 3, 2)$

FIGURE 1

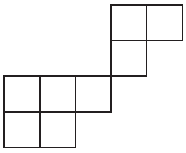
Given T, a SSYT of shape λ/μ and type $(t_1, t_2, \ldots)$, we define its *weight*, denoted $w(T)$, to be the monomial obtained by replacing each i in T by x_i and taking the product over all boxes, i.e., $w(T) = x_1^{t_1} x_2^{t_2} \cdots$. The skew Schur function $s_{\lambda/\mu}$ is defined combinatorially by the formal power series

$$s_{\lambda/\mu} = \sum_T w(T),$$

where the sum runs over all SSYTs of shape λ/μ. To obtain the usual Schur function one sets $\mu = \emptyset$.

For any positive integer n, the Schur function indexed by the partition (n) is called the *n-th homogeneous symmetric function* and will be denoted by h_n. That is, $h_n := s_{(n)}$.

For any two Young diagrams λ and μ, we let $\lambda \times \mu$ denote the diagram obtained by joining the corners of the leftmost, lowest box in λ, i.e the box in position $(\ell(\lambda), 1)$, with the rightmost, highest box of μ, i.e., the box in position $(1, \mu_1)$.



$\lambda \times \mu$ where $\lambda = (2,1)$ and $\mu = (3,2)$.

FIGURE 2

It follows directly from the combinatorial definition of Schur functions that $s_{\lambda\times\mu} = s_\lambda s_\mu$. One defines similarly $A \times B$, when A and B are skew shapes.

The *Littlewood-Richardson coefficients* are defined via the Hall inner product on symmetric functions (see [12, pg. 306]) as follows:

$$c^\lambda_{\mu\nu} := \langle s_\lambda, s_\mu s_\nu\rangle = \langle s_{\lambda/\mu}, s_\nu\rangle.$$

That is, $c^\lambda_{\mu\nu}$ is the coefficient of s_λ in the product $s_\mu s_\nu$. The Littlewood-Richardson rule gives a combinatorial interpretation for the coefficients $c^\lambda_{\mu\nu}$. Before we state the rule we recall some terminology. A *lattice permutation* is a sequence $a_1a_2\cdots a_n$ such that in any initial factor $a_1a_2\cdots a_j$, the number of i's is at least as great as the number of $(i+1)$'s for all i.

The *reverse reading word* of a tableau is the sequence of entries of T obtained by reading the entries from right to left and top to bottom, starting with the first row.

The *Littlewood-Richardson rule* states that the coefficient $c^\lambda_{\mu\nu}$ is equal to the number of SSYTs of shape λ/μ and type ν whose reverse reading word is a lattice permutation.

Example. The coefficient of $s_{(5,4,3)}$ in $s_{(4,3,2)}s_{(2,1)}$ is 2 since there are two Littlewood-Richardson tableaux of shape $(5,4,3)/(2,1)$ and type $(4,3,2)$:

$$\begin{array}{ccccc} & & 1 & 1 & 1 \\ & 2 & 2 & 2 & \\ 1 & 3 & 3 & & \end{array} \qquad \begin{array}{ccccc} & & 1 & 1 & 1 \\ & 1 & 2 & 2 & \\ 2 & 3 & 3 & & \end{array}$$

The Kronecker product of Schur functions is defined via the Frobenius characteristic map, ch, from the center of the group algebra of S_n to the ring of symmetric functions. For a definition of the Frobenius map, see [12, pg. 351]. The map ch is a ring homomorphism and an isometry. It is known that for any irreducible character χ^λ of the symmetric group

$$ch(\chi^\lambda) = s_\lambda.$$

Let χ^λ and χ^μ be two irreducible characters of S_n. The *Kronecker product* $\chi^\lambda\chi^\mu$ is defined for every $\sigma \in S_n$ by $\chi^\lambda\chi^\mu(\sigma) = \chi^\lambda(\sigma)\chi^\mu(\sigma)$. Then

$$\chi^\lambda\chi^\mu = \sum_{\nu\vdash n} g_{\lambda,\mu,\nu}\chi^\nu.$$

Hence, using the Frobenius characteristic map, one defines the Kronecker product of Schur functions by

$$s_\lambda * s_\mu := \sum_{\nu\vdash n} g_{\lambda,\mu,\nu} s_\nu.$$

Littlewood [6] proved the following identity:

$$s_\lambda s_\mu * s_\eta = \sum_{\gamma \vdash |\lambda|} \sum_{\delta \vdash |\mu|} c^\eta_{\gamma\delta} (s_\lambda * s_\gamma)(s_\mu * s_\delta),$$

where $c^\eta_{\gamma\delta}$ is the Littlewood-Richardson coefficient. Garsia and Remmel [4] generalized this result for any skew shapes A, B and D:

$$(s_A s_B) * s_D = \sum_{\substack{D_1+D_2=D\\ |D_1|=|A|,|D_2|=|B|}} (s_A * s_{D_1})(s_B * s_{D_2}),$$

where the sum runs over all decompositions of the skew shape D. By induction one obtains:

$$(h_{n_1} h_{n_2} \cdots h_{n_k}) * s_D = \sum_{\substack{D_1+\cdots+D_k=D\\ |D_i|=n_i}} s_{D_1} \cdots s_{D_k},$$

where the sum runs over all decompositions of D of type $(n_1, n_2, \ldots, n_k)$.

2. A Schur Positivity Theorem

In this section we consider the Schur positivity of the symmetric function $s_{\lambda/\alpha} s_\alpha - s_{\lambda/\alpha^-} s_{\alpha^-}$, where $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}) \subseteq \lambda$ with $\alpha_1 > \alpha_2$ and $\alpha^- = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$. More explicitly, we show that this symmetric function is Schur positive if and only if $\lambda_1 \geq 2\alpha_1 - 1$. In order to prove this result, we need a variation of the Remmel–Whitney algorithm for expanding the skew Schur function $s_{\lambda/\mu}$. Recall that the *reverse lexicographic filling* of μ, $rl(\mu)$, is a filling of the Young diagram μ with the numbers $1, 2, \ldots, |\mu|$ so that the numbers are entered in order from right to left and top to bottom.

Skew Algorithm: The algorithm for computing

$$s_{\lambda/\mu} = \sum_{|\nu|=|\lambda|-|\mu|} c^\lambda_{\mu\,\nu} s_\nu$$

is as follows:

(1) Form the reverse lexicographic filling of μ, $rl(\mu)$.
(2) Starting with the Young diagram λ, label $|\mu|$ of its outermost boxes with the numbers $|\mu|, |\mu| - 1, \ldots, 2, 1$, starting with $|\mu|$, so that the following conditions are satisfied:
   (a) After labelling each box, the unlabelled boxes form a Young diagram.
   (b) Suppose that in $rl(\mu)$ the box in position (i, j) has label x, where $x \leq |\mu|$. If $j > 1$, let x^- be the label in position $(i, j - 1)$ in $rl(\mu)$. If $i < \ell(\mu)$, let x^+ be the label in position $(i + 1, j)$ in $rl(\mu)$. Then in λ, x will be placed to the left and weakly below (to the SW) of x^- and above and weakly to the right (to the NE) of x^+.
(3) From each of the diagrams obtained (with $|\mu|$ labelled boxes), remove all labelled boxes. The resulting unlabelled diagrams correspond to the summands in the decomposition of $s_{\lambda/\mu}$.

Remark. Suppose (i, j) is the position of x in $rl(\mu)$ and (l, m) is the new position of x in λ. The conditions (a) and (b) impose constraints on l and m. It can be easily verified that $l \geq i$ and $m \geq \mu_i - j + 1$, where μ_i is the number of boxes in the i-th row of μ.

Example. The decomposition of $s_{\lambda/\mu}$, where $\lambda = (4, 4, 2, 2)$, $\mu = (3, 3)$:

$$\lambda = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} \end{array}, \qquad rl(\mu) = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 6 & 5 & 4 \\ \hline \end{array}.$$

First we establish the constraints on the position of each label in λ using the Remark.

label	position (i, j) in μ	position (l, m) in λ	position relative to x^- and x^+
6	$(2, 1)$	$l \geq 2$ and $m \geq 3 - 1 + 1 = 3$	
5	$(2, 2)$	$l \geq 2$ and $m \geq 3 - 2 + 1 = 2$	SW of 6
4	$(2, 3)$	$l \geq 2$ and $m \geq 3 - 3 + 1 = 1$	SW of 5
3	$(1, 1)$	$l \geq 1$ and $m \geq 3 - 1 + 1 = 3$	NE of 6
2	$(1, 2)$	$l \geq 1$ and $m \geq 3 - 2 + 1 = 2$	SW of 3 and NE of 5
1	$(1, 3)$	$l \geq 1$ and $m \geq 3 - 3 + 1 = 1$	SW of 2 and NE of 4

The algorithm is carried out in Fig. 3. Thus, $s_{\lambda/\mu} = s_{(2,2,1,1)} + s_{(3,2,1)} + s_{(3,3)}$.

The Skew Algorithm above follows from the Remmel–Whitney algorithm [10] and the fact that skewing is the adjoint operation of multiplication, i.e., $\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle$. In some sense we are reversing the steps taken in [10] when expanding the product $s_\mu s_\nu$.

In order to state our first result we need to recall the definition of *lexicographic order* on the set of all partitions. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}) \vdash m$, we say that λ is less than μ *in lexicographic order*, and write $\lambda <_l \mu$, if there is a non-negative integer k such that $\lambda_i = \mu_i$ for all $i = 1, 2, \ldots, k$ and $\lambda_{k+1} < \mu_{k+1}$. The lexicographic order is a total order on the set of all partitions.

Lemma 2.1. *Consider the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ such that $\alpha \subseteq \lambda$. The smallest partition ν in lexicographic order such that s_ν appears in the expansion of $s_{\lambda/\alpha}$ is the partition obtained by reordering the parts of λ/α in decreasing order, i.e., the parts of ν are $\lambda_1 - \alpha_1, \lambda_2, -\alpha_2, \ldots, \lambda_{\ell(\lambda)} - \alpha_{\ell(\lambda)}$ ($\alpha_i = 0$ if $i > \ell(\alpha)$) reordered such that ν is a partition. Moreover, the multiplicity of s_ν in the expansion of $s_{\lambda/\alpha}$ is equal to 1.*

Proof. Using the Skew Algorithm, we obtain the smallest partition in lexicographic order when the labels in $rl(\alpha)$ are each placed in the highest possible row of λ (since we are removing the largest possible number of boxes from the top rows of λ). We will show that the partition obtained in this way is precisely the partition obtained by reordering the rows of λ/α. We argue inductively by the number of rows of α.

Assume $\alpha = (\alpha_1)$. We form the reverse lexicographic order of α. According to the Skew Algorithm, the label $|\alpha|$ can be placed in the first row of λ if $\lambda_1 > \lambda_2$. In general, the highest position where we can place $|\alpha|$ is (k, λ_k), where k is the positive integer such that $\lambda_k = \lambda_1$ and $\lambda_{k+1} < \lambda_1$. We will place the other labels of α to the SW of this position respecting the rules of the algorithm. We will remove the highest possible horizontal strip (a skew shape so that no two boxes are in the same column) with α_1

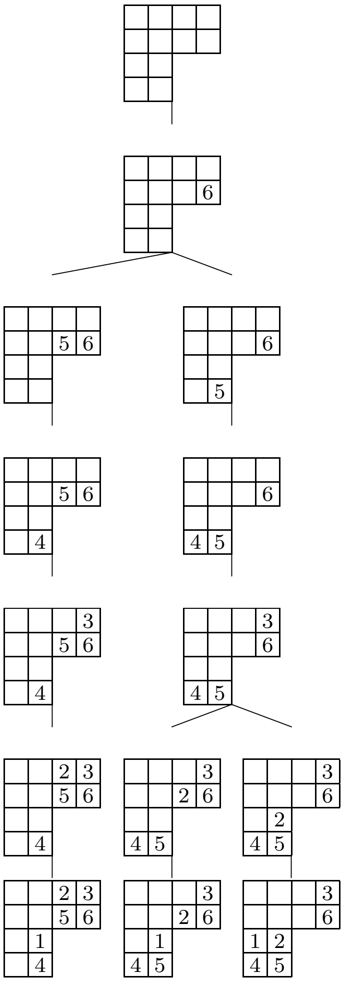


FIGURE 3

boxes starting with position (k, λ_k) and continuing SW. This also follows from Pieri's rule [12, Corollary 7.15.9].

Now suppose t is the positive integer such that $\lambda_k - \lambda_{k+t} \geq \alpha_1$ and $\lambda_k - \lambda_{k+t-1} < \alpha_1$ (i.e., label 1 will be placed in the $(k+t)$-th row). Then the smallest shape in lexicographic order appearing in the Skew Algorithm is

$$\begin{aligned}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k - (\lambda_k - \lambda_{k+1}), \lambda_{k+1} - (\lambda_{k+1} - \lambda_{k+2}), \ldots, \lambda_{k+t-1} - (\lambda_{k+t-1} - \lambda_{k+t}),\\ \lambda_{k+t} - (\alpha_1 - (\lambda_k - \lambda_{k+t})), \lambda_{k+t+1}, \ldots, \lambda_{\ell(\lambda)})\\ = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_{k+t}, \lambda_k - \alpha_1, \lambda_{k+t+1}, \ldots, \lambda_{\ell(\lambda)}).\end{aligned}$$

Since $\lambda_k = \lambda_1$, this is precisely the partition obtained by reordering the rows of $\lambda/(\alpha_1) = (\lambda_1 - \alpha_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$.

