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A NOTE ON SOME MAHONIAN STATISTICS

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Abstract. We construct a class of mahonian statistics on words, related to the classical statistics maj and inv. These statistics are constructed via Foata's second fundamental transformation.

1. Introduction

Consider the alphabet $\mathcal{X} = [r] = \{1, 2, \ldots, r\}$.

A *word* $w = w_1 w_2 \cdots w_n$ on $\mathcal{X}$ is a finite string of not-necessarily distinct elements of $\mathcal{X}$. The set of all words on $\mathcal{X}$ is written $\mathcal{X}^*$. The *rearrangement class* $R(w)$ of a word w is the set of all words that can be obtained by permuting the letters of w. If the letters of w are distinct, then w is a permutation and $R(w)$ is the set of elements of the symmetric group $\mathcal{S}_n$.

The statistics inv and maj are defined on $\mathcal{X}^*$ by

$$\operatorname{inv} w = \#\{\, (i,j) \mid 1 \le i < j \le n, w_i > w_j \,\},$$
$$\operatorname{maj} w = \sum \{\, i \mid 1 \le i < n, w_i > w_{i+1} \,\}.$$

It is a result of MacMahon [5] that inv and maj are equidistributed on any rearrangement class $R(c)$. Foata [2, 3] gave a bijective proof using his *second fundamental transformation.* A statistic equidistributed with inv (or maj) is called *mahonian.*

Let $a, b \in \mathcal{X}$. The *cyclic interval* $\rrbracket a, b \rrbracket$ has been defined by Han [4] as

$$\rrbracket a, b \rrbracket = \begin{cases} (a, b], & \text{if } a \le b; \\ \mathcal{X} \setminus (b, a], & \text{otherwise.} \end{cases}$$

In particular, $\rrbracket a, a \rrbracket = \emptyset$.

Han has redefined the statistics maj and inv in terms of cyclic intervals as follows. If $1 \le j \le n$, define the *j-factor* of w as $\operatorname{Fact}_j w =$

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$w_1w_2\dots w_{j-1}$. Write $w_{n+1}=\infty$. Then

$$\operatorname{inv} w = \sum_{j=2}^{n} \left|\operatorname{Fact}_j w \cap \rrbracket w_j, \infty \rrbracket\right|,$$

$$\operatorname{maj} w = \sum_{j=2}^{n} \left|\operatorname{Fact}_j w \cap \rrbracket w_j, w_{j+1} \rrbracket\right|.$$

We define the partial statistics s_j and t_j by

$$s_j = s_j(w) = \left|\operatorname{Fact}_j w \cap \rrbracket w_j, w_{j+1} \rrbracket\right|,$$
$$t_j = t_j(w) = \left|\operatorname{Fact}_j w \cap \rrbracket w_j, \infty \rrbracket\right|,$$

$2 \le j \le n$. Then $s_n = t_n$ and

$$\operatorname{inv} w = t_2(w) + \dots + t_n(w),$$
$$\operatorname{maj} w = s_2(w) + \dots + s_n(w).$$

Our main result is the following.

Theorem 1. *Let* $\mathbf{e} = (e_2, \dots, e_{n-1}) = \mathbb{Z}_2^{n-2}$. *For each* j, $1 < j \le n$, *let*

$$u_j = \begin{cases} s_j, & \text{if } e_j = 0; \\ t_j, & \text{if } e_j = 1. \end{cases}$$

Put $u_n = s_n = t_n$. *Then the statistic*

$$\operatorname{inmaj}_{\mathbf{e}} w = u_2 + \dots + u_n$$

is mahonian.

Note that $\operatorname{maj} w = \operatorname{inmaj}_{(0,\dots,0)} w$, $\operatorname{inv} w = \operatorname{inmaj}_{(1,\dots,1)} w$. This theorem defines 2^{n-2} mahonian statistics, all but two of which seem to be new (although implicit in Foata's second fundamental transformation).

Although this result suggests that for each j, the partial statistics s_j and t_j are equidistributed, this is not in general true — on the rearrangement class $R(1\,1\,2\,3)$, the statistics s_3 and t_3 are differently distributed.

2. Foata's second fundamental transformation

Let $w = w_1w_2\dots w_n$ be a word on $\mathcal{X}$ and let $a \in \mathcal{X}$. If $w_n \le a$, the *a-factorization* of w is $w = v_1b_1\dots v_pb_p$, where each b_i is a letter less than or equal to a, and each v_i is a word (possibly empty), all of whose letters are greater than a. Similarly, if $w_n > a$, the *a-factorization* of w is $w = v_1b_1\dots v_pb_p$, where each b_i is a letter greater than a, and each

v_i is a word (possibly empty), all of whose letters are less than or equal to a. In each case we define

$$\gamma_a(w) = b_1 v_1 \dots b_p v_p.$$

With the above notations, let $a = w_n$ and let $w' = w_1 \dots w_{n-1}$. The second fundamental transformation Φ is defined recursively by $\Phi(w) = w$, if w has length 1, and

$$\Phi(w) = \gamma_a(\Phi(w'))a,$$

if w has length $n > 1$. Then

$$\operatorname{inv}\Phi(w) = \operatorname{maj} w,$$

see [3].

Let us define the mapping Φ_1 as the juxtaposition product

$$\Phi_1(w) = \gamma_a(w')a.$$

The following result follows easily from the proof of the result in [3].

