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BERGMAN COMPLEXES, COXETER ARRANGEMENTS, AND GRAPH ASSOCIAHEDRA

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Abstract. Tropical varieties play an important role in algebraic geometry. The Bergman complex $\mathcal{B}(M)$ and the positive Bergman complex $\mathcal{B}^+(M)$ of an oriented matroid M generalize to matroids the notions of the tropical variety and positive tropical variety associated to a linear ideal. Our main result is that if $\mathcal{A}$ is a Coxeter arrangement of type Φ with corresponding oriented matroid M_Φ, then $\mathcal{B}^+(M_\Phi)$ is dual to the graph associahedron of type Φ, and $\mathcal{B}(M_\Phi)$ equals the nested set complex of $\mathcal{A}$. In addition, we prove that for any orientable matroid M, one can find $|\mu(M)|$ different reorientations of M such that the corresponding positive Bergman complexes cover $\mathcal{B}(M)$, where $\mu(M)$ denotes the Möbius function of the lattice of flats of M.

1. Introduction

In this paper we study the Bergman complex and the positive Bergman complex of a Coxeter arrangement. We relate them to the nested set complexes that arise in De Concini and Procesi's wonderful arrangement models [11, 12], and to the graph associahedra introduced by Carr and Devadoss [8], by Davis, Januszkiewicz, and Scott [10], and by Postnikov [19].

The *Bergman complex* of a matroid is a pure polyhedral complex which can be associated to any matroid. It was first defined by Sturmfels [23] in order to generalize to matroids the notion of a tropical variety associated to a linear ideal. The Bergman complex can be described in terms of the lattice of flats of the matroid, and is homotopy equivalent to a wedge of spheres, as shown by Ardila and Klivans [1].

The *positive Bergman complex* $\mathcal{B}^+(M)$ of an oriented matroid M is a subcomplex of the Bergman complex of the underlying unoriented matroid $\underline{M}$. It generalizes to oriented matroids the notion of the positive tropical variety associated to a linear ideal. $\mathcal{B}^+(M)$ depends on a choice of acyclic orientation of M, and as one varies this acyclic orientation, one gets a covering of the Bergman complex of $\underline{M}$; we will prove this in Section 3. The positive Bergman complex can be described in terms of the Las Vergnas face lattice of M and it is homeomorphic to a sphere, as shown by Ardila, Klivans, and Williams [2].

Graph associahedra are polytopes which generalize the associahedron, which were discovered independently by Carr and Devadoss [8], by Davis, Januszkiewicz, and Scott [10], and by Postnikov [19]. There is an intrinsic tiling by associahedra of the Deligne–Knudsen–Mumford compactification of the real moduli space of curves

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$\overline{M_0^n}(\mathbb{R})$, a space which is related to the Coxeter complex of type A. The motivation for Carr and Devadoss' work was the desire to generalize this phenomenon to all simplicial Coxeter systems.

Let $\mathcal{A}_\Phi$ be the Coxeter arrangement corresponding to the (possibly infinite, possibly non-crystallographic) root system Φ associated to a Coxeter system (W, S) with diagram Γ; see Section 6 below. Choose a region R of the arrangement, and let M_Φ be the oriented matroid associated to $\mathcal{A}_\Phi$ and R. In this paper we prove:

Theorem 1.1. *The positive Bergman complex $\mathcal{B}^+(M_\Phi)$ of the arrangement $\mathcal{A}_\Phi$ is dual to the graph associahedron $P(\Gamma)$.*

In particular, the cellular sphere $\mathcal{B}^+(M_\Phi)$ is actually a simplicial sphere, and a *flag* (or *clique*) complex.

This result is also related to the *wonderful model of a hyperplane arrangement* and to *nested set complexes*. The wonderful model of a hyperplane arrangement is obtained by blowing up the non-normal crossings of the arrangement, leaving its complement unchanged. De Concini and Procesi [11] introduced this model in order to study the topology of this complement. They showed that the *nested sets* of the arrangement encode the underlying combinatorics. Feichtner and Kozlov [12] gave an abstract notion of the *nested set complex* for any meet-semilattice, and Feichtner and Müller [13] studied its topology. Recently, Feichtner and Sturmfels [14] studied the relation between the Bergman complex and the nested set complexes (see Section 7 below).

In this paper we also prove for *finite* root systems Φ:

Theorem 1.2. *The Bergman complex $\mathcal{B}(M_\Phi)$ of $\mathcal{A}_\Phi$ equals its nested set complex.*

In particular, the cell complex $\mathcal{B}(M_\Phi)$ is actually a simplicial complex.

2. The Bergman complex and the positive Bergman complex

Our goal in this section is to explain the notions of the Bergman complex of a matroid and the positive Bergman complex of an oriented matroid which were studied in [1] and [2]. In order to do so we must review a certain operation on matroids and oriented matroids.

Definition 2.1. Let M be a matroid or oriented matroid of rank r on the ground set $[n]$, and let $\omega \in \mathbb{R}^n$. Regard ω as a weight function on M, so that the weight of a basis $B = \{b_1, \ldots, b_r\}$ of M is given by $\omega_B = \omega_{b_1} + \omega_{b_2} + \cdots + \omega_{b_r}$. Let B_ω be the collection of bases of M having minimum ω-weight. (If M is oriented, then bases in B_ω inherit orientations from bases of M.) This collection is itself the set of bases of a matroid (or oriented matroid) which we call M_ω.

It is not obvious that M_ω is well-defined. However, when M is an unoriented matroid, we can see this by considering the matroid polytope of M: the face that minimizes the linear functional ω is precisely the matroid polytope of M_ω. For a proof that M_ω is well-defined when M is oriented, see [2].

Notice that M_ω will not change if we translate ω or scale it by a positive constant. We can therefore restrict our attention to the sphere

$$S^{n-2} := \{\, \omega \in \mathbb{R}^n \ :\ \omega_1 + \cdots + \omega_n = 0\,,\, \omega_1^2 + \cdots + \omega_n^2 = 1\}.$$

The Bergman complex of M will be a certain subset of this sphere.

The matroid M_ω depends only on a certain flag associated to ω.

Definition 2.2. Given $\omega \in \mathbb{R}^n$, let $\mathcal{F}(\omega)$ denote the unique flag of subsets

$$\emptyset = F_0 \subset F_1 \subset \cdots \subset F_k \subset F_{k+1} = [n] \tag{1}$$

such that ω is constant on each set $F_i \setminus F_{i-1}$ and satisfies $\omega|_{F_i \setminus F_{i-1}} < \omega|_{F_{i+1} \setminus F_i}$. We call $\mathcal{F}(\omega)$ the *flag* of ω, and we say that the *weight class* of ω or of the flag $\mathcal{F}$ is the set of vectors ν such that $\mathcal{F}(\nu) = \mathcal{F}$.

It is shown in [1] that M_ω depends only on the flag $\mathcal{F} := \mathcal{F}(\omega)$; specifically

$$M_\omega = \bigoplus_{i=1}^{k+1} F_i/F_{i-1}, \tag{2}$$

where F_i/F_{i-1} is obtained from the matroid restriction of M to F_i by quotienting out the flat F_{i-1}. Hence we also refer to this oriented matroid M_ω as $M_\mathcal{F}$.

Definition/Theorem 2.3. [1] *The Bergman complex of a matroid M on the ground set $[n]$ is the set*

$$\begin{aligned} \mathcal{B}(M) &= \{\omega \in S^{n-2} \ :\ M_{\mathcal{F}(\omega)} \text{ has no loops}\} \\ &= \{\omega \in S^{n-2} \ :\ \mathcal{F}(\omega) \text{ is a flag of flats of } M\}. \end{aligned}$$

Since the matroid M_ω depends only on the weight class that ω is in, the Bergman complex of M is the disjoint union of the weight classes of flags $\mathcal{F}$ such that $M_\mathcal{F}$ has no loops. We say that the weight class of a flag $\mathcal{F}$ is *valid* for M if $M_\mathcal{F}$ has no loops.

There are two polyhedral subdivisions of $\mathcal{B}(M)$, one of which is clearly finer than the other.

Definition 2.4. The *fine subdivision* of $\mathcal{B}(M)$ is the subdivision of $\mathcal{B}(M)$ into valid weight classes: two vectors ω and ν of $\mathcal{B}(M)$ are in the same class if and only if $\mathcal{F}(\omega) = \mathcal{F}(\nu)$. The *coarse subdivision* of $\mathcal{B}(M)$ is the subdivision of $\mathcal{B}(M)$ into M_ω-equivalence classes: two vectors ω and ν of $\mathcal{B}(M)$ are in the same class if and only if $M_\omega = M_\nu$. We call these equivalence classes *fine cells* and *coarse cells*; however, by default, any reference to a *cell* of $\mathcal{B}(M)$ will refer to a coarse cell.

The fine subdivision gives the following corollary of Theorem 2.3.

Corollary 2.5. [1] *Let M be a matroid of rank r. The fine subdivision of the Bergman complex $\mathcal{B}(M)$ is a geometric realization of $\Delta(L_M - \{\,\hat{0}\,,\hat{1}\,\}\,)$, the order complex of the proper part of the lattice of flats of M. It follows that $\mathcal{B}(M)$ is homotopy equivalent to a wedge of $|\mu(M)|$ spheres of dimension $r-2$, where $\mu(M)$ denotes the Möbius function from the bottom to the top element in L_M.*

There are positive analogues of all of the above definitions and theorems. First we must give the definition of *positive covectors* and *positive flats*.