Suppose the lemma is true for all partitions with ℓ parts that are contained in λ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \alpha_{\ell+1})$ be a partition with $\ell + 1$ parts such that $\alpha \subseteq \lambda$. Thus $\ell(\alpha) = \ell + 1$.

When we place the labels from the reverse lexicographic order of α into the boxes of λ according to the Skew Algorithm such that each label is placed in the highest possible

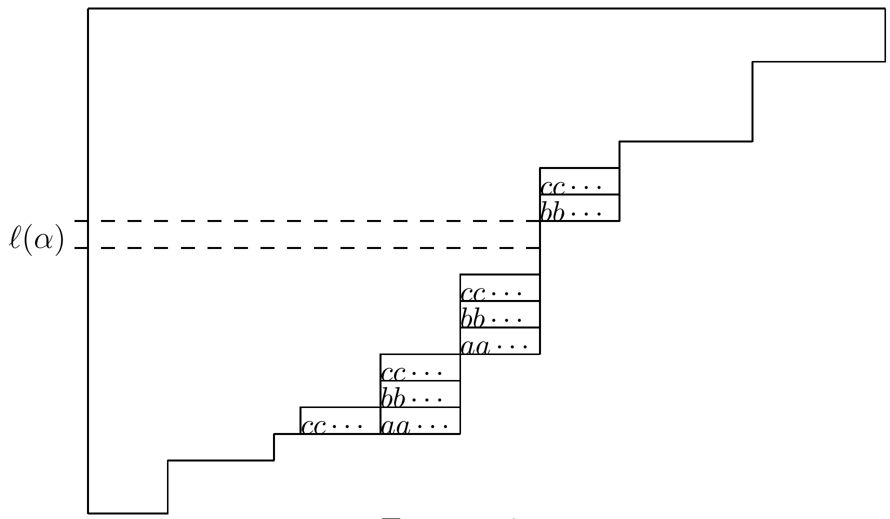


FIGURE 4

row, we first place the labels of the last row of $rl(\alpha)$. They are placed as in the case $\alpha = (\alpha_1)$ above, starting with placing $|\alpha|$ in position (k, λ_k), where k is the positive integer such that $\lambda_k = \lambda_{\ell(\alpha)}$ and $\lambda_{k+1} < \lambda_{\ell(\alpha)}$ (note that $k \geq \ell(\alpha)$). The remaining labels of the last row of α are placed to the SW of this position forming a horizontal strip with boxes in the highest possible rows of λ. Observe that this is equivalent to labelling the highest possible horizontal strip of $\alpha_{\ell(\alpha)}$ boxes in $(\lambda_{\ell(\alpha)}, \lambda_{\ell(\alpha)+1}, \ldots, \lambda_{\ell(\lambda)})$ such that the unlabelled boxes yield a Young diagram, and then adding back the rows $\lambda_1, \ldots, \lambda_{\ell(\alpha)-1}$ above the shape $(\lambda_{\ell(\alpha)}, \lambda_{\ell(\alpha)+1}, \ldots, \lambda_{\ell(\lambda)})$ (with the labelled boxes). Notice that the unlabelled boxes form the partition which is a rearrangement of $(\lambda_1, \ldots, \lambda_{\ell(\alpha)} - \alpha_{\ell(\alpha)}, \ldots, \lambda_{\ell(\lambda)})$. According to the Skew Algorithm, we now place the labels in row $\ell(\alpha)-1$ of $rl(\alpha)$. Requiring that the labels be placed in the highest rightmost position at each step will automatically satisfy the requirements of the Skew Algorithm. Therefore, in order to obtain the smallest partition in lexicographic order, we just need to label the highest horizontal strips with $\alpha_{\ell(\alpha)}$ boxes starting at row $\ell(\alpha)$ and continuing SW, then the highest horizontal strip with $\alpha_{\ell(\alpha)-1}$ boxes starting at row $\ell(\alpha) - 1$ and continuing SW in the remaining unlabelled boxes, and so on until we label the highest horizontal strip with α_1 boxes starting at row 1 such that, at each step, removing all labelled boxed yields a Young diagram. See Fig. 4 for an example of placing the labels of the last 3 rows of $rl(\alpha)$ in λ; the horizontal strip containing a's is the one corresponding to row $\ell(\alpha)$, the strip containing b's corresponds to row $\ell(\alpha) - 1$ and the strip containing c's corresponds to row $\ell(\alpha) - 2$.

If we follow this procedure of placing each label as high as possible we obtain the smallest partition in lexicographic order appearing in the expansion of $s_{\lambda/\alpha}$. The labels in rows $2, 3, \ldots, \ell(\alpha)$ of $rl(\alpha)$ can only be placed in $\bar{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$. The unlabelled boxes form the partition $\mu = (\lambda_1, \mu_2, \ldots, \mu_{\ell(\mu)})$ where $(\mu_2, \ldots, \mu_{\ell(\mu)})$ is the smallest partition in lexicographic order occurring in $s_{\bar{\lambda}/\bar{\alpha}}$. By induction hypothesis, $(\mu_2, \ldots, \mu_{\ell(\mu)})$ is obtained by rearranging $\lambda_2 - \alpha_2, \lambda_3 - \alpha_3, \ldots, \lambda_{\ell(\alpha)} - \alpha_{\ell(\alpha)}, \lambda_{\ell(\alpha)+1}, \ldots \lambda_{\ell(\lambda)}$. If we continue the labelling in λ with the labels in row α_1 of $rl(\alpha)$, the labels are placed in $\mu = (\lambda_1, \mu_2, \ldots, \mu_{\ell(\mu)})$. By the discussion in the previous paragraph, we will be labelling the highest possible horizontal strip in μ. Hence, removing this strip from μ yields the partition ν which is a rearrangement of $\lambda_1 - \alpha_1, \mu_2, \ldots, \mu_{\ell(\mu)}$. By induction we have that

ν is the smallest partition and it is a rearrangement of $\lambda_1 - \alpha_1, \lambda_2 - \alpha_2, \ldots \lambda_{\ell(\lambda)} - \alpha_{\ell(\lambda)}$, where $\alpha_i = 0$ for $i > \ell(\alpha)$.

Since there is only one way of placing the labels of the reverse lexicographic order of α in the highest possible rows of λ, the multiplicity of s_ν in $s_{\lambda/\alpha}$, where ν is the smallest partition in lexicographic order, equals 1. □

Corollary 2.2. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{\ell(\beta)})$ be arbitrary partitions. The smallest partition ν in lexicographic order appearing in the expansion of $s_\alpha s_\beta$ is the partition obtained by concatenating the parts of α and β and reordering them to form a partition.*

Proof. The proof is a direct consequence of the Littlewood-Richardson rule. It also follows directly from the Remmel–Whitney algorithm for multiplying two Schur functions [10] and Lemma 2.1. □

Definition. A symmetric function is said to be *Schur positive* (or *s-positive*) if, when expanded as a linear combination of Schur functions, all the coefficients are positive.

Consider the product $s_{\lambda/\alpha} s_\alpha$. The combinatorial definition of Schur functions implies that $s_{\lambda/\alpha} s_\alpha$ is the skew Schur function corresponding to the skew shape μ/η, where

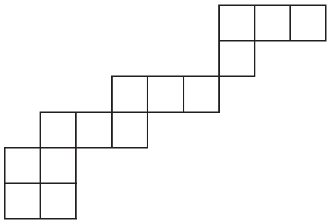
$$\mu = (\lambda_1 + \alpha_1, \lambda_1 + \alpha_2, \ldots \lambda_1 + \alpha_{\ell(\alpha)}, \lambda_1, \lambda_2, \ldots, \lambda_{\ell(\alpha)}, \lambda_{\ell(\alpha)+1}, \ldots, \lambda_{\ell(\lambda)})$$

and

$$\eta = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\ell(\alpha)\text{ times}}, \alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}).$$

This is the skew shape $\alpha \times \lambda/\alpha$.

Example. Let $\lambda = (6,4,2,2)$ and $\alpha = (3,1)$. Then $s_{\lambda/\alpha} s_\alpha$ is the skew Schur function corresponding to the skew shape $\alpha \times \lambda/\alpha$ below.



Definition. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ be any sequence of non-negative integers. A sequence $a_1 a_2 \cdots a_n$ is an *α-lattice permutation* if in any initial factor $a_1 a_2 \cdots a_j$, $1 \le j \le n$, we have for any positive integer i:

$$\text{the number of } i's + \alpha_i \ge \text{ the number of } (i+1)'s + \alpha_{i+1}.$$

Here $\alpha_i = 0$ if $i > \ell(\alpha)$.

Then, if $\nu = (\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}) \vdash n$, the multiplicity of s_ν in $s_{\lambda/\alpha} s_\alpha$ is given by the number of SSYT of shape λ/α and type $\nu/\alpha := (\nu_1 - \alpha_1, \nu_2 - \alpha_2, \ldots, \nu_{\ell(\alpha)} - \alpha_{\ell(\alpha)}, \nu_{\ell(\alpha)+1}, \ldots, \nu_{\ell(\nu)})$ whose reverse reading word is an α-lattice permutation. (If $\alpha \not\subseteq \nu$, the multiplicity of s_ν in $s_{\lambda/\alpha} s_\alpha$ is 0.)

Theorem 2.3. *Let $\lambda \vdash n$, $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_{\ell(\alpha)}) \vdash p$ with $\alpha_1 > \alpha_2$ and $\alpha^- = (\alpha_1 - 1, \alpha_2, \ldots \alpha_{\ell(\alpha)}) \vdash p - 1$. Assume that $\alpha \subseteq \lambda$. The symmetric function $s_{\lambda/\alpha} s_\alpha - s_{\lambda/\alpha^-} s_{\alpha^-}$ is Schur positive if and only if $\lambda_1 \ge 2\alpha_1 - 1$.*

Proof. Assume that $\lambda_1 < 2\alpha_1 - 1$. Let ν denote the smallest partition in lexicographic order such that s_ν appears in the expansion of $s_{\lambda/\alpha}s_\alpha$ and let ν^* denote the smallest partition in lexicographic order such that s_{ν^*} appears in the expansion of $s_{\lambda/\alpha^-}s_{\alpha^-}$. Since $s_{\lambda/\alpha}s_\alpha$ is the skew Schur function corresponding to the skew shape $\alpha \times \lambda/\alpha$, it follows from Lemma 2.1 that the parts of ν are precisely the parts of $\alpha \times \lambda/\alpha$ reordered to form a partition. Since $\alpha_i = 0$ for $i > \ell(\alpha)$, the parts of ν are $\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}, \lambda_1 - \alpha_1, \lambda_2 - \alpha_2, \ldots \lambda_{\ell(\alpha)} - \alpha_{\ell(\alpha)}, \lambda_{\ell(\alpha)+1}, \ldots, \lambda_{\ell(\lambda)}$. Similarly, the parts of ν^* are precisely the parts of $\alpha^- \times \lambda/\alpha^-$, that is $\alpha_1 - 1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}, \lambda_1 - \alpha_1 + 1, \lambda_2 - \alpha_2, \ldots, \lambda_{\ell(\alpha)} - \alpha_{\ell(\alpha)}, \lambda_{\ell(\alpha)+1}, \ldots, \lambda_{\ell(\lambda)}$. The partitions ν and ν^* differ in only two parts: $\lambda_1 - \alpha_1$ vs. $\lambda_1 - \alpha_1 + 1$ and α_1 vs. $\alpha_1 - 1$. Since $\lambda_1 < 2\alpha_1 - 1$, we have $\lambda_1 - \alpha_1 < \alpha_1 - 1$ and $\lambda_1 - \alpha_1 + 1 < \alpha_1$. Thus, $\lambda_1 - \alpha_1 < \alpha_1$ and $\lambda_1 - \alpha_1 + 1 \leq \alpha_1 - 1$. Let k be the positive integer such that $\nu_k = \alpha_1$ and $\nu_{k+1} < \alpha_1$. Then $\nu_k^* = \alpha_1 - 1$. We have $\nu_i = \nu_i^*$ for all $i = 1, 2, \ldots, k-1$ and $\nu_k^* < \nu_k$. Thus ν^* is lexicographically smaller than ν and s_{ν^*} appears in the expansion of $s_{\lambda/\alpha}s_\alpha - s_{\lambda/\alpha^-}s_{\alpha^-}$ with coefficient -1.

Assume now that $\lambda_1 \geq 2\alpha_1 - 1$. We will show that, for any $\nu \vdash n$, the coefficient of s_ν in $s_{\lambda/\alpha^-}s_{\alpha^-}$ is smaller or equal than its coefficient in $s_{\lambda/\alpha}s_\alpha$. Let $\nu \vdash n$ be such that s_ν appears in the expansion of $s_{\lambda/\alpha^-}s_{\alpha^-}$. The coefficient of s_ν in the expansion of $s_{\lambda/\alpha^-}s_{\alpha^-}$ is equal to the number of SSYTs of shape λ/α^- and type ν/α^- whose reverse reading word is an α^--lattice permutation. Let T be a SSYT of shape λ/α^- and type ν/α^- whose reverse reading word is an α^--lattice permutation. Then, in the first row of λ/α^-,

$$\text{the number of } (i+1)'s + \alpha_{i+1} \leq \alpha_i, \qquad \text{if } i \geq 2 \text{ and}$$
$$\text{the number of } 2's + \alpha_2 \leq \alpha_1 - 1.$$

Recall that $\alpha_j = 0$ if $j > \ell(\alpha)$.