Lemma 2. *With the above notation,*

$$\begin{aligned} \operatorname{inv}\Phi_1(w) &= \begin{cases} \operatorname{inv} w' & \text{if } w_{n-1} \le w_n, \\ \operatorname{inv} w' + n - 1 & \text{if } w_{n-1} > w_n \end{cases} \\ &= \operatorname{inv} w' + \operatorname{maj} w - \operatorname{maj} w'. \end{aligned}$$

□

Since Φ_1 is a bijection, this shows that the statistic

$$\operatorname{inv} w' + \operatorname{maj} w - \operatorname{maj} w'$$

is mahonian. Now it is routine to verify that, in the notation of the previous section,

$$\begin{aligned} \operatorname{inv} w' + \operatorname{maj} w - \operatorname{maj} w' &= t_2 + \cdots + t_{n-2} + s_{n-1} + s_n \\ &= \operatorname{inmaj}_{(1,\dots,1,0)} w. \end{aligned}$$

Hence $\operatorname{inmaj}_{(1,\dots,1,0)}$ is mahonian.

More generally, let $1 \le j \le n$. Write $w = u_1 u_2$, where u_1 is a word of length j. Then we can show in the same way as before that there is a bijection Φ' satisfying

$$\begin{aligned} \operatorname{inv}\Phi'(w) = \operatorname{inv} u_1 + \operatorname{maj} w - \operatorname{maj} u_1 &= t_2 + \cdots + t_{j-1} + s_j + \cdots + s_n \\ &= \operatorname{inmaj}_{(1,\dots,1,0,\dots,0)} \end{aligned}$$

for all words w on n letters. Hence the statistic $\operatorname{inmaj}_{(1,\dots,1,0,\dots,0)}$ is mahonian (where the subscript contains $j-2$ ones and $n-j$ zeros).

Proof of Theorem 1. Let $\mathbf{e} = (e_2, \dots, e_{n-1}) \in \mathbb{Z}^{n-1}$ as before. We will show that $\operatorname{inmaj}_{\mathbf{e}}$ is mahonian, by showing that there is a bijection $\Phi_{\mathbf{e}}$ satisfying $\operatorname{inv} \Phi_{\mathbf{e}}(w) = \operatorname{inmaj}_{\mathbf{e}} w$. We have shown this above for $\mathbf{e}$ of the form $(1, \dots, 1, 0, \dots, 0)$.

We use induction on the number of zeros in the vector $\mathbf{e}$. The result is true if $\mathbf{e}$ contains no zeros, as in this case $\operatorname{inmaj}_{\mathbf{e}} = \operatorname{inv}$.

Suppose the result is true for all vectors $\mathbf{e}'$ containing fewer zeros than $\mathbf{e}$. Let k be the smallest index such that $e_k = 0$ and let j be the largest index such that $e_i = 0$ for $k \le i \le j$. If $j = n-1$ then the result follows, so suppose that $j < n - 1$. Then $e_{j+1} = 1$. Thus

$$\operatorname{inmaj}_{\mathbf{e}} w = t_2 + \cdots + t_{k-1} + s_k + \cdots + s_j + t_{j+1} + u_{j+2} + \cdots + u_n,$$

where $u_i = t_i$ or s_i.

Let $\mathbf{f}$ be the vector of length $j - 1$ whose components are $e_2, \dots, e_j$, i.e., $\mathbf{f} = (1, \dots, 1, 0, \dots, 0)$ (with $k - 2$ ones). Then there is a bijection $\Phi_{\mathbf{f}}$ satisfying

$$\operatorname{inv} \Phi_{\mathbf{f}} v = \operatorname{inmaj}_{\mathbf{f}} v$$

for all words v of length $j + 1$. Define a bijection θ on words of length n by

$$\theta(v_1 v_2) = \Phi_{\mathbf{f}}(v_1) v_2,$$

where v_1 and v_2 are words of lengths $j + 1$ and $n - j - 1$ respectively. Let $\mathbf{g} = (g_i)$ be the vector defined by

$$g_i = \begin{cases} 1, & \text{if } 2 \le i \le j; \\ e_i, & \text{if } i > j. \end{cases}$$

Then

$$\begin{aligned} \operatorname{inmaj}_{\mathbf{g}} \theta(v_1 v_2) &= (t_2 + \cdots + t_{j+1}) \Phi_{\mathbf{f}}(v_1) + (u_{j+2} + \cdots + u_n) \Phi_{\mathbf{f}}(v_1) v_2 \\ &= (u_2 + \cdots + u_{j+1}) v_1 + (u_{j+2} + \cdots + u_n) v_1 v_2 \\ &= \operatorname{inmaj}_{\mathbf{e}} v_1 v_2. \end{aligned}$$

Now, as $\mathbf{g}$ contains fewer zeros than $\mathbf{e}$, there is a bijection $\Phi_{\mathbf{g}}$ such that

$$\operatorname{inv} \Phi_{\mathbf{g}}(w) = \operatorname{inmaj}_g w$$

for all words w on n letters. Hence, putting $\Phi_{\mathbf{e}} = \Phi_{\mathbf{g}} \circ \theta$,

$$\operatorname{inv} \Phi_{\mathbf{e}}(w) = \operatorname{inv} \Phi_{\mathbf{g}}(\theta(w)) = \operatorname{inmaj}_{\mathbf{g}} \theta(w) = \operatorname{inmaj}_{\mathbf{e}} w.$$

□

Finally, we refer the reader to [1] for a rather different application of Foata's second fundamental transformation.

References

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