Definition 2.6. Let M be an acyclic oriented matroid on the ground set $[n]$. We say that a covector $v \in \{+,-,0\}^n$ of M is *positive* if each of its entries is $+$ or 0. We say that a flat of M is *positive* if it is the 0-set of a positive covector.

Observation 2.7. If M is the acyclic oriented matroid corresponding to a hyperplane arrangement $\mathcal{A}$ whose orientation is determined by a choice of region R, then the positive flats are in correspondence with the faces of R. In this case we will also say that the flats which are positive are "positive with respect to R."

For example, consider the braid arrangement A_3, consisting of the six hyperplanes $x_i = x_j, 1 \leq i < j \leq 4$ in $\mathbb{R}^4$. Figure 1 illustrates this arrangement, when intersected with the hyperplane $x_4 = 0$ and the sphere $x_1^2 + x_2^2 + x_3^2 = 1$. Let R be the region specified by the inequalities $x_1 \geq x_2 \geq x_3 \geq x_4$, and let M_{A_3} be the oriented matroid corresponding to the arrangement A_3 and the region R. Then the positive flats are $\emptyset, 1, 4, 6, 124, 16, 456$ and 123456.

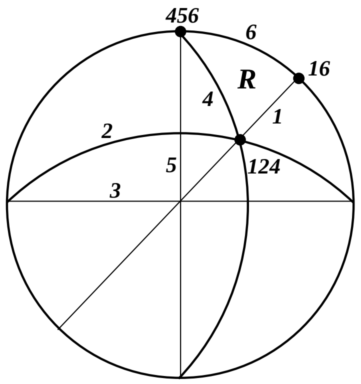


FIGURE 1. The braid arrangement A_3.

The positive Bergman complex counterpart to Definition/Theorem 2.3 is the following.

Definition/Theorem 2.8. [2] *The positive Bergman complex of M is*

$$\begin{aligned}\mathcal{B}^+(M) &= \{\omega \in S^{n-2} \ : \ M_{\mathcal{F}(\omega)} \text{ is acyclic}\}\\ &= \{\omega \in S^{n-2} \ : \ \mathcal{F}(\omega) \text{ is a flag of positive flats of } M\}.\end{aligned}$$

Within each equivalence class of the coarse subdivision of $\mathcal{B}(M)$, the vectors ω give rise to the same unoriented M_ω. Since the orientation of M_ω is inherited from that of M, they also give rise to the same oriented matroid M_ω. Therefore each coarse cell of $\mathcal{B}(M)$ is either completely contained in or disjoint from $\mathcal{B}^+(M)$. Thus $\mathcal{B}^+(M)$ inherits the coarse and the fine subdivisions from $\mathcal{B}(M)$, and each subdivision of $\mathcal{B}^+(M)$ is a subcomplex of the corresponding subdivision of $\mathcal{B}(M)$.

Recall that the *Las Vergnas face lattice* $\mathcal{F}_{\ell v}(M)$ is the lattice of positive flats of M, ordered by containment. Note that the lattice of positive flats of the oriented matroid M sits inside L_M, the lattice of flats of M. By Observation 2.7, if M is the oriented matroid of the arrangement $\mathcal{A}$ and the region R, then $\mathcal{F}_{\ell v}(M)$ is the face poset of R.

Corollary 2.9. [2] *Let M be an oriented matroid of rank r. Then the fine subdivision of $\mathcal{B}^+(M)$ is a geometric realization of $\Delta(\mathcal{F}_{\ell v}(M) - \{\hat{0}, \hat{1}\})$, the order complex of the proper part of the Las Vergnas face lattice of M. It follows that the positive Bergman complex of an oriented matroid is homeomorphic to an $(r-2)$-sphere.*

Example 2.10. Let M be the oriented matroid from Figure 1. The positive flats of M are $\{\emptyset, 1, 4, 6, 16, 124, 456, 123456\}$. The lattice of positive flats of M is shown in bold in Figure 2, within the lattice of flats of M.

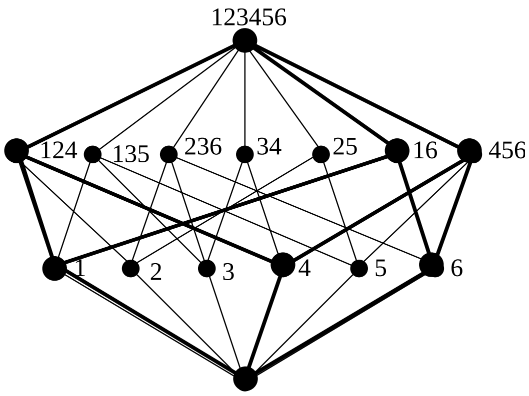


FIGURE 2. The lattice of positive flats within the lattice of flats.

3. FURTHER THEORY OF BERGMAN AND POSITIVE BERGMAN COMPLEXES

This section develops some further theory of Bergman complexes in the setting of both unoriented and oriented matroids. These results will be used later, in the proofs of Theorems 1.1 and 1.2, but are also of independent interest.

3.1. **Covering the Bergman complex with positive Bergman complexes.** We know that, for any acyclic orientation of a matroid M of rank r, the corresponding positive Bergman complex is homeomorphic to an $(r-2)$-sphere, while the Bergman complex $\mathcal{B}(M)$ of the (unoriented) matroid M is homotopy equivalent to $|\mu(M)|$ such $(r-2)$-spheres. In fact, as we vary the acyclic reorientations (that is, the *topes* or maximal covectors) of M, the corresponding positive Bergman complexes cover $\mathcal{B}(M)$. The first goal of this section is to give a polyhedral realization of this statement: for any orientable matroid M, we exhibit $|\mu(M)|$ reorientations of M whose positive Bergman complexes cover $\mathcal{B}(M)$.

The motivating example is the matroid M of a real central hyperplane arrangement $\mathcal{A}$. Let H be an affine hyperplane which is generic with respect to $\mathcal{A}$. Consider the regions of $\mathcal{A}$ which have a non-empty and bounded intersection with H; we will see that there are $|\mu(M)|$ of them. We claim that the positive Bergman complexes corresponding to these regions cover the Bergman complex of M.

For general oriented matroids, one can mimic the previous construction. Let $M = (E, \mathcal{L})$ be an oriented matroid of rank r with ground set E and collection of covectors $\mathcal{L} \subseteq \{+, -, 0\}^E$. Let $\mathcal{L}_g \subseteq \{+, -, 0\}^{E \cup \{g\}}$ be an extension of $\mathcal{L}$ by a generic element $g \notin E$; this means that g is not in the closure of any set $A \subset E$ with $r(A) < r$. Let N be the affine oriented matroid $N = (E \cup \{g\}, \mathcal{L}_g, g)$ with distinguished element g (which is not a loop).

Let $\mathcal{L}_g^+ = \{X \in \mathcal{L}_g \mid X_g = +\}$ and let $\widehat{\mathcal{L}}_g^+ = \mathcal{L}_g^+ \cup \{\widehat{0}, \widehat{1}\}$ be the *affine face lattice* of N. Let $\mathcal{L}_g^{++} = \{X \in \mathcal{L}_g^+ \mid (\mathcal{L}_g)_{\leq X} \subseteq \widehat{\mathcal{L}}_g^+\}$ be the *bounded complex* of N. The maximal elements of $\mathcal{L}_g^+$ are the topes, and the maximal elements of $\mathcal{L}_g^{++}$ are the bounded topes with respect to g. Each bounded tope of $\mathcal{L}_g$ with respect to g determines a tope of $\mathcal{L}$ by deletion of g; we call these the *bounded topes of $\mathcal{L}$ with respect to g*, and write B^{++} for the set of such topes.

In the realizable case, M is the oriented matroid of $\mathcal{A}$ with respect to a chosen region. Instead of the generic affine hyperplane H, we consider the translate g of H through the origin, declaring H to be on the positive side of g. The arrangement $\mathcal{A} \cup \{g\}$ determines the generic extension N of M. The faces in $\mathcal{L}_g^+$ correspond to the faces of $\mathcal{A}$ that intersect H, and the faces in $\mathcal{L}_g^{++}$ correspond to the faces of $\mathcal{A}$ that have a non-empty and bounded intersection with H. The topes in B^{++} are in one-to-one correspondence with the bounded regions of the arrangement $\mathcal{A} \cup H$.

The *beta invariant* $\beta(N)$ of a matroid N is given by

$$\beta(N) = (-1)^{r(N)} \sum \mu_N(X) r(X),$$

summing over all flats X of N. Here μ_N denotes the Möbius function of the lattice L_N.