Thus, the number of boxes in the first row of λ/α^- with labels different from 1 is at most

$$\sum_{i\geq 2}(\alpha_i - \alpha_{i+1}) + \alpha_1 - 1 - \alpha_2 = \alpha_1 - 1.$$

Since $\lambda_1 \geq 2\alpha_1 - 1$, the number of boxes in the first row of λ/α^- satisfies

$$\lambda_1 - \alpha_1 + 1 \geq \alpha_1 > \alpha_1 - 1.$$

Thus, in T, the leftmost box in the first row of λ/α^-, $(1, \alpha_1)$, must be filled with 1.

Let T^* be the tableau obtained from T by removing the leftmost box in the first row of T, i.e., the box labelled 1 in position $(1, \alpha_1)$. Then T^* is a SSYT of shape λ/α and type ν/α whose reverse reading word is an α-lattice permutation.

For every SSYT of shape λ/α^- and type ν/α^- whose reverse reading word is an α^--lattice permutation we obtained a SSYT of shape λ/α and type ν/α whose reverse reading word is an α-lattice permutation. Thus the multiplicity of s_ν in $s_{\lambda/\alpha}s_\alpha$ is greater or equal to the multiplicity of s_ν in $s_{\lambda/\alpha^-}s_{\alpha^-}$ and $s_{\lambda/\alpha}s_\alpha - s_{\lambda/\alpha^-}s_{\alpha^-}$ is Schur-positive. $\square$

3. Combinatorial rule for the Kronecker coefficient

In this section we use Theorem 2.3 to give a combinatorial rule for computing the coefficient $g_{(n-p,p),\lambda,\nu}$ in the Kronecker product $s_{(n-p,p)} * s_\lambda$, whenever $\lambda_1 \geq 2p - 1$ or $\ell(\lambda) \geq 2p - 1$. Our combinatorial rule works for all partitions λ that do not fit in the $(2p-2) \times (2p-2)$ square. If $n > (2p-2)^2$, then the rule applies to all partitions λ of n.

Definition. A SSYT T of shape λ/α and type ν/α whose reverse reading word is an α-lattice permutation is called a *Kronecker Tableau* of shape λ/α and type ν/α if

(I) $\alpha_1 = \alpha_2$ or
(II) $\alpha_1 > \alpha_2$ and any one of the following two conditions is satisfied:
  (i) The number of 1's in the second row of λ/α is exactly $\alpha_1 - \alpha_2$.
  (ii) The number of 2's in the first row of λ/α is exactly $\alpha_1 - \alpha_2$.

Denote by $k^{\lambda}_{\alpha\nu}$ the number of Kronecker tableaux of shape λ/α and type ν/α.

Note. There are $\alpha_1 - \alpha_2$ 1's in the second row of λ/α if and only if there is a 1 in box $(2, \alpha_1)$ of λ/α.

Lemma 3.1. *Let $\lambda \vdash n$ and $\alpha \vdash p$ be partitions such that $\alpha_1 > \alpha_2$, $\alpha \subseteq \lambda$ and $\lambda_1 \geq 2\alpha_1 - 1$ and let $\alpha^- = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$. Then the coefficient of s_ν in $s_\alpha s_{\lambda/\alpha} - s_{\alpha^-} s_{\lambda/\alpha^-}$ equals the number of Kronecker tableaux of shape λ/α and type ν/α, i.e., $k^{\lambda}_{\alpha\nu}$.*

Proof. By Theorem 2.3 we have that $s_\alpha s_{\lambda/\alpha} - s_{\alpha^-} s_{\lambda/\alpha^-}$ is Schur positive if and only if $\lambda_1 \geq 2\alpha_1 - 1$. Furthermore, we have observed that the coefficient of s_ν in $s_\alpha s_{\lambda/\alpha}$ is equal to the number of SSYTs of shape λ/α and type ν/α whose reverse reading word is an α-lattice permutation. Denote the set of these SSYTs by $\mathcal{T}^{\nu}_{\lambda/\alpha}$ and let $K^{\nu}_{\lambda/\alpha}$ denote the set of Kronecker tableaux of shape λ/α and type ν/α. Denote by $NK^{\nu}_{\lambda/\alpha}$ the set of elements in $\mathcal{T}^{\nu}_{\lambda/\alpha}$ that are not Kronecker tableaux. Hence, $\mathcal{T}^{\nu}_{\lambda/\alpha} = K^{\nu}_{\lambda/\alpha} \cup NK^{\nu}_{\lambda/\alpha}$ (disjoint union).

Let α be such that $\alpha_1 > \alpha_2$ and let $\mathcal{T}^{\nu}_{\lambda/\alpha^-}$ be the set of SSYTs of shape λ/α^- and type ν/α^- whose reverse reading word is an α^--lattice permutation. We show that there exists a bijection

$$\mathcal{T}^{\nu}_{\lambda/\alpha^-} \longrightarrow NK^{\nu}_{\lambda/\alpha}.$$

In Theorem 2.3, we showed that in every $T \in \mathcal{T}^{\nu}_{\lambda/\alpha^-}$ the box in position $(1, \alpha_1)$ is filled with 1. The bijection is given by removing this box from T to obtain a tableau $T^* \in \mathcal{T}^{\nu}_{\lambda/\alpha}$. Furthermore, the number of 2's in the first row in T^* can be at most $\alpha_1 - \alpha_2 - 1$ since T yielded an α^--lattice permutation. Also, the label in position $(2, \alpha_1)$ of T^* must be greater than 1, since in T the box in position $(1, \alpha_1)$ was filled with 1. Thus $T^* \in NK^{\nu}_{\lambda/\alpha}$. On the other hand, to any element $T \in NK^{\nu}_{\lambda/\alpha}$ we can add a box labelled 1 in position $(1, \alpha_1)$ to obtain a tableau in $\mathcal{T}^{\nu}_{\lambda/\alpha^-}$. Thus, the coefficient of s_ν in $s_\alpha s_{\lambda/\alpha} - s_{\alpha^-} s_{\lambda/\alpha^-}$ equals $k^{\lambda}_{\alpha\nu}$. □

Let $\lambda, \nu \vdash n$. The *intersection* of λ and ν, denoted $\lambda \cap \nu$, is the Young diagram consisting of the boxes (i, j) that belong to both λ and ν.

Example. If $\lambda = \ydiagram{5,3,1}$ and $\nu = \ydiagram{3,2,2,1,1}$, then $\lambda \cap \nu = \ydiagram{3,2,1}$.

Theorem 3.2. *Let n and p be positive integers such that $n \geq 2p$. Let $\lambda, \nu \vdash n$, $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ and let λ' denote the partition conjugate to λ.*

(a) *If $\lambda_1 \geq 2p - 1$, the multiplicity of s_ν in $s_{(n-p,p)} * s_\lambda$ equals* $\displaystyle\sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda \cap \nu}} k^{\lambda}_{\alpha\nu}$.

(b) *If $\ell(\lambda) \geq 2p-1$, the multiplicity of s_ν in $s_{(n-p,p)} * s_\lambda$ equals* $\displaystyle\sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda' \cap \nu'}} k^{\lambda'}_{\alpha\nu'}$.

Proof. (a) From the Garsia–Remmel formula it easily follows that

$$s_{(n-p,p)} * s_\lambda = \sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda}} s_\alpha s_{\lambda/\alpha} - \sum_{\substack{\beta \vdash p-1 \\ \beta \subseteq \lambda}} s_\beta s_{\lambda/\beta}.$$

There is a $1-1$ correspondence between the partitions $\alpha \vdash p$ with $\alpha_1 > \alpha_2$ and the partitions $\alpha^- \vdash p-1$ given by

$$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_{\ell(\alpha)}) \longleftrightarrow \alpha^- = (\alpha_1 - 1, \alpha_2, \ldots \alpha_{\ell(\alpha)}).$$

The Young diagram of α^- is obtained from the Young diagram of α by removing the last box of the first row. Hence, we have

$$s_{(n-p,p)} * s_\lambda = \sum_{\substack{\alpha \vdash p, \alpha_1 > \alpha_2 \\ \alpha \subseteq \lambda}} (s_\alpha s_{\lambda/\alpha} - s_{\alpha^-} s_{\lambda/\alpha^-}) + \sum_{\substack{\alpha \vdash p, \alpha_1 = \alpha_2 \\ \alpha \subseteq \lambda}} s_\alpha s_{\lambda/\alpha}.$$

If $\alpha_1 = \alpha_2$, by definition, every SSYT whose reverse reading word is an α-lattice permutation is a Kronecker tableau. If $\alpha_1 > \alpha_2$, by Lemma 3.1 the coefficient of s_ν in $s_\alpha s_{\lambda/\alpha} - s_{\alpha^-} s_{\lambda/\alpha^-}$ equals $k^\lambda_{\alpha\nu}$. Note that since $\alpha_1 \leq p$, we have $\lambda_1 \geq 2\alpha_1 - 1$ for all partitions $\alpha \vdash p$.

(b) If $\ell(\lambda) \geq 2p-1$, then the first part of λ' is greater than or equal to $2p-1$. The identity $s_{\lambda'} = s_{(1^n)} * s_\lambda$ and the commutativity and associativity of the Kronecker product give $s_{(n-p,p)} * s_\lambda = (s_{(n-p,p)} * s_{\lambda'}) * s_{(1^n)} = \left(\sum g_{(n-p,p),\lambda',\nu'} s_{\nu'}\right) * s_{(1^n)} = \sum g_{(n-p,p),\lambda',\nu'} (s_{\nu'} * s_{(1^n)}) = \sum g_{(n-p,p),\lambda',\nu'} s_\nu$. Part (a) of the theorem applied to λ' implies the result of part (b). $\square$

Example: If $n = 15$, $p = 3$, $\lambda = (6,4,4,1)$ and $\nu = (5,4,3,3)$, the Kronecker tableaux are:

$$\begin{array}{cccccc} & & & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & & \\ 2 & 3 & 4 & 4 & & \\ 4 & & & & & \end{array} \qquad \begin{array}{cccccc} & & 1 & 1 & 1 & 2 \\ & 2 & 3 & 3 & & \\ 2 & 3 & 4 & 4 & & \\ 4 & & & & & \end{array} \qquad \begin{array}{cccccc} & & 1 & 1 & 2 & 3 \\ & 1 & 2 & 3 & & \\ 2 & 3 & 4 & 4 & & \\ 4 & & & & & \end{array} \qquad \begin{array}{cccccc} & & 1 & 1 & 2 & 3 \\ & 2 & 2 & 3 & & \\ 1 & 3 & 4 & 4 & & \\ 4 & & & & & \end{array}$$

Hence, $g_{(12,3),(6,4,4,1),(5,4,3,3)} = 4$.

Note. If s_ν appears in $s_{(n-p,p)} * s_\lambda$, the length of ν satisfies

$$\ell(\nu) \leq \ell(\lambda) + \min\{p, \ell(\lambda)\}.$$

In particular we obtain the known fact that, if s_ν appears in $s_{(\lambda_1,\lambda_2)} * s_{(\mu_1,\mu_2)}$, then ν has at most four parts.

Corollary 3.3. *Let $\lambda, \nu \vdash n$, then for a positive integer p*

$$g_{(n-p,p),\lambda,\nu} \leq \sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda \cap \nu}} k^\lambda_{\alpha\nu}.$$

This corollary is a direct consequence of the proof of Theorem 3.2.

Example. Consider the Kronecker product $s_{(4,3)} * s_{(4,3)}$. Using MAPLE one checks that the coefficient of each $s_{(4,2,1)}$ and $s_{(3,2,2)}$ in $s_{(4,3)} * s_{(4,3)}$ is 1. However, there are two Kronecker tableaux of shape $(4,3)/(2,1)$ and type $(4,2,1)/(2,1)$ and two Kronecker tableaux of shape $(4,3)/(2,1)$ and type $(3,2,2)/(2,1)$. These are the only two Schur functions appearing in the decomposition of $s_{(4,3)} * s_{(4,3)}$ with coefficient strictly less than the number of Kronecker tableaux.

4. Applications of the Combinatorial Rule

In this section we illustrate the usefulness of Theorem 3.2. We apply the theorem to address the question of characterizing multiplicity free Kronecker products when one partition has two parts. We also consider special cases for λ and ν and obtain simple formulas for $g_{(n-p,p),\lambda,\nu}$ just by counting Kronecker tableaux. We believe that, when compared with known results [7], [8], [9], [11], the formulas of Theorem 3.2 provides elegant combinatorial solutions to the Kronecker product coefficients for the cases in which it works.

4.1. **Multiplicity-Free Kronecker Products.** In this subsection we use our main theorem to determine the partitions λ for which all s_ν in the decomposition of $s_{(n-p,p)}*s_\lambda$ have coefficients either 0 or 1, i.e., $s_{(n-p,p)}*s_\lambda$ is multiplicity-free. Recall that a partition is called *rectangular* or a *rectangle partition* if all its rows have the same size. That is, $\lambda = (a^k) = (a, a, \ldots, a)$ is a rectangle. Define $C(\lambda) := |\{i \mid \lambda_i > \lambda_{i+1}, 1 \le i \le \ell(\lambda)-1\}|$.

