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A FOATA BIJECTION FOR THE ALTERNATING GROUP AND FOR q-ANALOGUES

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Abstract. The Foata bijection $\Phi : S_n \to S_n$ is extended to the bijections $\Psi : A_{n+1} \to A_{n+1}$ and $\Psi_q : S_{n+q-1} \to S_{n+q-1}$, where S_m, A_m are the symmetric and the alternating groups. These bijections imply bijective proofs for recent equidistribution theorems, by Regev and Roichman, for A_{n+1} and for S_{n+q-1}.

1. Introduction

In [MM16] MacMahon proved his remarkable theorem about the equidistribution in the symmetric group S_n of the *length* (or the *inversion-number*) and the *major index* statistics. This raised the natural question of constructing a canonical bijection on S_n, for each n, that would correspond these *length* and *major index* statistics, and would thus yield a bijective proof of that theorem of MacMahon. That problem was solved by Foata [Foa68], who constructed such a canonical bijection, see Section 3 for a discussion of the Foata bijection Φ. Throughout the years, MacMahon's equidistribution theorem has received far reaching refinements and generalizations, see for example [BW91], [Car54], [Car75], [FS78], [GG79], [Kra95] and [Sta02].

We remark that MacMahon's equidistribution theorem fails when the S_n statistics are restricted to the alternating subgroups A_n. However, by introducing new A_n statistics which are natural analogues of the S_n statistics, in [RR04], analogous equidistribution theorems were proved for A_n. This was done by first formulating the above S_n statistics in terms of the Coxeter ganerators $s_i = (i, i+1), \quad 1 \leq i \leq n-1$. By choosing the "Mitsuhashi" generators $a_i = s_1 s_{i+1}$ for A_{n+1} [Mit01], analogous statistics on A_{n+1} were obtained, via canonical presentations by these generators. These canonical presentations allow the introduction of the map $f : A_{n+1} \to S_n$, which is one of the main tools in [RR04], see Section 2 for a discussion of these presentations and of f.

MacMahon-type theorems were obtained in [RR04] by introducing the *delent* statistics for these groups (for S_n, this statistic already appeared in [BW91]). Via the above map $f : A_{n+1} \to S_n$, equidistribution theorems were then lifted from S_n to A_{n+1}, thus yielding (new) equidistribution theorems for A_{n+1}. In particular, equidistribution theorems for the A_{n+1}-analogues of the *length* and *(reverse) major-index* statistics were obtained in this way, see for example Theorem 4.9 below.

These theorems naturally raise the question of constructing an A_{n+1}-analogue of the Foata bijection — with analogous properties. This problem is solved in Section 5, where indeed we construct such a map Ψ. That map Ψ is composed of a reflection of Foata's original bijection Φ, together with the map $f : A_{n+1} \to S_n$ and of certain "local" inversions of f. This of course gives a new — bijective — proof of Theorem 4.9.

Statistics on symmetric groups which are q-analogues of the classical S_n statistics were introduced in [RR05], via S_n canonical presentations and the maps $f_q : S_{n+q-1} \to S_n$, see Section 7 below. This map f_q sends the q statistics on S_{n+q-1} to the corresponding classical statistics on S_n. As in the case of A_{n+1}, this allows the lifting of equidistribution theorems from S_n to S_{n+q-1}. This was done in [RR05], where in that process an interesting connection with dashed patterns in permutations has appeared, see Theorems 7.7 and 7.8 below.

Again, these equidistribution theorems naturally raise the question of finding the (q-)analogue of the Foata bijection. In Section 7 we indeed construct such a bijection Ψ_q. As in the case of Ψ, the bijection Ψ_q is composed of f_q, of a reflection of the original Foata bijection Φ, and of certain "local" inversions of f_q. As an application, Ψ_q yields new — bijective — proofs of Theorems 7.7 and 7.8.

The paper is organized as follows. Sections 2, 3 and 4 contain preliminary material, mostly from [Foa68], [RR04] and [RR05], which is necessary for defining and studying the bijections Ψ and Ψ_q. A reader who is familiar with these three papers can skip these preliminary sections. In Section 5 we introduce and study Ψ, and Section 6 is an example, showing the properties of Ψ. Finally, the bijections Ψ_q are introduced and studied in Section 7.