Proposition 3.1. [15, 18] *An affine oriented matroid N with distinguished element g has exactly $\beta(N)$ topes that are bounded with respect to g.*

Lemma 3.2. *(cf.* [15, Theorem 3.2]*) If N is a generic extension of M by g, then*

$$\beta(N) = (-1)^{r(N)} \mu(M).$$

Proof. For any nonloop, noncoloop element g in a matroid N, one has [26, Theorem 7.3.2(c)]

$$\beta(N) = \beta(N - g) + \beta(N/g). \tag{3}$$

Since g is generic, the lattice $L_{N/g}$ is simply the truncation of L_{N-g} in which one removes the entire rank $r-1$ (but keeps the element $\widehat{1}$), where $r = r(N)$. Hence starting with equation (3), one has on the right-hand side two sums of the quantities $\mu(X)r(X)$ with X ranging over the two lattices L_{N-g} and $L_{N/g}$, with opposite signs in front of the two sums because the ranks of $N - g$ and N/g differ by one. Thus the terms with X of rank at most $r-2$ all cancel, and one is left with the terms of rank at least $r-1$ in the two sums:

$$\begin{aligned}(-1)^r\beta(N) &= r\cdot\mu(N-g)+(r-1)\left(\sum_{X \text{ of rank } r-1 \text{ in } L_{N-g}} \mu(X)\right)-(r-1)\mu(N/g)\\ &= r\cdot\mu(N-g)+(r-1)\left(\mu(N/g)-\mu(N-g)\right)-(r-1)\mu(N/g)\\ &= \mu(N-g),\end{aligned}$$

as we wished to show. □

Theorem 3.3. *Let M be an oriented matroid, and N an extension by a generic element g. Let $T_1,\ldots,T_{|\mu(M)|}$ be the bounded topes in M with respect to g. Then the $|\mu(M)|$ positive Bergman complexes corresponding to the T_is cover the Bergman complex of the unoriented matroid $\underline{M}$.*

Proof. There is no harm in assuming that M is simple and loop and coloop-free. In view of Definition/Theorem 2.8, it suffices to show that, for any flag of flats $\mathcal{F} = \{\emptyset \subset F_1 \subset \cdots \subset F_{r-1} \subset E\}$, we can find a tope T_i with $1 \le i \le |\mu(M)|$ such that all the F_js are positive with respect to T_i. This means that T_i has a flag of subfaces (covectors with some entries of T_i replaced by zeroes) $X_1 > \ldots > X_{r-1}$ such that X_i spans F_i.

We proceed by induction, where the base case is trivial. Now consider the rank $r-1$ oriented matroid M/F_1 (which, in the realizable case, corresponds to the arrangement that $\mathcal{A}$ determines on the hyperplane F_1). The set F_1 is also a flat in N, and N/F_1 is also a generic extension of M/F_1 by g. Consider the flag of flats

$$\mathcal{F}' = \{\emptyset \subset F_2 - F_1 \subset \cdots \subset F_{r-1} - F_1 \subset E - F_1\}$$

of M/F_1. By the induction hypothesis, we can find a tope T', bounded in M/F_1 with respect to g, which has a flag of faces $T' = Y_1 > Y_2 > \ldots > Y_{r-1}$ such that Y_i spans $F_i - F_1$ in M/F_1.

Since M is simple, the flat F_1 consists of a single element; call it e. Then the covector Y_i of M/F_1 comes from the covector X_i of M which is identical to Y_i, except for the extra entry $(X_i)_e = 0$. Clearly $X_1 > \cdots > X_{r-1}$ and X_i spans F_i.

Since e is not a loop of M, some covectors Z^+ and Z^- of M have $Z^+_e = +$ and $Z^-_e = -$. The topes $T^+ = X_1 \circ Z^+$ and $T^- = X_1 \circ Z^-$ are identical to $Y_1 = T'$, except for the extra entries $(T^+)_e = +$ and $(T^-)_e = -$. The X_is are faces of both T^+ and T^-, so it remains to show that at least one of T^+ and T^- is bounded with respect to g in M.

Suppose this is not the case. Then we can find non-zero covectors $A \le T^+ \cup \{g\}$ and $B \le T^- \cup \{g\}$ of N such that $A_g = B_g = 0$. If $A_e = 0$, then $A - e$ would be a non-zero covector in N/F_1, smaller than $T' \cup \{g\}$ and satisfying $(A-e)_g = 0$; this would contradict the boundedness of T' in M/F_1 with respect to g. Therefore we have $A_e = +$ and, similarly, $B_e = -$.

Consider now the covectors A and B of N and their separator e. In fact, e is the only separator of A and B, because

$$A \le T^+ \cup \{g\} = T' \cup \{e\} \cup \{g\}$$
$$B \le T^- \cup \{g\} = T' \cup \{\overline{e}\} \cup \{g\}.$$

Using the covector axiom (L3) [4, Theorem 4.1.1], we will find a covector C of N such that $C_e = 0$ and $C_f = (A \circ B)_f = (B \circ A)_f$ for all $f \neq e$. Thus $C - e$ is a covector of N/F_1 which is smaller than $T' \cup \{g\}$ and satisfies $(C - e)_g = 0$. This contradicts the boundedness of T' in M/F_1 with respect to g, unless $C - e = 0$. But if this were the case, then e would be a coloop of N. In the presence of the generic element g, this is impossible: $E - e$ has corank at most 1 in M, so $(E - e) \cup \{g\}$ is spanning in N, without containing e. This completes the proof. $\square$

Theorem 3.3 is closely related to recent work of Björner and Wachs. In [5], they construct a basis for the homology of the geometric lattice of an orientable matroid M, which is indexed by the bounded topes of M with respect to an extension by a generic element g.

3.2. **The forest of a flag, and coarse cells in the Bergman complex.** Recall from Definition 2.4 that the Bergman complex $\mathcal{B}(M)$ has two subdivisions into cells. Its fine subdivision has cells indexed by all flags $\mathcal{F}$ of flats of M. These fine cells then group themselves into the cells of the coarse subdivision, according to their associated matroids $M_{\mathcal{F}}$. It turns out that one can always determine $M_{\mathcal{F}}$, and hence the coarse cell to which a flag $\mathcal{F}$ corresponds, based on a certain labelled forest $T_{\mathcal{F}}$ associated to $\mathcal{F}$. These forests also turn out (see Section 7) to be closely related to the complex of nested sets[1].

Recall that the *connected components* of a matroid M are the equivalence classes for the following equivalence relation on the ground set E of M: say $e \sim e'$ for two elements e, e' in E whenever they lie in a common circuit of M, and then take the transitive closure of $\sim$. Recall also that every connected component is a flat of M, and M decomposes (uniquely) as the direct sum of its connected components.

Definition 3.4. To each flag $\mathcal{F}$ of flats of a matroid M indexed as in (1), associate a forest $T_{\mathcal{F}}$ of rooted trees, in which each vertex v is labelled by a flat $F(v)$, as follows:

- For each connected component F of the matroid M, create a rooted tree (as specified below) and label its root vertex with F.
- For each vertex v already created, and already labelled by some flat $F(v)$ which is a connected component of some flat F_j in the flag $\mathcal{F}$, create children of v labelled by each of the connected components of F_{j-1} which are contained *properly* in $F(v)$.

[1]The material in this subsection is closely related to results of Feichtner and Sturmfels [14, Section 4 and end of Section 3]; in particular, see our Remark 3.10. However, the crucial notion of a *circuitous base* (Definition 3.6, Proposition 3.9) does not appear in their work, and we have chosen to explain this subsection in our language so as to keep the paper more self-contained.

Alternatively, one can construct the forest $T_{\mathcal{F}}$ by listing all the connected components of all the flats in $\mathcal{F}$, and partially ordering them by inclusion.

Proposition 3.5. *For any flag $\mathcal{F}$ of flats in a matroid M, the labelled forest $T_{\mathcal{F}}$ determines the matroid $M_{\mathcal{F}}$.*

Proof. Recall the expression (2) for $M_{\mathcal{F}}$. By construction of $T_{\mathcal{F}}$, every component of F_i is $F(v)$ for some unique vertex v, and every component of F_{i-1} lying in $F(v)$ is $F(v')$ for some child v' of v. Since quotients commute with direct sums, this gives

$$M_{\mathcal{F}} = \bigoplus_{\text{vertices } v \text{ of } T_{\mathcal{F}}} \left(F(v)/ \bigoplus_{\text{children } v' \text{ of } v} F(v') \right). \tag{4}$$

□

In general, the converse of this proposition does *not* hold; one can have $M_{\mathcal{F}} = M_{\mathcal{F}'}$ without $T_{\mathcal{F}} = T_{\mathcal{F}'}$. For example (cf. [14, Example 1.2]), in the matroid M on ground set $E = \{1, 2, 3, 4, 5\}$ having rank 3 and circuits $\{123, 145, 2345\}$, the two flags

$$\mathcal{F} := (\emptyset \subset 1 \subset 123 \subset 12345)$$
$$\mathcal{F}' := (\emptyset \subset 1 \subset 145 \subset 12345)$$

exhibit this possibility.

However, there is at least one nice hypothesis that allows one to reconstruct $T_{\mathcal{F}}$ from $M_{\mathcal{F}}$. Given a base B of a matroid M on ground set E, and any element $e \in E \backslash B$, there is a unique circuit of M contained in $B \cup \{e\}$, called the *basic circuit* $\mathrm{circ}(B, e)$. Note that the flat spanned by $\mathrm{circ}(B, e)$ will always be a connected flat.

Definition 3.6. Say that a base B of a matroid M is *circuitous* if every connected flat spanned by a subset of B is spanned by the basic circuit $\mathrm{circ}(B, e)$ for some $e \in E \backslash B$.

Note that the basic circuit $\mathrm{circ}(B, e)$ spanning the connected flat F must be $(F \cap B) \cup \{e\}$. Before we state our proposition, we prove two useful lemmas.

Lemma 3.7. *Let F be a flat in a matroid M, spanned by some independent set I. Then every connected component of F is spanned by some subset of I, namely, by the intersection of that component with I.*

Proof. Let r denote the rank function for M, and let F have components $F_1, \ldots, F_t$. Then

$$\sum_i r(F_i) = r(F) = |I| = \sum_i |F_i \cap I| = \sum_i r(F_i \cap I) \leq \sum_i r(F_i),$$

which means we must have an equality for each i: $r(F_i \cap I) = r(F_i)$. In other words, $F_i \cap I$ spans F_i. □

Given a subset $A \subset E$ of the ground set of a matroid, let $\mathrm{cl}(A)$ denote its *closure*, that is, the flat spanned by A.