Proposition 4.1. *Let n be a positive integer and $\lambda \vdash n$. Then $s_{(n-1,1)} * s_\lambda$ is multiplicity free if and only if λ is a rectangular partition, $\lambda = (a^k)$, or $\lambda = (a^{k_1}, b^{k_2})$.*

Proof. It is well-know that $s_{(n-1,1)} * s_\lambda = s_{(1)}s_{\lambda/(1)} - s_\lambda$, see for example [12, Exercise 7.81]. From this and Pieri's rule we can easily see that

$$s_{(n-1,1)} * s_\lambda = C(\lambda)s_\lambda + \sum s_\mu,$$

where $C(\lambda)$ is as defined above and the sum is over all partitions different from λ that can be obtained by removing one box and then adding a box to λ. Hence, the only partition that occurs with multiplicity greater than 1 is λ itself when $C(\lambda) \ge 1$. ☐

Corollary 4.2. *Let n be a positive integer and $\lambda \vdash n$. Then $s_{(2,1^{n-2})} * s_\lambda$ is multiplicity free if and only if λ is a rectangular partition, $\lambda = (a^k)$, or $\lambda = (a^{k_1}, b^{k_2})$.*

Proof. This follows from the identity $s_{(1^n)} * s_{(n-1,1)} = s_{(2,1^{n-2})}$ and Proposition 4.1. ☐

Lemma 4.3. *Let n, t be positive integers and $\lambda \vdash n$. If $\ell(\lambda) \ge t$, then $s_{(1^t)}s_{\lambda/(1^t)}$ is multiplicity free if and only if $\ell(\lambda) = t$ or $\lambda = (a^k)$ for some positive integers a, k.*

Proof. We first show that $s_{(1^t)}s_{\lambda/(1^t)}$ is not multiplicity free if $\ell(\lambda) > t$ and λ is not a rectangle partition. We will show that the multiplicity of s_λ in $s_{(1^t)}s_{\lambda/(1^t)}$ is greater than 1. Recall that the multiplicity of s_λ in $s_{(1^t)}s_{\lambda/(1^t)}$ is $k^\lambda_{(1^t)\lambda}$, the number of Kronecker tableaux of shape $\lambda/(1^t)$ and type $\lambda/(1^t)$. Below we obtain two different Kronecker tableaux of shape $\lambda/(1^t)$ and type $\lambda/(1^t)$:

(1) Fill row i of $\lambda/(1^t)$ with i's.

(2) Let s be the number of rows of size λ_1 in λ. Note that $s < \ell(\lambda)$ since λ is not a rectangular partition.

Case I. If $s \le t$. In box (s, λ_1) place $t+1$ and in box $(t+1, 1)$ place s. Then fill the remaining boxes with i if the boxes are in the i-th row.

Case II. If $s > t$. In boxes $(t, \lambda_1), (t+1, \lambda_1), \dots, (s, \lambda_1)$ place $t+1, t+2, \dots, s+1$ respectively. And in boxes $(t+1, 1), (t+2, 1), \dots, (s+1, 1)$ place $t, t+1, t+2, \dots, s$, respectively. Finally fill the remaining unlabelled boxes with i if the box is in the i-th row.

We now show that if $\ell(\lambda) \le t$, then $s_{(1^t)} s_{\lambda/(1^t)}$ is multiplicity free. If $\ell(\lambda) < t$ then $s_{(1^t)} s_{\lambda/(1^t)} = 0$, hence multiplicity free. If $\ell(\lambda) = t$, then $\lambda/(1^t) = (\lambda_1 - 1, \dots, \lambda_t - 1)$ is a partition. Hence $s_{(1^t)} s_{\lambda/(1^t)}$ is multiplicity free by Pieri's rule.

Finally, if $\lambda = (a^k)$, $k \ge t$, then $s_{\lambda/(1^t)} = s_{(a^{k-t},(a-1)^t)}$ by [14, Theorem 2.1]. This can also be seen by applying the Skew Algorithm of Section 2. In this case $s_{(1^t)} s_{\lambda/(1^t)}$ is multiplicity free by Pieri's rule. $\square$

Recall that $k^{\lambda}_{\alpha\nu}$ denotes the number of Kronecker tableaux of shape λ/α and type ν/α.

Lemma 4.4. *Let $n \ge 6$ and $\lambda = (\lambda_1, \lambda_2) \vdash n$ such that $\lambda_1 > \lambda_2 > 1$.*

(i) *If $\lambda_1 > \lambda_2 + 1$, then $k^{\lambda}_{(2),(\lambda_1-1,\lambda_2,1)} \ge 1$ and $k^{\lambda}_{(1,1),(\lambda_1-1,\lambda_2,1)} \ge 1$.*

(ii) *If $\lambda_1 = \lambda_2 + 1$, then $k^{\lambda}_{(2),(\lambda_1,\lambda_2-1,1)} \ge 1$ and $k^{\lambda}_{(1,1),(\lambda_1,\lambda_2-1,1)} \ge 1$.*

Proof. (i) Suppose $\lambda_1 > \lambda_2 + 1$, then $\lambda_1 - \lambda_2 \ge 2$. The following is a Kronecker tableaux of shape $\lambda/(2)$ and type $(\lambda_1 - 1, \lambda_2, 1)/(2)$:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline & & 1 & \cdots & 1 & 1 & \cdots & 1 & 2 & 2 \\ \hline 1 & 2 & 2 & \cdots & 2 & 3 \\ \cline{1-6}\end{array}$$

Hence, $k^{\lambda}_{(2),(\lambda_1-1,\lambda_2,1)} \ge 1$. The following is a Kronecker tableaux of shape $\lambda/(1,1)$ and type $(\lambda_1 - 1, \lambda_2, 1)/(1,1)$:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline & 1 & 1 & \cdots & 1 & \cdots & 1 & 3 \\ \hline & 2 & 2 & \cdots & 2 \\ \cline{1-5}\end{array}$$

Hence, $k^{\lambda}_{(1,1),(\lambda_1-1,\lambda_2,1)} \ge 1$.

(ii) Let $\lambda_1 = \lambda_2 + 1$. The following is a Kronecker tableaux of shape $\lambda/(2)$ and type $(\lambda_1, \lambda_2 - 1, 1)/(2)$:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline & & 1 & \cdots & 1 & 2 & 2 \\ \hline 1 & 1 & 2 & \cdots & 2 & 3 \\ \cline{1-6}\end{array}$$

Thus, $k^{\lambda}_{(2),(\lambda_1,\lambda_2-1,1)} \ge 1$. The following is a Kronecker tableaux of shape $\lambda/(1,1)$ and type $(\lambda_1, \lambda_2 - 1, 1)/(1,1)$:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline & 1 & 1 & \cdots & 1 & 1 & 1 \\ \hline & 2 & 2 & \cdots & 2 & 3 \\ \cline{1-6}\end{array}$$

Therefore, $k^{\lambda}_{(1,1),(\lambda_1,\lambda_2-1,1)} \ge 1$. $\square$

Theorem 4.5. *Let $n \geq 6$ and $\lambda \vdash n$. Then $s_{(n-2,2)} * s_\lambda$ is multiplicity free if and only if $\lambda = (n-1,1), (n), (1^n), (2,1^{n-2})$ or (m^k) for m,k positive integers.*

Proof. Since $n \geq 6$, we have $\lambda_1 \geq 3$ or $\ell(\lambda) \geq 3$. We assume without loss of generality that $\lambda_1 \geq 3$, since if $\lambda_1 < 3$, then $\ell(\lambda) \geq 3$ and in this case the first row of λ' is greater or equal to 3 and $s_{(1^n)} * (s_{(n-2,2)} * s_{\lambda'}) = s_{(n-2,2)} * s_\lambda$.

By Theorem 3.2, $g_{(n-2,2),\lambda,\nu} = k^\lambda_{(2),\nu} + k^\lambda_{(1,1),\nu}$. By Lemma 4.3, if $\ell(\lambda) > 2$ or if λ is not a rectangular partition then there exists a $\nu \vdash n$ such that $k^\lambda_{(1,1),\nu} \geq 2$. Therefore, $s_{(n-2,2)} * s_\lambda$ is not multiplicity free for these λ's. If $\ell(\lambda) = 2$ and $\lambda_1 > \lambda_2 > 1$, then by Lemma 4.4 $s_{(n-2,2)} * s_\lambda$ is not multiplicity free. Thus, the only partitions left for consideration are those in the statement of the theorem.

Since $s_{(n-2,2)} * s_\lambda$ is multiplicity free if and only if $s_{(n-2,2)} * s_{\lambda'}$ is multiplicity free, it suffices to show that $s_{(n-2,2)} * s_\lambda$ is multiplicity free when $\lambda = (n), (n-1,1)$, and $\lambda = (m^k)$ where $m \geq k$. For $\lambda = (n)$ or (1^n) the result is trivial. For $\lambda = (n-1,1)$ or $(2,1^{n-2})$ the result follows from Proposition 4.1.

Let m,k be positive integers such that $n = mk$, $m \geq k$ and $k \geq 2$. Define the multiset

$$K_{(m^k),\alpha} := \{\nu \vdash n \,|\, k^{(m^k)}_{\alpha\nu} \neq 0\}.$$

By [14, Theorem 2.1] or by the Skew Algorithm we have that $s_{(m^k)/(1,1)} = s_{(m^{k-2},(m-1)^2)}$. Thus, by Pieri's rule we have

$$K_{(m^2),(1,1)} = \{(m,m),(m,m-1,1),(m-1,m-1,1,1)\}.$$
$$K_{(m^3),(1,1)} = \{(m+1,m,m-1),(m+1,m-1,m-1,1),(m^3),(m^2,m-1,1),$$
$$(m,m-1,m-1,1,1)\}.$$

If $k > 3$:

$$K_{(m^k),(1,1)} = \{((m+1)^2,m^{k-4},(m-1)^2),(m+1,m^{k-2},m-1),(m^k),$$
$$(m^{k-1},m-1,1),(m+1,m^{k-3},(m-1)^2,1),(m^{k-2},(m-1)^2,1^2)\}.$$

Claim. Let $k \geq 2$.

(a) If $m > 3$, then

$$K_{(m^k),(2)} = \{(m+2,m^{k-2},m-2),(m+1,m^{k-2},m-2,1),(m^{k-1},m-2,2)\}.$$

(b) $K_{(3^k),(2)} = \{(5,3^{k-2},1),(4,3^{k-2},1^2)\}$.

Assuming the claim, we have determined the diagrams that occur in each $K_{\alpha,(m^k)}$ when $\alpha = (1,1)$ and $\alpha = (2)$ and they are disjoint, by Theorem 3.2, $s_{(n-2,2)} * s_{(m^k)}$ is multiplicity free. □

Proof of Claim. (a) From [2] we have that $g_{(n-2,2),(m^k),\nu} = 0$ if $\nu_1 > m+2$. Hence $k^{(m^k)}_{(2),\nu} = 0$ and $k^{(m^k)}_{(1,1),\nu} = 0$ if $\nu_1 > m+2$. Notice that if $\nu_1 < m$ the conditions of a Kronecker tableaux of shape $(m^k)/(2)$ and type $\nu/(2)$ cannot be satisfied. For example, let $\nu_1 = m-1$ and consider the first two rows of a filling of $(m^k)/(2)$. We must place two 2's in the first row because there are not enough 1's to place two 1's in the second row. This forces the following situation:

		1	$\cdots$	1	2	2
1		2	$\cdots$	2	3	3

In the empty box $(2,2)$ we cannot place another 1, and placing a 2 violates the α-lattice permutation condition since there will be more 2's than the number of 1's plus 2 in the initial subword of length $2m-3$. A similar argument can be applied if $\nu_1 < m-1$. Therefore, if $k^{(m^k)}_{(2),\nu} \in K_{(m^k),(2)}$, we must have $m \leq \nu_1 \leq m+2$.

If $\nu_1 = m+2$, then the first row of $(m^k)/(2)$ and the boxes $(2,1)$ and $(2,2)$ must be filled with 1's. This completely forces the following tableau:

		1	$\cdots$	1
1	1	2	$\cdots$	2
2	2	3	$\cdots$	3
			$\vdots$	
k-2	k-2	k-1	$\cdots$	k-1
k-1	k-1	k	$\cdots$	k

Hence, $k^{(m^k)}_{(2),(m+2,m^{k-2},m-2)} = 1$.

Now suppose that $\nu_1 = m+1$. In this case, we cannot place two 2's in the first row because there will not be enough room for all the 1's. Hence we must place two 1's in the second row. This condition forces the following tableau:

		1	$\cdots$	1	2
1	1	2	$\cdots$	2	3
2	2	3	$\cdots$	3	4
			$\vdots$		
k-2	k-2	k-1	$\cdots$	k-1	k
k-1	k-1	k	$\cdots$	k	k+1

Therefore, $k^{(m^k)}_{(2),(m+1,m^{k-2},m-2,1)} = 1$.

Now if $\nu_1 = m$, we cannot place all 1's in the first row of $(m^k)/(2)$, otherwise the definition of Kronecker tableaux is not satisfied. Hence we must place two 1's in the second row and two 2's in the first row. Now it is easy to see that in order satisfy the condition of an α-lattice permutation we are forced to fill the rest of the diagram in the following unique way:

		1	$\cdots$	1	2	2
1	1	2	$\cdots$	2	3	3
2	2	3	$\cdots$	3	4	4
			$\vdots$			
k-1	k-1	k-1	$\cdots$	k-1	k	k
k-2	k-2	k	$\cdots$	k	k+1	k+1

Thus, $k^{(m^k)}_{(2),(m^{k-1},m-2,2)} = 1$.