2. Canonical presentations and the covering map

2.1. **The S- and A-canonical presentations.** In this subsection we review the presentations of elements in S_n and A_n by the corresponding generators and procedures for calculating them.

The Coxeter generators of S_n are $s_i = (i, i+1)$, $1 \le i \le n-1$. Recall the definition of the set R_j^S,

$$R_j^S = \{1,\ s_j,\ s_j s_{j-1},\ \ldots,\ s_j s_{j-1} \cdots s_1\} \subseteq S_{j+1},$$

and the following theorem.

Theorem 2.1 (see [Gol93, pp. 61–62]). *Let $w \in S_n$. Then there exist unique elements $w_j \in R_j^S$, $1 \le j \le n-1$, such that $w = w_1 \cdots w_{n-1}$. Thus, the presentation $w = w_1 \cdots w_{n-1}$ is unique. Call that presentation* the S-canonical presentation of w.

The number of s_i in the S-canonical presentation of $\sigma \in S_n$ is its *S-length*, $\ell_S(\sigma)$. The *descent set* of $\sigma \in S_n$ is $\mathrm{Des}_S(\sigma) = \{\, i \mid \sigma(i) > \sigma(i+1) \,\}$; the *$S$ major-index* is $\mathrm{maj}_S(\sigma) = \sum_{i \in \mathrm{Des}_S(\sigma)} i$, and the *reverse S_n major-index* is $\mathrm{rmaj}_{S_n}(\sigma) = \sum_{i \in \mathrm{Des}_S(\sigma)} (n-i)$. Note that $i \in \mathrm{Des}_S(\sigma)$ iff $\ell_S(\sigma) > \ell_S(\sigma s_i)$.

The S-procedure is a simple way to calculate the S-canonical presentation of a given element in S_n. Let $\sigma \in S_n$, $\sigma(r) = n$, $\sigma = [\ldots, n, \ldots]$. Then by definition of the s_is, n can be 'pulled to its place on the right': $\sigma s_r s_{r+1} \cdots s_{n-1} = [\ldots, n]$. This gives $w_{n-1} = s_{n-1} \cdots s_{r+1} s_r \in R_{n-1}^S$. Looking at $\sigma w_{n-1}^{-1} = \sigma s_r s_{r+1} \cdots s_{n-1} = [\ldots, n-1, \ldots, n]$ now, pull $n-1$ to its right place (second from the right) by a similar product $s_t s_{t+1} \cdots s_{n-2}$, yielding $w_{n-2} = s_{n-2} \cdots s_t \in R_{n-2}^S$. Continue this way until finally $\sigma = w_1 \cdots w_{n-1}$.

Example 2.2. Let $\sigma = [5, 6, 3, 2, 1, 4]$, then $w_5 = s_5 s_4 s_3 s_2$; $\sigma w_5^{-1} = [5, 3, 2, 1, 4, 6]$, so in order to 'pull 5 to its place' we need $w_4 = s_4 s_3 s_2 s_1$; now $\sigma w_5^{-1} w_4^{-1} = [3, 2, 1, 4, 5, 6]$, so no need to move 4, hence $w_3 = 1$; continuing the same way, check that $w_2 = s_2 s_1$ and $w_1 = s_1$, so $\sigma = w_1 w_2 w_3 w_4 w_5 = (s_5 s_4 s_3 s_2)(s_4 s_3 s_2 s_1)(1)(s_2 s_1)(s_1)$. Thus $\ell_S(\sigma) = 11$. Here $\mathrm{Des}_S\, \sigma = \{2, 3, 4\}$, so $\mathrm{maj}_S(\sigma) = \mathrm{rmaj}_{S_6}(\sigma) = 9$.

For A_n, the "Mitsuhash" generators are $a_i = s_1 s_{i+1}$, $1 \le i \le n-2$. Recall the definition

$$R_j^A = \{1,\, a_j,\, a_j a_{j-1},\, \ldots,\, a_j \cdots a_2,\, a_j \cdots a_2 a_1,\, a_j \cdots a_2 a_1^{-1}\} \subseteq A_{j+2}$$

(for example, $R_3^A = \{1, a_3, a_3a_2, a_3a_2a_1, a_3a_2a_1^{-1}\}$), and the following theorem.