Lemma 3.8. *Let $F \subset G$ be flats of a matroid that are spanned by subsets of a circuitous base B. If G is connected, then G/F is also connected.*

Proof. Let $I_F = F \cap B$ and $I_G = G \cap B$; these are bases for F and G, respectively. Also, $I_F \subset I_G$, and $I_G - I_F$ is a base for the quotient G/F. Since G is a connected flat spanned by a subset of the circuitous base B, there exists e in $G - B$ such that $\mathrm{cl}(\mathrm{circ}(B, e)) = G$, and $\mathrm{circ}(B, e) = I_G \cup \{e\}$.

We now claim that

$$\mathrm{circ}_{G/F}(I_G - I_F, e) = I_G - I_F \cup \{e\}.$$

We need to check that $I_G - I_F \cup \{e\} - \{g\}$ is independent in G/F for any $g \in I_G - I_F \cup \{e\}$. Since I_F is a basis of F, this follows from the fact that $I_G \cup \{e\} - \{g\}$ is independent in G. We conclude by observing that G/F is the flat spanned by $\mathrm{circ}(I_G - I_F, e)$, so it is connected. □

Proposition 3.9. *Let B be a circuitous base of a matroid M. Then for any two flags $\mathcal{F}, \mathcal{F}'$ of flats spanned by subsets of B, one has $M_{\mathcal{F}} = M_{\mathcal{F}'}$ if and only if $T_{\mathcal{F}} = T_{\mathcal{F}'}$.*

Proof. We start by making two observations about the matroid $M_{\mathcal{F}}$ and the tree $T_{\mathcal{F}}$.

First we observe that, under these hypotheses, the expression (4) is actually the decomposition of $M_{\mathcal{F}}$ into its irreducible components. By Lemma 3.7, the $F(v)$s are connected flats spanned by subsets of B. The direct sums $\oplus_{v'} F(v')$ are also spanned by subsets of B. Lemma 3.8 then guarantees that $F(v)/\oplus_{v'} F(v')$ is connected for each vertex v of the tree.

Secondly we show that, among the sets $\mathrm{cl}(\mathrm{circ}(B, e))$ with e in $F(v) \backslash \cup F(v')$ and not in B, there is a maximum one under containment, which is precisely $F(v)$.

Take any e in $F(v) \backslash \cup F(v')$ and not in B. The flat $F(v)$ is spanned by a subset I of B, and $I \cup \{e\}$ is dependent. Therefore $\mathrm{circ}(B, e) \subseteq I \cup \{e\} \subseteq F(v)$, which implies $\mathrm{cl}(\mathrm{circ}(B, e)) \subseteq F(v)$.

Now, since $F(v)$ is a connected flat spanned by a subset of B, $F(v) = \mathrm{cl}(\mathrm{circ}(B, e))$ for some $e \in E \backslash B$. Clearly $e \in F(v)$. If e was in $F(v')$ for some child v' of v, the argument of the previous paragraph would imply that $\mathrm{cl}(\mathrm{circ}(B, e)) \subseteq F(v')$. Therefore $e \in F(v) \backslash \cup F(v')$.

The two previous observations give us a procedure to recover the tree $T_{\mathcal{F}}$ from the matroid $M_{\mathcal{F}}$. The first step is to decompose $M_{\mathcal{F}}$ into its connected components $M_1, \ldots, M_t$, having accompanying ground set decomposition $E = E_1 \sqcup \cdots \sqcup E_t$. The second step is to recover the flat corresponding to each M_i, as the maximum $\mathrm{cl}(\mathrm{circ}(B, e))$ with $e \in E_i \backslash B$. The labelled forest $T_{\mathcal{F}}$ is simply the poset of inclusions among these flats. □

It will turn out that the simple roots Δ of a root system Φ always form a circuitous base for the associated matroid M_Φ; see Proposition 6.1(iii) below.

Remark 3.10. When the matroid M is connected, the forest $T_{\mathcal{F}}$ constructed above is a rooted tree. It coincides with the tree constructed by Feichtner and Sturmfels in [14, Proposition 3.1] when they choose the minimal building set for their lattice. In this way, Proposition 3.5 follows from [14, Theorem 4.4].

4. Graph associahedra

To any graph one can associate a polytope called a graph associahedron; the graph associahedron associated to a chain is the usual associahedron. These polytopes were discovered independently by Carr and Devadoss [8], Davis, Januszkiewicz, and Scott [10], and Postnikov [19]. There is an intrinsic tiling by associahedra of the Deligne–Knudsen–Mumford compactification of the real moduli space of curves $\overline{M_0^n(\mathbb{R})}$, a space which is related to the Coxeter complex of type A. The motivation for the work of Carr and Devadoss was the desire to generalize this phenomenon to other Coxeter systems.

In order to define graph associahedra, we must introduce the notions of tubes and tubings. We follow the presentation of [8].

Definition 4.1. Let Γ be a graph. A *tube* is a nonempty set of nodes of Γ whose induced graph is a proper, connected subgraph of Γ. There are three ways that two tubes can interact on the graph:

- Tubes are *nested* if $t_1 \subset t_2$.
- Tubes *intersect* if $t_1 \cap t_2 \neq \emptyset$ and $t_1 \not\subset t_2$ and $t_2 \not\subset t_1$.
- Tubes are *adjacent* if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2$ is a tube in Γ.

Tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* T of Γ is a set of tubes of Γ such that every pair of tubes in T is compatible. A k*-tubing* is a tubing with k tubes.

Graph-associahedra are defined via a construction which we will now describe.

Definition 4.2. Let Γ be a graph on n nodes. Let Δ_Γ be the $n-1$ simplex in which each facet corresponds to a particular node. Note that each proper subset of nodes of Γ corresponds to a unique face of Δ_Γ, defined by the intersection of the faces associated to those nodes. The empty set corresponds to the face which is the entire polytope Δ_Γ. For a given graph Γ, truncate faces of Δ_Γ which correspond to 1-tubings in increasing order of dimension (i.e. first truncate vertices, then edges, then 2-faces, …). The resulting polytope $P(\Gamma)$ is the *graph associahedron* of Carr and Devadoss.

Figure 3 illustrates the construction of the graph associahedron of a Coxeter diagram of type D_4. We start with a simplex, whose four facets correspond to the nodes of the diagram. In the first step, we truncate three of the vertices, to obtain the second polytope shown. We then truncate three of the edges, to obtain the third polytope shown. In the final step, we truncate the four facets which all correspond to tubes. This step is not shown in Figure 3, since it does not affect the combinatorial type of the polytope.

When the graph Γ is the n-element chain, the polytope $P(\Gamma)$ is the associahedron A_{n-1}. One can see this by considering an easy bijection between valid tubings and parenthesizations of a word of length $n-1$, as illustrated in Figure 4.

Carr and Devadoss proved that the face poset of $P(\Gamma)$ can be described in terms of valid tubings.

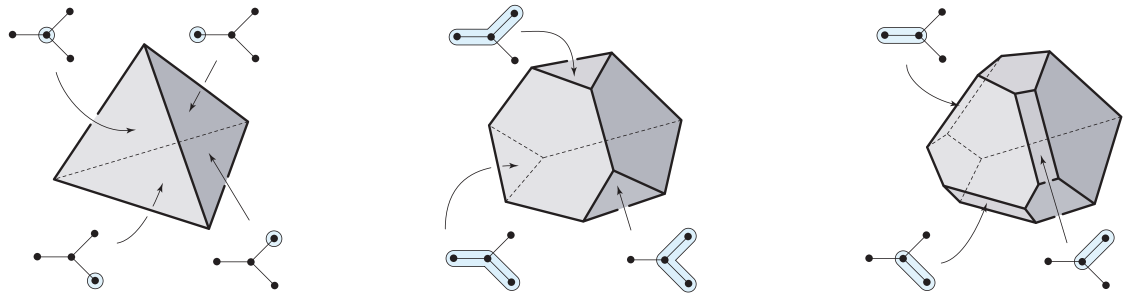
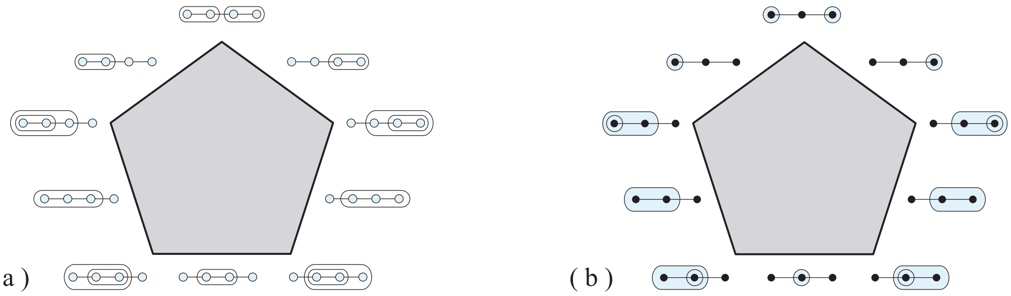


FIGURE 3. $P(D_4)$ ©Satyan Devadoss



(a) (b)

FIGURE 4. The associahedron A_2 is the graph associahedron of a 3-element chain. ©Satyan Devadoss

Theorem 4.3. [8] *The face poset of $P(\Gamma)$ is isomorphic to the set of valid tubings of Γ, ordered by reverse containment: $T < T'$ if T is obtained from T' by adding tubes.*

Corollary 4.4. [8] *When Γ is a path with $n-1$ nodes, $P(\Gamma)$ is the associahedron A_n of dimension n. When Γ is a cycle with $n-1$ nodes, $P(\Gamma)$ is the cyclohedron W_n.*

5. Coxeter systems, the Tits cone, and parabolic flats

In this section we review the notion of a Coxeter system (W, S), and explain two ways of thinking about the associated matroid. The first way is to consider the vector configuration of positive roots Φ^+ of the corresponding root system Φ in $V := \mathbb{R}^{|S|}$. The second way is to consider a certain arrangement $\mathcal{A}_\Phi$ of hyperplanes in V^* intersecting a W-invariant convex cone known as the Tits cone. We attempt to give a careful discussion of the issues that arise when W is infinite, for example, how to define the Bergman complex, and what kinds of flats of the associated matroid are relevant for Bergman complexes and wonderful compactifications.