Part (b) is shown in the same manner with the exception that it is not possible to obtain Kronecker tableaux of shape $(m^k)/(2)$ and type $(m^{k-1}, m-2, 2)/(2)$ when $m = 3$. $\square$

Remark. For $n = 4$, every partition is of the type listed in Theorem 4.5. In this case, although not covered by Theorem 4.5, the result still holds that $s_{(2,2)} * s_\lambda$ is multiplicity free for all $\lambda \vdash 4$. For $n = 5$, Theorem 4.5 does not hold because in addition to the cases listed in the statement we also have that $s_{(3,2)} * s_{(3,2)}$ and $s_{(3,2)} * s_{(2,2,1)}$ are multiplicity free.

The following corollary follows directly from the proof of Theorem 4.5. Recall that $\chi(S)$ is the function that has value 1 if S is true and 0 if S is false.

Corollary 4.6. *Let $n \geq 6$ and $n = mk$. Then*

$$\begin{aligned} s_{(n-2,2)} * s_{(m^k)} = s_{(m^k)} &+ s_{(m^{k-1},m-1,1)} + s_{(m^{k-2},(m-1)^2,1^2)} + \chi(k \geq 4) s_{((m+1)^2,m^{k-4},(m-1)^2)} \\ &+ \chi(k \geq 3) s_{(m+1,m^{k-2},m-1)} + \chi(k \geq 3) s_{(m+1,m^{k-3},(m-1)^2,1)} \\ &+ s_{(m+2,m^{k-2},m-2)} + s_{(m+1,m^{k-2},m-2,1)} + \chi(m \geq 4) s_{(m^{k-1},m-2,2)}. \end{aligned}$$

Lemma 4.7. *Let n and p be positive integers such that $p \geq 3$ and let $\lambda \vdash n$ be such that $\lambda_1 \geq 2p-1$. If $3 \leq \ell(\lambda)$ or if $\ell(\lambda) = 2$ and $\lambda_1 > \lambda_2 > 1$, then*

$$s_{(p-1,1)} s_{\lambda/(p-1,1)} - s_{(p-2,1)} s_{\lambda/(p-2,1)}$$

is not multiplicity free.

Proof. Since $\lambda_1 \geq 2p-1$, by Lemma 3.1. the coefficient of s_ν in the Schur function expansion of $s_{(p-1,1)} s_{\lambda/(p-1,1)} - s_{(p-2,1)} s_{\lambda/(p-2,1)}$ equals the number of Kronecker tableaux of shape
$\lambda/(p-1,1)$ and type $\nu/(p-1,1)$. To prove the lemma it suffices to show that it is possible to construct two Kronecker tableaux of the same type. We proceed by cases.

Case 1. Let $\ell(\lambda) \geq 3$ and $\lambda_2 \geq p-1$. Fill the rows of $\lambda/(p-1,1)$ as follows:

Row1: Label box $(1, \lambda_1)$ with 2 and all other boxes with 1's.

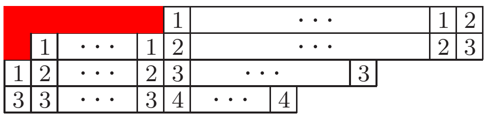
Row 2: Place 1's in boxes $(2, i)$ for $i = 2, \ldots, p-1$. For $j = p, \ldots, \lambda_2$, if box $(1, j)$ is labelled x place $x+1$ in box $(2, j)$.

Row 3: Place 1 in box $(3, 1)$. Then, if box $(2, i)$ is labelled x place $x+1$ in box $(3, i)$ for all $i = 2, \ldots, \lambda_3$.

If $\ell(\lambda) \geq 4$, then place a 3 in box $(4, 1)$. For $i = 2, \ldots, \lambda_4$, place $x+1$ in box $(4, i)$ if box $(3, i)$ contains x.

For all other rows, place an $x+1$ under a box labelled x.

Below we show a sample tableaux. In our example $\lambda_1 = \lambda_2$ to illustrate the case when we need to add a 3 in the second row. If in addition $\lambda_2 = \lambda_3$, then we would also have a 4 at the end of the third row and so on.

$T =$ 

The tableau T is a SSYT with $p-2$ boxes labelled 1 in the second row. It is straightforward to check that its reverse reading word is a $(p-1,1)$-lattice permutation. Hence, T is a Kronecker tableau of shape $\lambda/(p-1,1)$.

We can switch the 2 in box $(1, \lambda_1)$ with the 1 in box $(3, 1)$ to obtain a different Kronecker tableau of the same shape and type as T. Hence, $k^\lambda_{(p-1,1),type(T)} \geq 2$.

Case 2. Let $\ell(\lambda) \geq 3$ and $2 \leq \lambda_2 < p-1$. In this case fill the diagram $\lambda/(p-1,1)$ as follows:

Row 1: Place 2's in the last $p-2$ boxes and 1's in all other boxes. Since $\lambda_1 \geq 2p-1$, we always have at least two 1's in the first row.

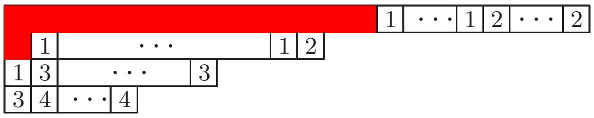
Row 2: Place 1's in boxes $(2,i)$ for $i = 2, \ldots, \lambda_2 - 1$ (if $\lambda_2 > 2$) and place 2 in box $(2, \lambda_2)$. (If $\lambda_2 = 2$, there are no 1's in the second row.) Notice that placing the 2 in this box does not violate the definition of Kronecker tableaux because in the initial subword of length $\lambda_1 - p + 2$ we have $\#1's + (p-1) \geq p+1$ and $\#2's + 1 = p$.

Row 3: Place a 1 in box $(3,1)$ and 3's in the remaining boxes.

If $\ell(\lambda) \geq 4$, then in row 4 place a 3 in box $(4,1)$ and 4's in the remaining boxes.

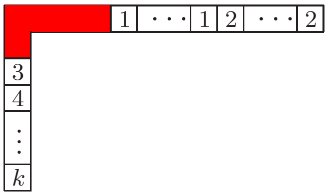
For all remaining rows place an $x+1$ under a box labelled x.

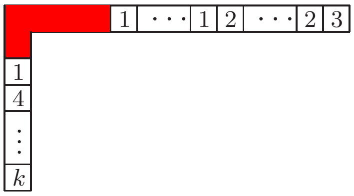
We illustrate the first four rows of such a tableau:

$T' =$ 

The tableau T' is a Kronecker tableaux of shape $\lambda/(p-1,1)$. Switching the 2 in box $(2,\lambda_2)$ with the 1 in box $(3,1)$ yields another Kronecker tableaux of the same type and shape as T'. Hence, $k^{\lambda}_{(p-1,1),type(T')} \geq 2$.

Case 3. Let $\ell(\lambda) \geq 3$ and $\lambda_2 = 1$. In this case λ is a hook partition. We illustrate two Kronecker tableaux of shape $\lambda/(p-1,1)$ and type $(\lambda_1 - p + 2, p-1, 1^{n-\lambda_1-1})/(p-1,1)$:



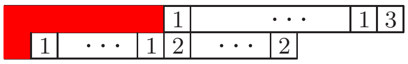


Here $\ell(\lambda) = k$ and if there are m 1's in the tableau on the left hand side, then there are $m-1$ 1's in the first row of the tableau on the right. The number of 2's in the first row is $p-2$.

By cases (1)-(3) we have that $s_{(p-1,1)} * s_{\lambda/(p-1,1)} - s_{(p-2,1)} * s_{\lambda/(p-2,1)}$ is not multiplicity free if $\ell(\lambda) \geq 3$.

Case 4. Let $\ell(\lambda) = 2$ and $\lambda_1 \neq \lambda_2$ and $\lambda_2 > 1$. We have two subcases.

Case (i): $\lambda_2 > p-1$. In this case the following is a Kronecker tableau of shape $\lambda/(p-1,1)$ and type $\nu/(p-1,1)$, where $\nu = (\lambda_1 + (p-3), \lambda_2 - (p-2), 1)$:



We obtain another Kronecker tableau of the same shape and type by switching the 3 in box $(1, \lambda_1)$ with the 2 in box $(2, \lambda_2)$. Since $\lambda_1 \neq \lambda_2$, this is always possible. Hence, $k^{\lambda}_{(p-1,1),\nu} \geq 2$.

Case (ii): $2 \leq \lambda_2 \leq p-1$. In this case the following are Kronecker tableaux of shape $\lambda/(p-1,1)$ and type $\nu/(p-1,1)$, where $\nu = (\lambda_1 + \lambda_2 - p, p-1, 1)$:

$$\begin{array}{cccccccccc} \blacksquare & \cdots & \blacksquare & 1 & \cdots & 1 & 2 & \cdots & 2 \\ \blacksquare & 1 & \cdots & 1 & 3 & & & & \end{array} \qquad \begin{array}{cccccccccc} \blacksquare & \cdots & \blacksquare & 1 & \cdots & 1 & 2 & \cdots & 2 & 3 \\ \blacksquare & 1 & \cdots & 1 & 1 & & & & & \end{array}$$

There are $p-2$ boxes labelled 2 in the first row of these diagrams. In the first row, in the left tableau there is one more 1 than in the right tableau. Hence, $k^{(\lambda_1,\lambda_2)}_{(p-1,1),\nu} \geq 2$ whenever $\lambda_1 > \lambda_2 > 1$. This concludes the proof. $\square$

Theorem 4.8. *Let $n > 16$ and $\lambda \vdash n$. Then $s_{(n-3,3)} * s_\lambda$ is multiplicity free if and only if $\lambda = (n), (1^n), (n-1,1), (2,1^{n-2})$ and if n is even also $\lambda = (n/2, n/2)$ or $(2^{n/2})$.*

Proof. Since $n > 16$, we have $\ell(\lambda) \geq 5$ or $\lambda_1 \geq 5$. Without loss of generality, for the remainder of the proof we can assume that $\lambda_1 \geq 5$. The results for (1^n), $(2, 1^{n-2})$ and $(2^{n/2})$ will follow by the fact that they are conjugate to $(n), (n-1, 1)$ and $(n/2, n/2)$ respectively.

Since $\lambda_1 \geq 5$, by Theorem 3.2 we have $g^{\lambda}_{(n-3,3),\nu} = k^{\lambda}_{(3),\nu} + k^{\lambda}_{(2,1),\nu} + k^{\lambda}_{(1,1,1),\nu}$. By Lemma 4.3, if $\ell(\lambda) > 3$ and λ is not a rectangular partition, then there exists a ν such that $k^{\lambda}_{(1,1,1),\nu} \geq 2$. Also, by Lemma 4.7, if $\ell(\lambda) \geq 3$ or if $\ell(\lambda) = 2$ and $\lambda_1 > \lambda_2 > 1$, then there exists a ν such that $k^{\lambda}_{(2,1),\nu} \geq 2$. Hence, for all these cases $s_{(n-3,3)} * s_\lambda$ is not multiplicity free. It remains to investigate the cases in which $\ell(\lambda) \leq 2$ and either $\lambda_2 = 0, 1$ or $n/2$ (when n is even). If $\lambda_2 = 0$, it is trivially true that $s_{(n-3,3)} * s_{(n)}$ is multiplicity free. If $\lambda_2 = 1$, since $C(n-3,3) = 1$, it follows by Proposition 4.1 that $s_{(n-3,3)} * s_{(n-1,1)}$ is multiplicity free.

Now assume that n is even and $\lambda = (n/2, n/2)$. Recall that $K_{\alpha,\lambda}$ is the multiset containing all ν's for which $k^{\lambda}_{\alpha\nu} \neq 0$. The multiplicity of each ν in $K_{\alpha,\lambda}$ is $k^{\lambda}_{\alpha\nu}$. We have that $K_{(1,1,1),(n/2,n/2)} = \emptyset$,

$$K_{(2,1),(\frac{n}{2},\frac{n}{2})} = \left\{ \left(\frac{n}{2}+1, \frac{n}{2}-1\right), \left(\frac{n}{2}+1, \frac{n}{2}-2, 1\right), \left(\frac{n}{2}, \frac{n}{2}-1, 1\right), \left(\frac{n}{2}, \frac{n}{2}-2, 1, 1\right), \right.$$
$$\left. \left(\frac{n}{2}, \frac{n}{2}-2, 2\right), \left(\frac{n}{2}-1, \frac{n}{2}-1, 2\right), \left(\frac{n}{2}-1, \frac{n}{2}-2, 2, 1\right) \right\},$$

and

$$K_{(3),(n/2,n/2)} = \left\{ \left(\frac{n}{2}+3, \frac{n}{2}-3\right), \left(\frac{n}{2}+2, \frac{n}{2}-3, 1\right), \right.$$
$$\left. \left(\frac{n}{2}+1, \frac{n}{2}-3, 2\right), \left(\frac{n}{2}, \frac{n}{2}-3, 3\right) \right\}.$$

By inspection we see that $K_{(2,1),(\frac{n}{2},\frac{n}{2})}$ and $K_{(3),(\frac{n}{2},\frac{n}{2})}$ are disjoint. Furthermore, we have

$$s_{(n-3,3)} * s_{(\frac{n}{2},\frac{n}{2})} = s_{(\frac{n}{2}+1,\frac{n}{2}-1)} + s_{(\frac{n}{2}+1,\frac{n}{2}-2,1)} + s_{(\frac{n}{2},\frac{n}{2}-1,1)} + s_{(\frac{n}{2},\frac{n}{2}-2,1,1)} + s_{(\frac{n}{2},\frac{n}{2}-2,2)} + s_{(\frac{n}{2},\frac{n}{2}-3,3)}$$
$$+ s_{(\frac{n}{2}-1,\frac{n}{2}-1,2)} + s_{(\frac{n}{2}-1,\frac{n}{2}-2,2,1)} + s_{(\frac{n}{2}+3,\frac{n}{2}-3)} + s_{(\frac{n}{2}+2,\frac{n}{2}-3,1)} + s_{(\frac{n}{2}+1,\frac{n}{2}-3,2)}.$$

$\square$

Theorem 4.9. *Let $p \geq 4$ and $n > (2p-2)^2$. Then $s_{(n-p,p)} * s_\lambda$ is multiplicity free if and only if $\lambda = (n), (1^n), (n-1,1)$, or $(2, 1^{n-2})$.*

Proof. Since $n > (2p-2)^2$ we have λ_1 or $\ell(\lambda)$ greater or equal to $2p-1$. Without loss of generality we assume that $\lambda_1 \geq 2p-1$. It is well known that if $\lambda = (n)$ or (1^n), then $s_{(n-p,p)} * s_\lambda$ is multiplicity free. Since $C(n-p,p) \leq 1$, by Proposition 4.1 and Corollary 4.2, $s_{(n-p,p)} * s_\lambda$ is also multiplicity free when $\lambda = (n-1,1)$ or $(2,1^{n-2})$.