A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations of the form

$$(ss')^{m(s,s')} = 1,$$

where $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in S. In case no relation occurs for a pair (s, s'), we make the convention that $m(s, s') = \infty$. We will always assume that S is finite.

Note that to specify a Coxeter system (W, S), it is enough to draw the corresponding *Coxeter diagram* Γ: this is a graph on vertices indexed by elements of S, with vertices s and s' joined by an edge labelled $m(s, s')$ whenever this number (∞ allowed) is at least 3.

Remark 5.1. In what follows, the reader should note that the *positive* Bergman complex and the graph associahedron associated with Γ will turn out *not* to depend on the edge labels $m(s, s')$ of Γ, and only depend upon the undirected graph underlying Γ. However, the Bergman complex *will* turn out to depend upon the edge labels $m(s, s')$.

Although an arbitrary Coxeter system (W, S) need not have a faithful representation of W as a group generated by orthogonal reflections (for a positive definite inner product), there exists a reasonable substitute, called its *geometric representation* [6, Sec. V.4], [16, Sec. 5.3, 5.13], which we recall here. Let $V := \mathbb{R}^{|S|}$ with a basis of *simple roots* $\Delta := \{\alpha_s : s \in S\}$. Define an $\mathbb{R}$-valued bilinear form $(\cdot, \cdot)$ on V by

$$(\alpha_s, \alpha_{s'}) := -\cos\left(\frac{\pi}{m(s, s')}\right),$$

and let s act on V by the "reflection" that fixes $\alpha_s^\perp$ and negates α_s:

$$s(v) := v - 2(v, \alpha_s)\alpha_s.$$

This turns out to extend to a faithful representation of W on V, and one defines the *root system* Φ and *positive roots* Φ^+ by

$$\Phi := \{w(\alpha_s) : w \in W, s \in S\},$$
$$\Phi^+ := \{\alpha \in \Phi : \alpha = \sum_{s \in S} c_s \alpha_s \text{ with } c_s \geq 0\}.$$

It turns out that $\Phi = \Phi^+ \sqcup \Phi^-$ where $\Phi^- := -\Phi^+$, and that W will be infinite if and only if Φ is infinite.

Definition 5.2. Given a root $\beta \in \Phi$, expressed uniquely in terms of the simple roots Δ as $\beta = \sum_{s \in S} c_s \alpha_s$, define the *support* of β (written $\operatorname{supp} \beta$) to be the vertex-induced subgraph of the Coxeter diagram Γ on the set of vertices $s \in S$ for which $c_s \neq 0$.

We will need the following lemma about supports of roots. It is well-known when W is finite and crystallographic [6, No. VI.1.6, Cor. 3], and a proof of its first assertion for the Coxeter systems associated to Kac–Moody Lie algebras can be found in [17, Lemma 1.6]; we will need the assertion in general.

Lemma 5.3. *Let (W, S) be an arbitrary Coxeter system with Coxeter graph Γ. Then for any root $\beta \in \Phi$ the graph $\operatorname{supp} \beta$ is connected, and conversely, every connected subgraph Γ' of Γ occurs as $\operatorname{supp} \beta$ for some positive root β.*

Proof. For the first assertion, let β be a root, which we may assume is positive without loss of generality. It is known [3, Sec. 4.6] that there exists a chain of (distinct) positive roots

$$\alpha = \beta_1 \lessdot \beta_2 \lessdot \cdots \lessdot \beta_k = \beta$$

in which α is a simple root, and where each relation $\lessdot$ is a covering relation in what Björner and Brenti call the *root poset*. This is the poset on positive roots defined as follows: $\beta \leq \gamma$ if there exists $s_1, s_2, \ldots, s_k \in S$ such that

(1) $\gamma = s_k s_{k-1} \ldots s_1 \beta$, and
(2) $\mathrm{dp}(s_i s_{i-1} \ldots s_1 \beta) = \mathrm{dp}(\beta) + i$, for all $1 \leq i \leq k$.

Here, the *depth* dp of a positive root is defined to be

$$\mathrm{dp}(\beta) = min\{k : w(\beta) \in \Phi^- \text{ for some } w \in W \text{ with } \ell(w) = k\}.$$

In particular, when two positive roots

$$\gamma = \sum_{t \in S} c_t \alpha_t$$
$$\gamma' = \sum_{t \in S} c'_t \alpha_t$$

satisfy $\gamma \lessdot \gamma'$, then

$$\gamma' = s(\gamma) = \gamma - 2(\gamma, \alpha_s)\alpha_s \tag{5}$$

for some $s \in S$, and the nonnegative coefficients c_t, c'_t satisfy $c_t \leq c'_t$ for all $t \in S$, so that $\mathrm{supp}\,\gamma \subseteq \mathrm{supp}\,\gamma'$; see [3, Corollary 4.6.5]. By induction on k, it suffices to show that if $\mathrm{supp}(\gamma)$ is connected and $\gamma \lessdot \gamma'$, then $\mathrm{supp}(\gamma')$ is connected. From expression (5) we conclude that $\mathrm{supp}\,\gamma \subseteq \mathrm{supp}\,\gamma' \subseteq \mathrm{supp}\,\gamma \cup \{s\}$. Hence either

- $\mathrm{supp}\,\gamma' = \mathrm{supp}\,\gamma$, which is connected, so we are done, or
- $s \notin \mathrm{supp}\,\gamma$ and $\mathrm{supp}\,\gamma' = \mathrm{supp}\,\gamma \sqcup \{s\}$. If $\mathrm{supp}\,\gamma \sqcup \{s\}$ is connected, we are done. If not, then $(\gamma, \alpha_s) = 0$, so the expression (5) forces the contradiction $\gamma' = \gamma$.

For the second assertion, let Γ' be a connected subgraph of Γ, and we will exhibit a positive root γ' with $\mathrm{supp}\,\gamma' = \Gamma'$ using induction on the number of vertices of Γ'. Let $s \in S$ be a vertex lying in Γ' whose removal leaves a connected subgraph $\Gamma'' = \Gamma - \{s\}$. By induction there exists a positive root γ having $\mathrm{supp}\,\gamma = \Gamma''$, and we claim that $\gamma' := s(\gamma)$ has $\mathrm{supp}(\gamma') = \Gamma'$. To see this, note that $\gamma = \sum_{t \in \Gamma''} c_t \alpha_t$ with each $c_t > 0$. Hence

$$(\alpha_s, \gamma) = \sum_{t \in \Gamma''} c_t (\alpha_s, \alpha_t) < 0,$$

since each (α_s, α_t) is nonpositive, and at least one is negative due to $\Gamma'' \cup \{s\} = \Gamma'$ being connected. Therefore the expression (5) for γ' shows that $\mathrm{supp}(\gamma') = \Gamma'$. $\square$

We use M_Φ to denote the matroid represented by the vector configuration of positive roots Φ^+ in V. Thus M_Φ is a matroid of finite rank $r = |S|$, but has ground set $E = \Phi^+$ of possibly (countably) infinite cardinality.

Remark 5.4. When the ground set E is infinite, we need to be careful about how we define the objects that we are studying: it is no longer clear what is meant by a weight vector ω or the bases of minimum ω-weight. Therefore we will not refer to M_ω in this case; only to the matroid $M_\mathcal{F}$ associated to a flag of flats $\mathcal{F}$. We will not think of the positive Bergman complex as a subset of weight vectors (as in

Definition/Theorem 2.8), but as a coarsening of the order complex of the lattice of positive flats (which we can do by Corollary 2.9). Although we can similarly consider the Bergman complex as a coarsening of the order complex of the lattice of flats, for technical reasons we will not deal with the Bergman complex of a matroid with an infinite ground set in this paper.

For an arbitrary Coxeter system (W, S), when one wants to think of the oriented matroid M_Φ as the oriented matroid of a hyperplane arrangement $\mathcal{A}_\Phi$ (as opposed to the oriented matroid of the configuration of vectors Φ^+), one must work with the contragredient representation V^*. Then M_Φ is simply the matroid of the reflecting hyperplanes in V^* for the positive roots Φ^+.

We now review the Tits cone. See [6, Sec. V.4], [16, Sections 1.15, 5.13], [7, Chapter I], and particularly [25] for a very detailed discussion. Let $\{\delta_s : s \in S\}$ denote the basis for V^* dual to the basis of simple roots Δ for V. Then the *(closed) fundamental chamber* R is the nonnegative cone spanned by $\{\delta_s : s \in S\}$ inside V^*. The *Tits cone* is the union $T := \bigcup_{w \in W} w(R)$, a (possibly proper, not necessarily closed nor polyhedral) convex cone inside V^*. Every positive root $\alpha \in \Phi^+$ has associated a hyperplane and two half-spaces, $H_\alpha, H_\alpha^+, H_\alpha^-$ in V^*, consisting of those functionals $f \in V^*$ for which $f(\alpha)$ is zero, positive, or negative, respectively. These hyperplanes and half-spaces decompose the Tits cone[2] into cells σ that turn out to be simplicial cones σ, each of them relatively open within the linear subspace that they span. The top-dimensional (open) cones are exactly the images $w(\mathrm{int}(R))$ as w runs through W, where $\mathrm{int}(R)$ denotes the *interior* of the fundamental chamber R. The tope (maximal covector) in the oriented matroid M_Φ associated to $w(\mathrm{int}(R))$ will have the sign $+$ on the roots $\Phi^+ \cap w^{-1}(\Phi^+)$ and the sign $-$ on the roots $\Phi^+ \cap w^{-1}(\Phi^-)$. More generally one has the following proposition (see [25, Section 3.2]) relating an arbitrary cone σ to a coset wW_J of a *standard parabolic subgroup* W_J (= the subgroup of W generated by J) for some subset $J \subseteq S$

Proposition 5.5. *The cones σ in the decomposition of T are naturally in bijection with the cosets wW_J of standard parabolic subgroups, with wW_J determined by the following equality:*

$$wW_J = \{u \in W : u^{-1}(\sigma) \subseteq R\}.$$

The cone σ will then have the following description as an intersection: for any u in the coset wW_J, one has

$$\sigma = \bigcap_{s \in J} u(H_{\alpha_s}) \cap \bigcap_{s \in S \setminus J} u(H_{\alpha_s}^+).$$

As a consequence of this proposition (see [25, Prop. 3.4]), the linear span of the cone σ in V^* is the hyperplane intersection $\bigcap_{s \in J} H_{w(\alpha_s)}$, which is the subspace $(V^*)^{wW_Jw^{-1}}$ fixed by the parabolic subgroup wW_Jw^{-1}.

[2]When W is infinite, note that only part of the hyperplane or its half-spaces lies inside the Tits cone T.

As pointed out in Remark 5.4, when W (equivalently Φ, or $E = \Phi^+$) is infinite, we want to consider the positive Bergman complex to be a coarsening of the order complex of the lattice of positive flats. However, in this situation we have a choice to make, because there are three different kinds of flats F of the (oriented) matroid M_Φ, not all of which are relevant to the De Concini–Procesi wonderful compactifications. These three kinds of flats are distinguished by how the associated intersection subspace

$$X_F := \bigcap_{\alpha \in F} H_\alpha$$

intersects the Tits cone T:

(1) An *arbitrary* flat F will have at least the zero subspace $\{0\}$ in the intersection $X_F \cap T$.
(2) A *parabolic flat* F is one for which $X_F \cap T$ is of maximum possible dimension, that is, $\dim(X_F)$. In this case, it must contain a cone σ of this same dimension, say indexed by the coset wW_J, whose linear span has pointwise stabilizer wW_Jw^{-1}. Hence $F = w\Phi_J^+$ is a W-conjugate[3] of a *standard parabolic* flat Φ_J^+, where Φ_J^+ is the subset of positive roots lying in the span of the simple roots $\{\alpha_s\}_{s\in J}$.
(3) A *finite parabolic flat* F is one for which $X_F \cap \mathrm{int}(T)$ has maximum possible dimension $\dim(X_F)$. This turns out [25, Cor. 3.8] to be equivalent to F being the parabolic flat $w\Phi_J^+$ where the parabolic subgroup W_J is *finite*. In other words, $\mathrm{int}(T)$ is the union of the cones σ in T whose associated parabolic subgroup is finite.

It turns out that W (or equivalently, Φ) is finite if and only if the Tits cone T coincides with the whole space V^* (and hence also coincides with the interior $\mathrm{int}(T)$). In this case, there is no distinction between the three kinds of flats: all flats are finite parabolic. The reader interested solely in the case of finite Coxeter groups W can therefore safely ignore the remainder of this section.

Example 5.6. The distinctions between the three kinds of flats are well-illustrated by the case where (W, S) is an irreducible *affine* Coxeter system, that is, when the bilinear form $(\cdot,\cdot)$ is positive semidefinite, but degenerate. In this case, the kernel of the bilinear form is a 1-dimensional subspace ℓ, and one can faithfully realize the group W as one generated by *Euclidean affine reflections* in an affine hyperplane of V^*: simply intersect the Tits cone with a (strictly) affine hyperplane normal to ℓ. See [6, Ch. V, Sec. 4.9].

As an example of a non-parabolic flat in this situation, pick any two roots α, β in Φ^+ whose corresponding affine reflections s_α, s_β have parallel reflecting hyperplanes. Then s_α, s_β generate an infinite subgroup $W' \subsetneq W$, whose fixed subspace in V^* corresponds to a flat $F = \mathrm{cl}(\{\alpha,\beta\})$ that is *not* parabolic. To see this, note that in the affine case, all proper parabolic subgroups W_J with $J \subsetneq S$ are finite, and hence all proper parabolic subgroups wW_Jw^{-1} are also finite. But W' is infinite.

[3]Strictly speaking, in order to insure that $w\Phi_J^+ \subseteq \Phi^+$, we should insist here that the coset representative w for wW_J is chosen to be of minimum length.

There is also a unique parabolic flat which is not finite parabolic in this situation, namely the improper flat $F = \Phi^+$. Its corresponding intersection subspace $X_F = \{0\}$ lies in the Tits cone T, but not in its interior $\mathrm{int}(T)$.

Remark 5.7. The geometry of the Tits cone when W is infinite, and in particular, its interior $\mathrm{int}(T)$, turn out to be important in the geometric group theory surrounding the *generalized (Artin) braid group* $B(W,S)$ associated to (W,S). Inside the complex vector space $V^* \otimes \mathbb{C}$, one has the open subset $V^* + i \cdot \mathrm{int}(T)$, from which one can remove the intersection with the complexified hyperplanes $\bigcup_{\alpha \in \Phi^+} H_\alpha$. This hyperplane complement carries a free action of W with interesting topology: it is conjectured (and proven in many cases) that it is an Eilenberg–MacLane $K(PB(W,S),1)$-space for the (pure) braid group $PB(W,S)$, and hence that its quotient by W is a $K(B(W,S),1)$ for the braid group $B(W,S)$; see Charney and Davis [9].

In principle one might therefore consider (at least) three different versions of the poset of flats of the oriented matroid M_Φ: the posets of arbitrary, parabolic, or finite parabolic flats

$$L^{\mathrm{arb}}(M_\Phi) \supset L^{\mathrm{par}}(M_\Phi) \supset L^{\mathrm{finpar}}(M_\Phi).$$

By the previous discussion, these posets of flats are isomorphic to the following posets of subgroups.

Proposition 5.8. *The map*

$$W' \mapsto \{\alpha \in \Phi^+ : s_\alpha \in W'\}$$

induces isomorphisms between the following posets of subgroups and posets of flats, all ordered by inclusion:

$$\begin{aligned} \{ \text{ reflection subgroups } \} &\cong L^{\mathrm{arb}}(M_\Phi) \\ \{ \text{ parabolic subgroups } \} &\cong L^{\mathrm{par}}(M_\Phi) \\ \{ \text{ finite parabolic subgroups } \} &\cong L^{\mathrm{finpar}}(M_\Phi). \end{aligned}$$

When W is finite, of course, all three notions of flats coincide and all reflection subgroups are finite parabolic. When W is infinite we must make a choice of which flats to consider.

Remark 5.9. When discussing the De Concini–Procesi wonderful compactifications of Coxeter arrangements, Carr and Devadoss [8] chose to consider only Coxeter systems (W,S) which they call *simplicial*, namely those in which every proper parabolic subgroup W_J with $J \subsetneq S$ is finite, or equivalently, the simplicial decomposition of the Tits cone T intersected with the unit sphere in V^* is a locally finite simplicial complex. Such Coxeter systems include those which are finite, affine and compact hyperbolic. When doing the wonderful compactification, they made the natural choice of compactifying the complement of the arrangement within the *interior* $\mathrm{int}(T)$ of the Tits cone, after intersecting with the sphere. This means that they only blew up along the *finite parabolic flats*, those in $L^{\mathrm{finpar}}(M_\Phi)$, and avoided the problem of how to define blow-ups along non-parabolic flats, where the normal structure is not that of a finite hyperplane arrangement.

We make a slightly different choice. If one is not so concerned with the blow-ups themselves, but rather with the truncations of the fundamental simplex R which would tile the hypothetical blow-up, then these truncated polytopes (the graph associahedra) are well-defined whether or not the arrangement is locally finite. In particular, we would like to consider graph-associahedra associated to graphs Γ for which (W, S) is *not* simplicial, such as the complete graphs $\Gamma = K_n$ for $n \geq 4$. For this reason we do not restrict ourselves to the finite parabolic flats; instead we consider all parabolic flats.

On the other hand, the relevant flats of the matroid which are relevant for these truncations and blow-ups in the wonderful compactification are those which intersect the Tits cone T in full dimension. For this reason, when discussing the positive Bergman complex $\mathcal{B}^+(M)$ in the next section, we will consider only the poset of parabolic flats $L^{\text{par}}(M_\Phi)$.

Remark 5.10. It is not clear that we should expect good behavior from the Bergman complex $\mathcal{B}(M_\Phi)$ or positive Bergman complex $\mathcal{B}^+(M_\Phi)$ defined with respect to the lattice $L^{\text{arb}}(M_\Phi)$ of *arbitrary* flats, when M_Φ is infinite.