By Lemma 4.3 and Lemma 4.7, if $\lambda_1 \geq 2p-1$, then $s_{(n-p,p)} * s_\lambda$ could be multiplicity free only when $\lambda_2 = 0, 1$ or, if n is even, when $\lambda = (\frac{n}{2}, \frac{n}{2})$. Hence, we only have to show that for $p \geq 4$, $s_{(n-p,p)} * s_{(\frac{n}{2},\frac{n}{2})}$ is not multiplicity free. We consider the case $p = 4$ separately. The following are two Kronecker tableaux of shape $(n/2, n/2)/\alpha$ and type ν/α where $\nu = (\frac{n}{2}, \frac{n}{2} - 2, 2)$ and $\alpha = (3,1)$ and $\alpha = (2,2)$ respectively.

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & 1 & \cdots & 1 & 2 & 2 \\ \hline & 1 & 1 & 2 & \cdots & 2 & 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & 1 & 1 & \cdots & 1 & 1 & 1 \\ \hline & & 2 & 2 & \cdots & 2 & 3 & 3 \\ \hline \end{array}$$

Hence, $k^{(\frac{n}{2},\frac{n}{2})}_{(3,1),\nu} + k^{(\frac{n}{2},\frac{n}{2})}_{(2,2),\nu} \geq 2$. Therefore, by Theorem 3.2, $g_{(n-4,4),(\frac{n}{2},\frac{n}{2}),\nu} \geq 2$.

If $p \geq 5$, the following are Kronecker tableaux of shape $(n/2, n/2)/\alpha$ and type ν/α, where $\nu = (\frac{n}{2} + p - 4, \frac{n}{2} - p + 2, 2)$ and $\alpha = (p-1,1)$ and $\alpha = (p-2,2)$ respectively:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & & 1 & \cdots & 1 & 2 & 2 \\ \hline & 1 & \cdots & 1 & 2 & \cdots & 2 & 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline & & & & & 1 & \cdots & 1 & 1 & 2 \\ \hline & & 1 & \cdots & 1 & 2 & \cdots & 2 & 3 & 3 \\ \hline \end{array}$$

Hence, we have $k^{(\frac{n}{2},\frac{n}{2})}_{(p-1,1),\nu} + k^{(\frac{n}{2},\frac{n}{2})}_{(p-2,2),\nu} \geq 2$. Therefore, by Theorem 3.2, $g_{(n-p,p),(\frac{n}{2},\frac{n}{2}),\nu} \geq 2$ for all $p \geq 5$. $\square$

4.2. **The Kronecker Product of a two row shape and a hook shape.** In this subsection we show how to use the combinatorial rule of Theorem 3.2 to obtain formulas for the coefficients in the Kronecker product $s_{(n-p,p)} * s_{(n-s,1^s)}$, where $n - s \geq 2p - 1$. The formulas obtained are equivalent to the results obtained by Remmel [7]. For this reason, we include only one example: the multiplicity of a hook in $s_{(n-p,p)} * s_{(n-s,1^s)}$ in the case $p \geq 2$ and $n - s \geq 2p - 1$.

We first need the following preliminary result.

Proposition 4.10. *Let s, p, n be positive integers such that $n - s \geq 2p - 1$ and let $\nu = (\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}) \vdash n$.*

(a) *If $g_{(n-p,p),(n-s,1^s),\nu} \neq 0$, then $n-s-p \leq \nu_1 \leq n-s+1$, $\max\{1, p-s\} \leq \nu_2 \leq p+1$, and $\nu_i \leq 2$ for $i \geq 3$.*

(b) $g_{(n-p,p),(n-s,1^s),\nu} = k^{(n-s,1^s)}_{(\nu_2,1^{p-\nu_2}),\nu} + k^{(n-s,1^s)}_{(\nu_2-1,1^{p-\nu_2+1}),\nu}$.

Proof. (a) Given $\nu = (\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}) \vdash n$, let $\alpha \vdash p$ such that $\alpha \subseteq (n-s, 1^s)$ and $\alpha \subseteq \nu$. Thus $\alpha = (p-u, 1^u)$, $u \leq \min\{s, \ell(\nu) - 1\}$, $p - u \leq \nu_1$. The maximum number of 1's in a Kronecker tableau occurs when $\alpha = (1^p)$ (if $p \leq \min\{s, \ell(\nu)-1\}$) and the first row is filled with 1's and there is a 1 in the first column of λ/α. Therefore $\nu_1 \leq n - s + 1$. The minimum number of 1's occurs when $\alpha = (p)$ and there are no 1's in the first column of λ/α. Therefore $\nu_1 \geq n - s - p$. The maximum number of 2's occurs when $\alpha = (p)$ and there is one 2 in the first column of λ/α. Thus $\nu_2 \leq p + 1$. If $p \leq s + 1$, the minimum number of 2's occurs when $\alpha = (1^p)$ and it equals 1. In this case there are no 2's in

λ/α. If $p > s+1$, the minimum number of 2's occurs when $\alpha = (p-s, s)$ and it equals $p-s$ (no 2's in the first column of λ/α). Thus $\nu_2 \geq \max\{1, p-s\}$. If $i \geq 3$, $\alpha_i \leq 1$. If $\alpha_i = 1$ at most one i can be placed in the first column of λ/α. If $\alpha_i = 0$ at most one i can be placed in the fist row of λ/α and at most one i can be placed in the first column of λ/α. Thus $\nu_i \leq 2$ for $i \geq 3$.

(b) When forming Kronecker tableaux of shape λ/α and type ν/α we cannot fill the box in position $(2, \alpha_1)$ with 1. If $\alpha_1 \neq 1$ or p, we must place 2's in exactly $\alpha_1 - 1$ boxes of the first row of λ/α and we can place at most one 2 in the first column of λ/α. Since $\alpha_2 = 1$, we have $\nu_2 = \alpha_1$ or $\nu_2 = \alpha_1 + 1$. If $\alpha_1 = p$, we must place 2's in exactly $\alpha_1 = p$ boxes of the first row of λ/α and we can place at most one 2 in the first column of λ/α. Since $\alpha_2 = 0$, we have $\nu_2 = \alpha_1$ or $\nu_2 = \alpha_1 + 1$. If $\alpha_1 = 1$, we can place at most one 2 in the first column of λ/α. Since $\alpha_2 = 1$, we have $\nu_2 = 1 = \alpha_1$ or $\nu_2 = 2 = \alpha_1 + 1$. $\square$

Note. If $n - s = 2p - 1$, then $k_{(p),\nu}^{(n-s,1^s)} = 0$ and if s_ν appears in the decomposition of $s_{(n-p,p)} * s_{(n-s,1^s)}$ then $\nu_2 \leq p$ and $\nu_1 \geq n - s - p + 1 = p$.

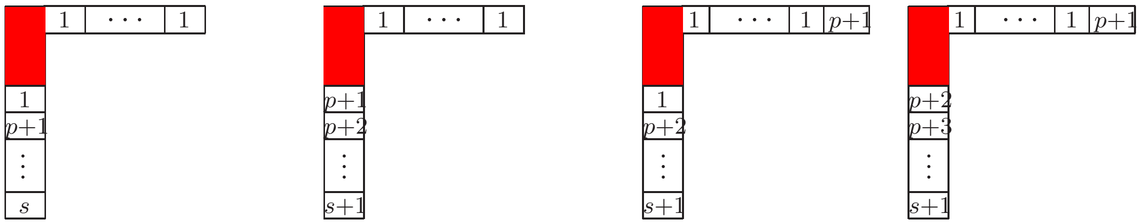
By analyzing Kronecker tableaux we have obtained the following well-known result [7].

Corollary 4.11. *If $g_{(n-p,p),(n-s,1^s),\nu} \neq 0$, then ν is a hook, $\nu = (\nu_1, 1^{n-\nu_1})$, or a double hook, $\nu = (\nu_1, \nu_2, 2^i, 1^j)$.*

To show the applicability of Theorem 3.2, we use the previous proposition to obtain the coefficient of a hook in the Kronecker product $s_{(n-p,p)} * s_{(n-s,1^s)}$.

It follows from Proposition 4.10 that $g_{(n-p,p),(n-s,1^s),(n-t,1^t)} = k_{(1^p),(n-t,1^t)}^{(n-s,1^s)}$.

To obtain the possible Kronecker tableaux we must have $p \leq s+1$ and no boxes of $(n-s, 1^s)/(1^p)$ can be filled with 2's.



$\nu = (n-s+1, 1^{s-1})$ $\quad \nu = (n-s, 1^s)$ $\quad \nu = (n-s, 1^s)$ $\quad \nu = (n-s-1, 1^{s+1})$

Hence, the highest possible coefficient in this case is 2 and we have

$$g_{(n-p,p),(n-s,1^s),(n-t,1^t)} = \begin{cases} 2 & \text{if } \ t = s \text{ and } s \geq p, \\ 1 & \text{if } \ (t = s \text{ and } s = p-1), \\ & \quad \text{or } (t = s+1 \text{ and } s \geq p-1), \\ & \quad \text{or } (t = s-1 \text{ and } s \geq p), \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, Proposition 4.10 leads to formulas for the coefficient of a double hook in $s_{(n-p,p)} * s_{(n-s,1^s)}$.

4.3. **Multiplicities in $s_{(n-p,p)} * s_\lambda$.** In this section we use Theorem 3.2 to compute formulas for the multiplicity of Schur functions corresponding to two row partitions in the decomposition of $s_{(n-p,p)} * s_\lambda$. Using this formula we show that under special conditions, if λ itself is a two row partition, these coefficients are unimodal. We have also computed a formula for the coefficient of $s_{(\nu_1,\nu_2,\nu_3,\nu_3)}$, $\nu_3 \neq 0$, in the decomposition of $s_{(n-p,p)} * s_\lambda$. The formulas are easy to program and can yield the values of the coefficients for arbitrarily large n.

4.3.1. *The multiplicity of $s_{(n-t,t)}$ in $s_{(n-p,p)} * s_\lambda$.* In this subsection we give a formula for the coefficient of $s_{(n-t,t)}$ in the product $s_{(n-p,p)} * s_\lambda$. As corollaries of this formula we obtain simple formulas for the case when λ is also a two row partition and we show that under special conditions the Kronecker coefficients are unimodal.