6. The positive Bergman complex of a Coxeter arrangement

In this section we prove that the positive Bergman complex of a Coxeter arrangement of type Φ is dual to the graph associahedron of type Φ. More precisely, both of these objects are homeomorphic to spheres of the same dimension, and their face posets are dual.

Caution. Throughout this section, whenever the Coxeter system (W, S) with root system Φ has W (or equivalently, Φ) infinite, the word *flat* used in the connection with the oriented matroid M_Φ will mean a *parabolic flat*, as discussed in the end of Section 5.

Proposition 6.1. *Let (W, S) be an arbitrary Coxeter system, with root system Φ and Coxeter diagram Γ.*

(i) *Positive flats in the oriented matroid M_Φ correspond to subsets $J \subset S$, that is, they are the standard parabolic flats Φ_J^+.*
(ii) *Connected positive flats in the oriented matroid M_Φ correspond to subsets $J \subset S$ such that the vertex-induced subgraph Γ_J is connected, that is, to tubes in Γ.*
(iii) *The simple roots Δ form a circuitous base for the matroid M_Φ.*
(iv) *If $F \subset G$ are flats in M_Φ with G connected, then the matroid quotient G/F is connected.*

Proof. (i): The hyperplanes bounding the base region/tope R are $\{H_{\alpha_s} : s \in S\}$, so positive flats are those spanned by sets of the form $\{\alpha_s : s \in J\}$ for subsets $J \subset S$. As in the previous section, we denote such a positive flat by Φ_J^+.
(ii): Let $J \subset S$ with subgraph Γ_J, and consider its associated positive flat Φ_J^+. The first assertion of Lemma 5.3 shows that Φ_J^+ will not be connected if Γ_J is disconnected.

To see this, represent the flat Φ_J^+ by a matrix in which the rows correspond to simple roots of Φ_J^+, i.e. vertices of Γ_J, and the columns express each positive root in Φ_J^+ as a combination of simple roots. By permuting columns, one can obtain a matrix which is a block-direct sum of two smaller matrices, and hence Φ_J^+ will not be connected.

On the other hand, if Γ_J is connected, then the second assertion of Lemma 5.3 shows that there is a positive root α with $\operatorname{supp}\alpha = \Gamma_J$, and consequently $\{\alpha_s : s \in J\} \cup \{\alpha\}$ gives a circuit in M_Φ spanning this flat, so it is connected.

(iii): This follows from the argument in (ii); given $J \subset S$ with Γ_J connected, the basic circuit $\operatorname{circ}(\Delta, \alpha)$ where $\operatorname{supp}\alpha = \Gamma_J$ spans the connected flat corresponding to J.

(iv): Let the flats F, G correspond (since they are assumed to be parabolic flats) to the parabolic subgroups uW_Ju^{-1}, vW_Kv^{-1}. Equivalently, assume they are equal to $u\Phi_J^+, v\Phi_K^+$. One can make the following reductions:

- Translating by v^{-1}, one can assume that v is the identity.
- Since (W_K, K) itself forms a Coxeter system with root system Φ_K, one can assume $M_\Phi = G$ and $K = S$. In particular, M_Φ is connected.
- Replacing the Coxeter system (W, S) by the system (W, uSu^{-1}), one can assume that u is the identity.

In other words, F is the positive flat corresponding to some subgraph Γ_J of Γ, and we must show M_Φ/F is a connected matroid. This is a consequence of (iii) and Lemma 3.8. ☐

We now give our main result.

Theorem 1.1. *Let (W, S) be an arbitrary Coxeter system, with root system Φ, Coxeter diagram Γ, and associated oriented matroid M_Φ. Then the face poset of the coarse subdivision of $\mathcal{B}^+(M_\Phi)$ is dual to the face poset of the graph associahedron $P(\Gamma)$.*

Proof. By Theorem 4.3, we need to show that the face poset of (the coarse subdivision of) $\mathcal{B}^+(M_\Phi)$ is equal to the poset of tubings of Γ, ordered by containment. We begin by describing a map Ψ from flags of positive flats to tubings of Γ.

By Proposition 6.1, positive flats of M_Φ correspond to subsets $J \subset S$ or subgraphs Γ_J of the Coxeter graph Γ. Furthermore, a positive flat is connected if and only if Γ_J is a tube, and hence an arbitrary positive flat corresponds to a disjoint union of compatible tubes, no two of which are nested. Since an inclusion of flats corresponds to an inclusion of the subsets J, a flag $\mathcal{F}$ of positive flats corresponds to a nested chain of such unions of non-nested compatible tubes, that is, to a tubing $\Psi(\mathcal{F})$. Furthermore, in this correspondence, inclusion of flags corresponds to containment of tubings.

We claim that the map from flags to tubings is surjective. Given some tubing of Γ, linearly order its tubes $J_1, \ldots, J_k$ by any linear extension of the inclusion partial ordering, and then the flag $\mathcal{F}$ of positive flats having F_i spanned by $\{\alpha_s : s \in J_1 \cup J_2 \cup \cdots \cup J_i\}$ will map to this tubing.

Lastly, we show that Ψ is actually a well-defined injective map when regarded as a map on cells of the coarse subdivision of $\mathcal{B}^+(M_\Phi)$. To do so, it is enough to show that two flags $\mathcal{F}, \mathcal{F}'$ of positive flats give the same tubing if and only if $M_{\mathcal{F}}$ and

$M_{\mathcal{F}'}$ coincide. By Lemma 6.1(iv) and Proposition 3.9, we need to show that $\Psi(\mathcal{F})$ and $\Psi(\mathcal{F}')$ coincide if and only if $T_{\mathcal{F}}$ and $T_{\mathcal{F}'}$ coincide. But this is clear, because by construction, the rooted forest $T_{\mathcal{F}}$ ignores the ordering within the flag, and only records the data of the tubes which appear, that is, the tubing. □

Corollary 6.2. *The positive Bergman complex of a Coxeter arrangement is simplicial, and is in fact, a flag simplicial sphere.*

Another corollary of our proof is a new realization for the positive Bergman complex of a Coxeter arrangement: we can obtain it from a simplex by a sequence of *stellar subdivisions* (since stellar subdivisions are dual to the truncations defining the graph-associahedra; see [27, Exercise 3.0]).

7. The Bergman complex of a Coxeter arrangement

In this section we will give a concrete description of the Bergman complex of a Coxeter arrangement, in terms of the nested set complex. We will also address a question of Eugene Tevelev [24] concerning the relationship of the positive Bergman complex to the Bergman complex in this setup.

Nested set complexes are simplicial complexes at the combinatorial heart of De Concini and Procesi's subspace arrangement models [11], and of the resolution of singularities in toric varieties [12]. We now recall the definition of the minimal nested set complex of a meet-semilattice L, which we will simply refer to as **the** nested set complex of L, and denote $\mathcal{N}(L)$. For the sake of avoiding the technicalities of infinite semilattices, matroids, and Coxeter groups, we will assume that **everything is finite** in this section.

Say an element y of L is *irreducible* if the lower interval $[\hat{0}, y]$ cannot be decomposed as the product of smaller intervals of the form $[\hat{0}, x]$. The nested set complex $\mathcal{N}(L)$ of L is a simplicial complex whose vertices are the irreducible elements of L. A set X of irreducibles is *nested* if for any nonempty antichain $\{x_1, \dots, x_k\}$ in X, $x_1 \vee \cdots \vee x_k$ is *not* irreducible. These nested sets are the simplices of $\mathcal{N}(L)$.

If M is a matroid and L_M is its lattice of flats, we will also call $\mathcal{N}(L_M)$ the nested set complex of M, and denote it $\mathcal{N}(M)$. It is easy to see that the irreducible elements of L_M are the connected flats of M.

Theorem 1.2. *For any finite Coxeter system (W, S) and associated finite root system Φ, the coarse subdivision of the Bergman complex $\mathcal{B}(M_\Phi)$ of the Coxeter arrangement of type Φ is equal to the nested set complex $\mathcal{N}(M_\Phi)$. In particular, the Bergman complex $\mathcal{B}(M_\Phi)$ is a simplicial complex.*

We offer two proofs of this result. The first is short, but not very self-contained in that it invokes a result of Feichtner and Sturmfels. They showed that, for any matroid M, the simplicial complex $\mathcal{N}(M)$ has a geometric realization which is intermediate in coarseness between the fine and coarse subdivisions of the Bergman complex $\mathcal{B}(M)$. Furthermore, they gave the following criterion for when $\mathcal{N}(M)$ and the coarse subdivision of $\mathcal{B}(M)$ coincide.

Theorem 7.1. [14, Theorem 5.3] *The nested set complex $\mathcal{N}(M)$ and the Bergman complex $\mathcal{B}(M)$ together with its coarse subdivision coincide if and only if the matroid G/F is connected for every pair of flats $F \subset G$ in which G is connected.*

Theorem 1.2 then follows immediately from this result together with our Proposition 6.1(iv).

On the other hand, one might suspect that in the case of a Coxeter arrangement, Theorem 1.1 describing the positive Bergman complex $\mathcal{B}^+(M)$, and Theorem 1.2 about the entire Bergman complex $\mathcal{B}(M)$, should be related by Theorem 3.3 showing that $\mathcal{B}(M)$ is covered by several copies of $\mathcal{B}^+(M)$. The goal of the remainder of this section is to develop this connection, in the context of arbitrary oriented matroids, partly in order to answer the question of Tevelev mentioned earlier, and partly for its own interest.

We begin in the setting of an (unoriented) matroid M, giving the relationship between nested sets and the labelled forest $T_{\mathcal{F}}$ associated to a flag of flats of M in Proposition 3.5. This next proposition can be gleaned implicitly from the material in [14, Sections 3 and 4], but we state it explicitly here, and include our own proof, for the sake of self-containment.

Proposition 7.2. *Let M be an (unoriented) matroid. Given a flag $\mathcal{F}$ of flats in M, consider the set of connected flats $\{G_i\}$ which label the vertices of the forest $T_{\mathcal{F}}$. Then the collection of all such sets $\{G_i\}$ of flats, as $\mathcal{F}$ ranges over all flags of flats in M, are precisely the nested sets of M.*

Proof. For any $\mathcal{F}$, the labels of $T_{\mathcal{F}}$ are connected flats of M by definition. Now let us show that they form a nested set. We need to show that for any antichain $\{G_1, \ldots, G_k\}$ of flats among the vertex labels $T_{\mathcal{F}}$, their L_M-join is not connected, so assume that it is. Let F_i be the smallest flat of $\mathcal{F}$ containing $H = G_1 \vee \cdots \vee G_k$. Since H is connected, it is a subset of a connected component G of F_i. But G lies above $G_1, \ldots, G_k$ in the tree; this means that the G_is are connected components of flats F_j with $j < i$, and are therefore contained in F_{i-1}. But then H, being their L_M-join, must also be contained in F_{i-1}, contradicting the minimality of i.

Now we show that every nested set N of M, when its elements are ordered by inclusion, can be obtained as the vertex labels of the forest $T_{\mathcal{F}}$ for some flag $\mathcal{F}$ of flats of M. The connected flats in N can be labelled $G_1, G_2, \ldots, G_k$ in such a way that $i < j$ implies $G_i \not\supseteq G_j$. Let the flag $\mathcal{F}$ consist of the flats $F_i = G_1 \vee \cdots \vee G_i$ for $1 \leq i \leq k$.

First we show that F_i is just the union of the maximal G_js with $j \leq i$. Suppose there was an element e in F_i which is not in one of these G_js. Then there must be a circuit containing e and elements of, say, $G_a, \ldots, G_z$. Then $e, G_a, \ldots, G_z$ are contained in the same connected component of M. Any other f in $G_a \vee \cdots \vee G_z$ is in this same component, so $G_a \vee \cdots \vee G_z$ is connected. This contradicts the assumption that N is nested.

By the same reasoning, we cannot have a circuit in F_i consisting of elements of more than one of the G_js. Therefore the maximal G_js with $j \leq i$ are actually the connected components of F_i. In particular, G_i is one of them. This shows that the flag $\mathcal{F}$ gives rise to the nested set N, as we wished to prove. □

We next wish to understand, in the setting of an (acyclically) *oriented* matroid M, how nested sets interact with the notion of positive flats. First, we need a small technical lemma.

Lemma 7.3. *If a flat of an oriented matroid M is positive, so are its connected components.*

Proof. Let G be a connected component of a positive flat F and assume for the sake of contradiction that G is not positive. By [4, Proposition 9.1.2], we can find a signed circuit X of M such that $X^+ \subseteq G$ and $X^- \nsubseteq G$. We then have that $X^+ \subseteq F$, which implies that $X^- \subseteq F$ since F is positive. Therefore X is a circuit in F, containing elements of more than one of its connected components. This is a contradiction. □

Proposition 7.4. *Let M be an acyclically oriented matroid whose positive tope is simplicial. As $\mathcal{F}$ ranges over all flags of positive flats in M, the sets of flats labelling vertices of the forests $T_{\mathcal{F}}$ are precisely those nested sets of M which consist of positive (connected) flats.*

Proof. By Proposition 7.2, if $\mathcal{F}$ is a flag of positive flats, then $T_{\mathcal{F}}$ is a nested set of M. The labels of $T_{\mathcal{F}}$ are positive by Lemma 7.3.

Now start with a nested set N of M consisting of positive connected flats, labelled $G_1, G_2, \ldots, G_k$ in such a way that $i < j$ implies $G_i \not\supseteq G_j$. As in the proof of Proposition 7.2, the flag $\mathcal{F}$ consisting of the flats $F_i = G_1 \vee \cdots \vee G_i$ for $1 \leq i \leq k$ satisfies $T_{\mathcal{F}} = N$. Finally, each F_i is a disjoint union of G_js, which are positive. Since the positive tope is simplicial, the F_is are also positive. □

Remark 7.5. Proposition 7.4 is closely related to Feichtner and Sturmfels' notion [14, Section 4] of the *localization* of the nested set complex $\mathcal{N}(M)$ to a basis B of the matroid M, if one chooses B to be the elements of the ground set which bound the simplicial positive tope of M.

Observation 7.6. Proposition 7.4 can fail if the positive tope is not simplicial. For example, consider the oriented matroid M of affine dependencies of the vertices of a square which are cyclically labelled $\{1, 2, 3, 4\}$. Now, $\{1, 3\}$ is a nested set of L_M consisting of positive flats. However, it does not arise as the forest of a flag of positive flats.

Observation 7.7. The nested sets of M which consist of positive (connected) flats are not the same as the nested sets of the lattice of positive flats $\mathcal{F}_{lv}(M)$. In the previous example, $\{1, 3\}$ is nested in L_M but not in $\mathcal{F}_{lv}(M)$. Even in Example 2.10, the (simplicial) oriented matroid of the braid arrangement A_3, $\{1, 4\}$ is not nested in L_M but it is nested in $\mathcal{F}_{lv}(M)$.

We now give the second proof of Theorem 1.2. In view of Proposition 7.2, Proposition 3.5 tells us that, for any matroid M, the nested set complex $\mathcal{N}(M)$ is a refinement of the coarse subdivision of the Bergman complex $\mathcal{B}(M)$ and a coarsening of the order complex $\Delta(\overline{L}_M)$ (i.e. the fine subdivision of $\mathcal{B}(M)$). Therefore, it suffices to show that, in the case of a (finite) Coxeter arrangement, every cell in the nested set complex is equal to the cell of the Bergman complex which contains it.

So consider an arbitrary cell C_N corresponding to a nested set N in the nested set complex $\mathcal{N}(M_\Phi)$, and the cell D of the Bergman complex $\mathcal{B}(M_\Phi)$ containing it. By Theorem 2.8, we can find a tope T of M such that the positive Bergman complex $\mathcal{B}^+(M_\Phi)$ corresponding to T contains the cell D, and therefore the cell C_N. Since M_Φ is simplicial, Proposition 7.4 implies that the flats in N are positive with respect to T.

Proposition 6.1 tells us that connected positive flats correspond to tubes in Γ. It is easy to see that a set of connected positive flats is nested if and only if the corresponding set of tubes is a tubing of Γ. Therefore C_N is precisely the cell of $\mathcal{B}^+(M_\Phi)$ labelled by the tubing corresponding to N, by (the proof of) Theorem 1.1. It follows that $C_N = D$, as we wished to show. $\square$

Recently Tevelev [24] asked whether every (coarse) cell in the Bergman complex $\mathcal{B}(M_\Phi)$ of a Coxeter arrangement of type Φ is Coxeter-group equivalent to a cell in $\mathcal{B}^+(M_\Phi)$, i.e. a cell obtained from a tubing of the corresponding Dynkin diagram. Using Theorem 3.3, we can give an affirmative answer to this question. We begin with the following observation about the group $\mathrm{Aut}(M)$ of all automorphisms $\phi : E \to E$ for a finite matroid M on ground set $E = [n]$.

Proposition 7.8. *The group* $\mathrm{Aut}(M)$*, acting on* $\mathbb{R}^n$ *by permuting coordinates, preserves the Bergman fan* $\mathcal{B}(M)$*, and acts by cellular automorphisms on its coarse subdivision.*

Proof. Recall that a weight vector ω in $\mathbb{R}^n$ has a uniquely associated flag $\mathcal{F}(\omega)$ of subsets as in (1), and any permutation $\phi : E \to E$ respects this association: $\mathcal{F}(\phi(\omega)) = \phi(\mathcal{F}(\omega))$. Since ω lies in $\mathcal{B}(M)$ if and only if this flag of subsets is a flag of flats, a matroid automorphism ϕ will preserve this property, and hence preserves $\mathcal{B}(M)$.

Recall also from (2) that the matroid M_ω induced by ω is exactly $\bigoplus_{i=1}^{k+1} F_i/F_{i-1}$. Since two weight vectors ω, ν lie in the same coarse cell if and only if $M_\omega = M_\nu$, the second assertion of the proposition follows. $\square$

Proposition 7.9. *Let* M_Φ *be the oriented matroid of a (finite) Coxeter arrangement of type* Φ *with Coxeter group* W*. Then any coarse cell in* $\mathcal{B}(M_\Phi)$ *is* W*-equivalent to a coarse cell in* $\mathcal{B}^+(M_\Phi)$*.*

Proof. Since W acts by matroid automorphisms on the ground set E of M_Φ, Proposition 7.8 implies that W permutes the coarse cells of $\mathcal{B}(M)$.

Now let C be any coarse cell of $\mathcal{B}(M)$, and choose a fine cell $c \subset C$ defined by a flag of flats $\mathcal{F}$. By Theorem 3.3 and the fact that W acts transitively on the regions of the arrangement, there exists some $w \in W$ such that $w(c)$ is a fine cell defined by a flag of positive flats, i.e. $w(c)$ lies inside a coarse cell D of $\mathcal{B}^+(M)$. But now it follows that $w(C) = D$: $w(C)$ must be a cell of $\mathcal{B}(M)$, and it contains $w(c)$, which lies in D. $\square$

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