Proposition 4.12. *Let n, t, p be non-negative integers, $p > 0$, and $\lambda \vdash n$ be such that $\lambda_1 \geq 2p-1$. Let*

$$\begin{aligned}
m_l &:= \min\{\lambda_1-\lambda_2, t-\lambda_2+p-2l-\max\{0,\lambda_3-l\}-\lambda_4, p-2l-1, \\
&\qquad\qquad \lambda_1+\lambda_4+p-2l-t\}\\
M_l &:= \max\{0, t-\lambda_2+p-2l-\lambda_3\}\\
m'_l &:= \min\{\lambda_2-\max\{l,\lambda_3\}, t-p+l-\max\{0,\lambda_3-l\}-\lambda_4, \lambda_1-2p+3l, \\
&\qquad\qquad \lambda_4-t+\lambda_1-p+l+\lambda_2\}\\
M'_l &:= \max\{0, t-p+l-\lambda_3, \lambda_2-p+l\}\\
a_p &:= \chi(p \text{ even})\chi\left(\lambda_3 \leq \frac{p}{2} \leq \min\{t,\lambda_2\}\right)\chi(\lambda_2+\lambda_4 \leq t \leq \min\{\lambda_2+\lambda_3, \lambda_1+\lambda_4\})\\
b(l) &:= \chi(\lambda_2-p+2l+\max\{0,\lambda_3-l\}+\lambda_4 \leq t \leq \lambda_2+\lambda_3-1)\\
c(l) &:= \chi(\lambda_1-\max\{p-l,\lambda_2\} \geq p-2l)\chi(p-l+\max\{0,\lambda_2-p+l\}+\max\{0,\lambda_3-l\}\\
&\qquad\qquad +\lambda_4 \leq t \leq \lambda_2+\lambda_3+p-2l-\max\{0,\lambda_3-l\})
\end{aligned}$$

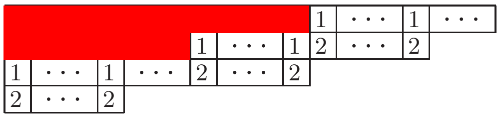
The coefficient of $s_{(n-t,t)}$ in the decomposition of $s_{(n-p,p)} * s_\lambda$ equals 0 if $\ell(\lambda) > 4$. If $\ell(\lambda) \leq 4$, then it equals

$$a_p + \sum_{l=\max\{\lambda_4, p-\lambda_2\}}^{\min\{\lfloor\frac{p+1}{2}\rfloor-1, t, \lambda_2, p-\lambda_3\}} b(l)\max\{0, m_l-M_l+1\} + \sum_{l=\lambda_4}^{\min\{\lfloor\frac{p+1}{2}\rfloor-1, t, \lambda_2, p-\lambda_3\}} c(l)\max\{0, m'_l-M'_l+1\}.$$

Proof. We count Kronecker tableaux of shape λ/α and type $(n-t,t)/\alpha$, where $\alpha \vdash p$, $\alpha \subseteq \lambda$ and $\alpha \subseteq (n-t,t)$. Thus $\alpha = (p-l,l)$ with $l \leq \lfloor\frac{p}{2}\rfloor$, $l \leq \lambda_2$, $l \leq t$. Since the type of the tableaux is $(n-t-p+l, t-l)$, the shape λ/α cannot contain three boxes directly above each other. Thus, we must have $\ell(\lambda) \leq 4$, $p-l \geq \lambda_3$ and $l \geq \lambda_4$. If $t < \lambda_4$, the coefficient of $s_{(n-t,t)}$ is zero. Hence, we consider the Kronecker product $s_{(n-p,p)} * s_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}$ with $\lambda_1 \geq 2p-1$, $\lambda_3 \leq p$ and $\lambda_4 \leq t$. We have $k^\lambda_{(p-l,l),(n-t,t)} = 0$ unless

$$\lambda_4 \leq l \leq \min\{\lfloor\frac{p}{2}\rfloor, t, \lambda_2, p-\lambda_3\}. \tag{1}$$

Moreover, whenever λ/α contains two boxes directly above each other, the top box must be labelled 1 and the bottom box must be labelled 2.



There must be enough 1's and 2's:

$$n - t \geq p - l + \max\{0, \lambda_2 - p + l\} + \max\{0, \lambda_3 - l\} + \lambda_4$$
$$t \geq l + \max\{0, \lambda_2 - p + l\} + \max\{0, \lambda_3 - l\} + \lambda_4$$

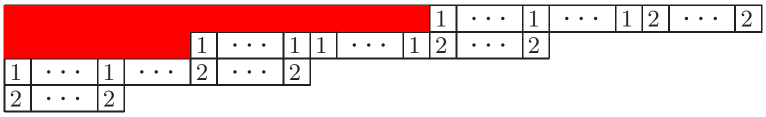
(I) Kronecker tableaux of shape $\lambda/(p/2, p/2)$ and type $(n-t, t)/(p/2, p/2)$. Here $l = p/2$. Thus, p must be even and since $p - l \geq \lambda_3$, then $p/2 \geq \lambda_3$. Condition (1) becomes $\lambda_3 \leq \frac{p}{2} \leq \min\{t, \lambda_2\}$. The first row of $\lambda/(p-l, l)$ must be filled with 1's and the second row must be filled with 2's.

- Enough 1's: $n - t \geq \lambda_1 + \lambda_4$.
- Enough 2's: $t \geq \lambda_2 + \lambda_4$.
- Lattice permutation: $t - \lambda_4 \leq \lambda_1$.

Hence,

$$k^{\lambda}_{(\frac{p}{2},\frac{p}{2}),(n-t,t)}$$
$$= \chi(p \text{ even})\chi\left(\lambda_3 \leq \frac{p}{2} \leq \min\{t, \lambda_2\}\right)\chi(\lambda_2 + \lambda_4 \leq t \leq \min\{\lambda_2 + \lambda_3, \lambda_1 + \lambda_4\}).$$

(II) Kronecker tableaux of shape $\lambda/(p - l, l)$ and type $(n - t, t)/(p - l, l)$ with the box in position $(2, p - l)$ labelled 1 and strictly less than $p - 2l$ boxes labelled 2 in the first row. The tableaux with exactly $p - 2l$ boxes labelled 2 in the first row will be counted in (III). We consider only $l \neq p/2$, i.e., $l \leq \lfloor \frac{p+1}{2} \rfloor - 1$.



- Position $(2, p - l)$ contained in $\lambda/(p - l, l)$: $l \geq p - \lambda_2$ (thus $\max\{0, \lambda_2 - p + l\} = \lambda_2 - p + l$).
- Enough 1's (at most $p-2l-1$ boxes labelled 2 in the first row): $n-t \geq \lambda_1 - p + 2l + 1 + p - 2l + \lambda_4$, i.e., $t \leq \lambda_2 + \lambda_3 - 1$.
- Enough 2's: $t \geq \lambda_2 - p + 2l + \max\{0, \lambda_3 - l\} + \lambda_4$.

We fill k boxes in the first row with 2's.

- Strictly less than $p - 2l$ boxes labelled 2 in first row: $k \leq p - 2l - 1$
- Room for 2's: $k \leq \lambda_1 - \lambda_2$.
- Total number of 2's: $k \leq t - \lambda_2 + p - 2l - \max\{0, \lambda_3 - l\} - \lambda_4$
- Lattice permutation in first row: $k \leq p - 2l$.
- Lattice permutation in third row: $k \leq \lambda_1 + \lambda_4 + p - 2l - t$.
- Minimum number of 2's in first row: $k \geq t - \lambda_2 + p - 2l - \lambda_3$

Set

$$m_l := \min\{\lambda_1 - \lambda_2, t - \lambda_2 + p - 2l - \max\{0, \lambda_3 - l\} - \lambda_4, p - 2l - 1,$$

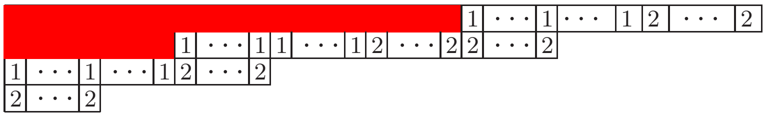
$$\lambda_1 + \lambda_4 + p - 2l - t\}$$

$$M_l := \max\{0, t - \lambda_2 + p - 2l - \lambda_3\}.$$

The number of Kronecker tableaux of shape $\lambda/(p-l,l)$ and type $(n-t,t)/(p-l,l)$ with the box in position $(2, p-l)$ labelled 1 and strictly less than $p-2l$ boxes labelled 2 in the first row equals

$$\sum_{l=\max\{\lambda_4, p-\lambda_2\}}^{\min\{\lfloor \frac{p+1}{2} \rfloor -1, t, \lambda_2, p-\lambda_3\}} \chi(\lambda_2 - p + 2l + \max\{0, \lambda_3 - l\} + \lambda_4 \le t \le \lambda_2 + \lambda_3 - 1) \cdot \max\{0, m_l - M_l + 1\}.$$

(III) Kronecker tableaux of shape $\lambda/(p-l,l)$ and type $(n-t,t)/(p-l,l)$ with exactly $p-2l$ boxes in the first row labelled 2. Again we consider only $l \neq p/2$, i.e., $l \le \lfloor \frac{p+1}{2} \rfloor - 1$.



- Room for 2's in first row: $\lambda_1 - \max\{p-l, \lambda_2\} \ge p - 2l$.
- Enough 2's: $t \ge p - l + \max\{0, \lambda_2 - p + l\} + \max\{0, \lambda_3 - l\} + \lambda_4$.
- Enough 1's: $n - t \ge \lambda_1 - p + 2l + \max\{0, \lambda_3 - l\} + \lambda_4$.

Fill k boxes in the second row with 2's.

- Room for 2's in second row and labels increasing in rows: $\max\{0, \lambda_2 - p + l\} \le k \le \lambda_2 - \max\{l, \lambda_3\}$.
- Total number of 2's: $k \le t - p + l - \max\{0, \lambda_3 - l\} - \lambda_4$
- Lattice permutation in second row: $k \le \lambda_1 - 2p + 3l$.
- Lattice permutation in third row: $k \le \lambda_4 - t + \lambda_1 - p + l + \lambda_2$.
- Minimum number of 2's in second row: $k \ge t - p + l - \lambda_3$

Set

$$m'_l := \min\{\lambda_2 - \max\{l, \lambda_3\}, t - p + l - \max\{0, \lambda_3 - l\} - \lambda_4, \lambda_1 - 2p + 3l, \lambda_4 - t + \lambda_1 - p + l + \lambda_2\}\}$$

$$M'_l := \max\{0, t - p + l - \lambda_3, \lambda_2 - p + l\}.$$

The number of Kronecker tableaux of shape $\lambda/(p-l,l)$ and type $(n-t,t)/(p-l,l)$ with exactly $p-2l$ boxes in the first row labelled 2 equals

$$\sum_{l=\lambda_4}^{\min\{\lfloor \frac{p+1}{2} \rfloor -1, t, \lambda_2, p-\lambda_3\}} \chi(\lambda_1 - \max\{p-l, \lambda_2\} \ge p - 2l) \cdot \chi(n - t \ge \lambda_1 - p + 2l + \max\{0, \lambda_3 - l\} + \lambda_4) \cdot \chi(t \ge p - l + \max\{0, \lambda_2 - p + l\} + \max\{0, \lambda_3 - l\} + \lambda_4) \cdot \max\{0, m'_l - M'_l + 1).$$

□

We now consider the special case when λ is also a two row partition in Theorem 4.12. Let $\lambda = (n-s, s) \vdash n$ such that $n - s \geq 2p - 1$. In this case we have $m_l = \min\{n - 2s, t - s + p - 2l, p - 2l - 1, n - s + p - 2l - t\}$. Since $n - t \geq t$, we have $m_l = \min\{n - 2s, t - s + p - 2l, p - 2l - 1\}$. Also $M_l = t - s + p - 2l$. We see that $M_l \geq m_l$. We obtain $m_l = M_l$ if and only if $t - s + p - 2l \leq \min\{n - 2s, p - 2l - 1\}$, i.e.,

$$m_l = M_l \text{ if and only if } t \leq s - 1 \text{ and } l \geq 1/2(t + s + p - n).$$

Similarly, $m'_l = \min\{s-l, t-p+l, n-s-2p+3l\}$ and $M'_l = \max\{t-p+l, \max\{0, s-p+l\}\}$. Again, $M'_l \geq m'_l$ and

$$M'_l = m'_l \text{ if and only if } l \geq p - t, t \geq s, l \leq 1/2(p + s - t), l \geq 1/2(t + p + s - n).$$

We also have

$$\begin{aligned}
a_p &= \chi(p \text{ even})\chi\left(0 \leq \frac{p}{2} \leq s\right)\chi(t = s)\\
b(l) &= \chi(t \leq s - 1)\chi\left(l \leq \frac{t - s + p}{2}\right)\\
c(l) &= \chi(l \geq p - s)\chi(t \geq s)\chi\left(\frac{2s + p - n}{2} \leq l \leq \frac{p + s - t}{2}\right)\\
&\qquad + \chi(l < p - s)\chi\left(\max\left\{\frac{2p + s - n}{3}, p - t\right\} \leq l \leq \frac{p + s - t}{2}\right)
\end{aligned}$$

We obtain the following corollary.

Corollary 4.13. *Let $n, p, s, t,$ be non-negative integers, such that $n - s \geq 2p - 1$ and $p > 0$. Let*

$$\begin{aligned}
m_1 &= \min\left\{t, \left\lfloor\frac{t - s + p}{2}\right\rfloor\right\} & M_1 &= \max\left\{0, p - s, \left\lceil\frac{t + s + p - n}{2}\right\rceil\right\}\\
m_2 &= \min\left\{s, \left\lfloor\frac{p + 1}{2}\right\rfloor - 1\right\} & M_2 &= \max\left\{0, p - s, \left\lceil\frac{2s + p - n}{2}\right\rceil\right\}\\
m_3 &= \min\left\{s, \left\lfloor\frac{p + s - t}{2}\right\rfloor\right\} & M_3 &= M_1\\
m_4 &= \min\left\{s, p - s, \left\lfloor\frac{p + s - t}{2}\right\rfloor\right\} & M_4 &= \max\left\{0, p - t, \left\lceil\frac{t + s + p - n}{2}\right\rceil,\right.\\
& & &\qquad\left.\left\lceil\frac{2p + s - n}{3}\right\rceil\right\}
\end{aligned}$$

Then the coefficient of $s_{(n-t,t)}$ in the decomposition of $s_{(n-p,p)} * s_{(n-s,s)}$ equals

$$\begin{cases}
\max\{0, m_1 - M_1 + 1\} & \textit{if } t < s,\\
\max\{0, m_2 - M_2 + 1\} + \chi(p \textit{ even})\chi(\frac{p}{2} \leq s) & \textit{if } t = s,\\
\max\{0, m_3 - M_3 + 1\} + \max\{0, m_4 - M_4 + 1\} & \textit{if } t > s.
\end{cases}$$

Corollary 4.14. *If $p \leq s$, $n - s \geq 2p - 1$ and $n - p \geq 2s - 1$. Then*

$$g_{(n-p,p),(n-s,s),(n-s,s)} = \begin{cases}
\lfloor\frac{p}{2}\rfloor + 1 & \textit{if } n - p \geq 2s,\\
\lfloor\frac{p}{2}\rfloor & \textit{if } n - p = 2s - 1.
\end{cases}$$

Note. If $p \leq s$, $0 \leq l \leq \lfloor p/2 \rfloor$, there are no Kronecker tableaux of shape $(n-s,s)/(p-l,l)$, and type $(n-t,t)/(p-l,l)$ if $t < s-p$ or $t > s+p$.

Corollary 4.15. *If $p \leq s-1$, $n-s \geq 2p-1$, $n-p \geq 2s-1$ and $n-t \geq 2s-1$. Then the coefficient of $s_{(n-t,t)}$ in $s_{(n-p,p)} * s_{(n-s,s)}$ is zero unless $s-p \leq t \leq s+p$. If $s-p \leq t \leq s+p$, then*

$$g_{(n-p,p),(n-s,s),(n-t,t)} = \begin{cases} \lfloor \frac{p+t-s}{2} \rfloor + 1 & \text{if } t \leq s-1, \\ \lfloor \frac{p+s-t}{2} \rfloor + 1 & \text{if } t \geq s \text{ and } n-p \geq 2s, \\ \lfloor \frac{p+s-t}{2} \rfloor & \text{if } t \geq s \text{ and } n-p = 2s-1, \end{cases}$$

i.e., the sequence of coefficients, as $t = s-p, s-p+1, \ldots, s+p-1, s+p$ is unimodal. It is:

$$\begin{array}{ll} 1,1,2,2,\ldots,\lfloor \frac{p}{2} \rfloor + 1, \ldots, 2,2,1,1 & \text{if } n-p \geq 2s, \\ 0,0,1,1,\ldots,\lfloor \frac{p}{2} \rfloor, \ldots, 1,1,0,0 & \text{if } n-p = 2s-1. \end{array}$$

4.3.2. *The multiplicity of $s_{(\nu_1,\nu_2,\nu_3,\nu_3)}$, in $s_{(n-p,p)} * s_{(n-s,s)}$.* In this subsection we obtain a formula for $g_{(n-p,p),(n-s,s),(\nu_1,\nu_2,\nu_3,\nu_3)}$, $\nu_3 \neq 0$, similar to the formulas given in [9, Theorem 3.1 and 3.2]. The advantage of our formula is that it does not involve cancellations.

Proposition 4.16. *Let $\nu = (\nu_1,\nu_2,\nu_3,\nu_3) \vdash n$ with $\nu_3 \neq 0$ and let n,p,s be positive integers such that $n \geq 2p$, $n \geq 2s$ and $n-s \geq 2p-1$. Let*

$$\begin{aligned} M_1 &:= \max\left\{p-\nu_1, \nu_3, p-s+\nu_3, \left\lceil \frac{1}{3}(2p-\nu_1) \right\rceil, \left\lceil \frac{1}{2}(p+s-\nu_3-\nu_1) \right\rceil \right\} \\ m_1 &:= \min\left\{ \left\lfloor \frac{p+1}{2} \right\rfloor - 1, \nu_2, s-\nu_3, \left\lfloor \frac{1}{2}(\nu_2+\nu_3+p-s) \right\rfloor \right\} \\ M_2 &:= \max\left\{ \nu_3, p-\nu_2, \left\lceil \frac{1}{2}(p+2s-n) \right\rceil \right\} \\ m_2 &:= \min\left\{ \left\lfloor \frac{p+1}{2} \right\rfloor - 1, \nu_2, s-\nu_3, \left\lfloor \frac{1}{2}(s+p-\nu_2-\nu_3) \right\rfloor \right\} \end{aligned}$$

The coefficient of s_ν in the decomposition of $s_{(n-p,p)} * s_{(n-s,s)}$ equals

$$\begin{aligned} g_{(n-p,p),(n-s,s),\nu} = {} & \chi(p \text{ even})\chi(\nu_2+\nu_3 = s)\chi(\nu_3 \leq p/2 \leq \nu_2) \\ & + \chi(\nu_2+\nu_3 \geq s)\max\{0, m_2-M_2+1\} \\ & + \chi(\nu_2+\nu_3 \leq s-1)\max\{0, m_1-M_1+1\}. \end{aligned}$$

Proof. Let $\alpha = (p-l,l)$ with $l \leq \lfloor p/2 \rfloor$, $l \leq s$, $l \leq \nu_2$, $l \geq p-\nu_1$. We determine the number of Kronecker tableaux of shape $(n-s,s)/(p-l,l)$ and type $\nu/(p-l,l)$. In this special case, the number of 3's equals the number of 4's. All 4's must be placed in the second row of λ/α and, because of the lattice permutation condition, all 3's must be placed in the first row. We need:

- Room for 4's: $s-l \geq \nu_3$
- Room for 3's: $n-s-p+l \geq \nu_3$
- Lattice permutation in the first row: $l \geq \nu_3$.

Since $n \geq 2p$ and $n \geq 2s$, the inequality $l \geq \nu_3$ implies $n-s-p+l \geq \nu_3$. Also, the inequality $\nu_3 \leq l \leq s-\nu_3$ implies $\nu_3 \leq s/2$.

We consider only

$$\max\{p-\nu_1,\nu_3\} \le l \le \min\{\lfloor p/2\rfloor, s-\nu_3,\nu_2\} \tag{2}$$

Note that the Kronecker coefficient of s_ν is non-zero only if $\nu_3 \le \min\{p/2, s/2\}$.

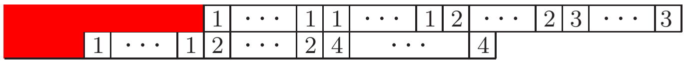
(I) Kronecker tableaux of shape $(n-p,p)/(p/2,p/2)$ and type $\nu/(p/2,p/2)$. In this case we must place all 1's in the first row and all 2's in the second row of $(n-s,s)/(p-l,l)$. Therefore we need $\nu_2+\nu_3=s$. Condition (2) becomes

$$\max\{p-\nu_1,\nu_3\} \le p/2 \le \min\{s-\nu_3,\nu_2\}$$

Now, $p-\nu_1 \le p/2$ implies $\nu_1 \ge p/2$. This condition is satisfied if $p/2 \le \nu_2$. Also, using $\nu_2+\nu_3=s$, the inequality $p/2 \le s-\nu_3$ becomes $p/2 \le \nu_2$. Thus the number of Kronecker tableaux of shape $(n-p,p)/(p/2,p/2)$ and type $\nu/(p/2,p/2)$ equals

$$\chi(p \text{ even})\chi(\nu_2+\nu_3=s)\chi(\nu_3 \le p/2 \le \nu_2).$$

(II) Kronecker tableaux of shape $(n-p,p)/(p-l,l)$ and type $\nu/(p-l,l)$, $l \ne p/2$, such that the box in position $(2,p-l)$ is labelled 1 and there are strictly less than $p-2l$ boxes labelled 2 in the first row. Since $l \ne p/2$, we have $l \le \lfloor\frac{p+1}{2}\rfloor - 1$. In order to place label 1 in position $(2,p-l)$ we must have $p-l \le s$.



There are ν_3 boxes labelled 4 and $p-2l$ boxes labelled 1 in the second row and ν_3 boxes labelled 3 in the first row. We need:

- Room for the 4's: $l \ge p-s+\nu_3$.
- Enough 1's: $l \ge 1/3(2p-\nu_1)$.
- Room for the $\nu_1-2p+3l$ remaining 1's in first row: $n-s-\nu_3-p+l \ge \nu_1-2p+3l$, i.e., $l \le 1/2(n-s+p-\nu_3-\nu_1) = 1/2(\nu_2+\nu_3+p-s)$.
- Number of 2's in the first row strictly less than $p-2l$: $\nu_2-(s-\nu_3-p+2l) \le p-2l-1$, i.e., $\nu_2+\nu_3 \le s-1$.
- Strictly increasing numbers in columns (no 2's above each other): $s-\nu_3 \le \nu_1-p+2l$, i.e., $l \ge 1/2(p+s-\nu_1-\nu_3)$.

Set

$$M_1 := \max\left\{p-\nu_1, \nu_3, p-s+\nu_3, \left\lceil\frac{1}{3}(2p-\nu_1)\right\rceil, \left\lceil\frac{1}{2}(p+s-\nu_3-\nu_1)\right\rceil\right\},$$
$$m_1 := \min\left\{\left\lfloor\frac{p+1}{2}\right\rfloor - 1, \nu_2, s-\nu_3, \left\lfloor\frac{1}{2}(\nu_2+\nu_3+p-s)\right\rfloor\right\}.$$

The number of Kronecker tableaux of shape $(n-p,p)/(p-l,l)$ and type $\nu/(p-l,l)$ such that the box in position $(2,p-l)$ is labelled 1 and there are strictly less than $p-2l$ boxes labelled 2 in the first row equals

$$\chi(\nu_2+\nu_3 \le s-1)\max\{0, m_1-M_1+1\}.$$

(III) Kronecker tableaux of shape $(n-p,p)/(p-l,l)$ and type $\nu/(p-l,l)$, $l \ne p/2$, such that exactly $p-2l$ boxes in the first row are labelled 2. Again, $l \le \lfloor\frac{p+1}{2}\rfloor - 1$.

				1	⋯	1	⋯	1	2	⋯	2	3	⋯ 3
	1	⋯	1	2	⋯	2	4	⋯	4				

There are ν_3 boxes labelled 3 and $p-2l$ boxes labelled 2 in the first row and ν_3 boxes labelled 4 in the second row. We need:

- Room for the 2's in first row: $n-s-\nu_3-p+l \leq p-2l$, i.e., $l \geq 1/3(2p+s-n+\nu_3)$. Since $n-s \geq 2p-1$, we have $1/3(2p+s-n+\nu_3) \leq 1/3(1+\nu_3) \leq \nu_3$ (here $\nu_3 \neq 0$). Since $l \geq \nu_3$, the condition $l \geq 1/3(2p+s-n+\nu_3)$ is satisfied.
- Enough 2's: $l \geq p-\nu_2$.
- Room for the ν_2-p+l remaining 2's in the second row: $s-\nu_3-l \geq \nu_2-p+l$, i.e., $l \leq 1/2(s+p-\nu_2-\nu_3)$.
- Strictly increasing numbers in columns.
  No 2's above each other: $s-\nu_3 \leq n-s-p+2l-\nu_3$, i.e., $l \geq 1/2(p+2s-n)$.
  No 1's above each other: $s-\nu_3-\nu_2+p-l \leq p-l$, i.e., $\nu_2+\nu_3 \geq s$.

Set

$$M_2 := \max\left\{\nu_3, p-\nu_2, \left\lceil\frac{1}{2}(p+2s-n)\right\rceil\right\},$$

$$m_2 := \min\left\{\left\lfloor\frac{p+1}{2}\right\rfloor - 1, \nu_2, s-\nu_3, \left\lfloor\frac{1}{2}(s+p-\nu_2-\nu_3)\right\rfloor\right\}.$$

The number of Kronecker tableaux of shape $(n-p,p)/(p-l,l)$ and type $\nu/(p-l,l)$ such that exactly $p-2l$ boxes in the first row are labelled 2 equals

$$\chi(\nu_2+\nu_3 \geq s)\max\{0, m_2-M_2+1\}.$$

□

Note. If $p=0$ or $p=1$ the multiplicity of ν in Proposition 4.16 equals 0.

References

[1] Ballantine, C.M. and Orellana, R.C.; "On the Kronecker Product $s_{(n-p,p)} * s_\lambda$.," *Elect. Journal of Comb.* Vol. 12, (2005), Article #R28, 1–26.

[2] Dvir, Y.; "On the Kronecker product of S_n characters", *J. Algebra* **154** (1993), 125–140.

[3] Gessel, I.M; "Multipartite P-partitions and inner products of Schur functions," *Contemp. Math.* **34** (1984), 289–302.

[4] Garsia, A.M., Remmel, J.B.; "Shuffles of Permutations and the Kronecker Product," *Graphs and Combin.* **1** (1985), 217–263..

[5] Lascoux, A.; "Produit de Kronecker des représentations du group symmétrique," *Lecture Notes in Mathematics*, no. **795**, 1980, Springer–Verlag pp. 319–329.

[6] Littlewood, D.E.; "The Kronecker Product of Symmetric Group Representations," *J. London Math. Soc.* **31** (1) 1956, 89–93.

[7] Remmel, J.B.; "Formulas for the expansion of the Kronecker products $s_{(m,n)} \otimes s_{(1^{p-r},r)}$ and $s_{(1^k,2^l)} \otimes s_{(1p-r,r)}$, *Discrete Math.* **99** (1992), 265–287.

[8] Remmel, J.B.; "A Formula for the Kronecker Products of Schur Functions of Hook Shapes," *J. Algebra* **120** (1989), 100–118.

[9] Remmel, J.B.,Whitehead, T.; "On the Kronecker product of Schur functions of two row shapes," *Bull. Belg. Math. Soc.* **1** (1994), 649–683.

[10] Remmel, J.B., Whitney, R.; "Multiplying Schur functions," *J. Algorithms*, **5** (1984), 471–487..

[11] Rosas, M.H.; "The Kronecker product of Schur functions indexed by two-row shapes or hook shapes," *J. Algebraic Combin.* **14** (2001), 153–173.

[12] Stanley, R.P.; *Enumerative Combinatorics Vol 2*, Cambridge Univ. Press, 1999.
[13] Stembridge, J.; "Multiplicity-Free Products of Schur Functions," *Annals Combin.* **5** (2001), 113–121.
[14] van Willigenburg, S.; "Equality of Schur and Skew Schur functions," *Proceedings of the Formal Power Series and Algebraic Combinatorics*, Taormina, 2005 .