Theorem 2.3 (see [RR04, Theorem 3.4])**.** *Let $v \in A_{n+1}$. Then there exist unique elements $v_j \in R_j^A$, $1 \le j \le n-1$, such that $v = v_1 \cdots v_{n-1}$, and this presentation is unique. Call that presentation* the A-canonical presentation of v.

The number of a_i in the A-canonical presentation of $\sigma \in A_{n+1}$ is defined to be its *A-length*, $\ell_A(\sigma)$. In analogy with S_n, the A-descent set of $\sigma \in A_{n+1}$ is defined as $\mathrm{Des}_A(\sigma) = \{\, i \mid \ell_A(\sigma) \ge \ell_A(\sigma a_i)\,\}$. Now define $\mathrm{maj}_A(\sigma) = \sum_{i \in \mathrm{Des}_A(\sigma)} i$, and $\mathrm{rmaj}_{A_{n+1}}(\sigma) = \sum_{i \in \mathrm{Des}_A(\sigma)} (n-i)$, see [RR05].

The A-procedure is a simple procedure for obtaining the A-canonical presentation of $\sigma \in A_n$.
A-procedure: Step 1: follow the *S-procedure* and obtain the S-canonical presentation of $\sigma \in A_n$. Step 2: pair the factors. Step 3: insert $s_1 s_1$ in the middle of each pair and obtain the A-canonical presentation.

Example 2.4. . Let $\sigma = [6, 4, 3, 7, 5, 2, 1]$.
Step 1: $\sigma = s_1s_2s_1s_3s_2s_1s_4s_3s_5s_4s_3s_2s_1s_6s_5s_4$.
Step 2: $\sigma = (s_1s_2)(s_1s_3)(s_2s_1)(s_4s_3)(s_5s_4)(s_3s_2)(s_1s_6)(s_5s_4)$.
Step 3: $\sigma = (s_1s_1s_1s_2)(s_1s_1s_1s_3)(s_2s_1s_1s_1)(s_4s_1s_1s_3)(s_5s_1s_1s_4)(s_3s_1s_1s_2)(s_1s_1s_1s_6)(s_5s_1s_1s_4) =$
$= (s_1s_2)(s_1s_3)(s_2s_1)(s_1s_4s_1s_3)(s_1s_5s_1s_4)(s_1s_3s_1s_2)(s_1s_6)(s_1s_5s_1s_4) =$
$= a_1a_2a_1^{-1}a_3a_2a_4a_3a_2a_1a_5a_4a_3 = (a_1)(a_2a_1^{-1})(a_3a_2)(a_4a_3a_2a_1)(a_5a_4a_3)$.
Thus $\ell_A(\sigma) = 12$ (while $\ell_S(\sigma) = 16$). It can be shown here that $\mathrm{Des}_A(\sigma) = \{1, 3, 4, 5\}$, hence $\mathrm{rmaj}_{A_7}(\sigma) = 10$.

2.2. **The covering map f.** We can now introduce the *covering map* f, which plays an important role in later sections in the constructions of the bijections Ψ and Ψ_q.

Definition 2.5 (see [RR04, Definition 5.1])**.** Define $f : R_j^A \to R_j^S$ by

(1) $f(a_j a_{j-1} \cdots a_\ell) = s_j s_{j-1} \cdots s_\ell$ if $\ell \ge 2$, and
(2) $f(a_j \cdots a_1) = f(a_j \cdots a_1^{-1}) = s_j \cdots s_1$.

Now extend $f : A_{n+1} \to S_n$ as follows: let $v \in A_{n+1}$, $v = v_1 \cdots v_{n-1}$ its A-canonical presentation, then

$$f(v) := f(v_1) \cdots f(v_{n-1}),$$

which is clearly the S-canonical presentation of $f(v)$.

3. The Foata bijection

The *second fundamental transformation on words* Φ was introduced in [Foa68] (for a full description, see [Lot83, §10.6]). It is defined on any finite word $r = x_1x_2 \ldots x_m$ whose letters $x_1, \ldots, x_m$ belong to a totally ordered alphabet.

Definition 3.1. Let X be a totally ordered alphabet, let $r = x_1x_2 \ldots x_m$ be a word whose letters belong to X, and let $x \in X$ such that $x_m \le x$ (respectively $x_m > x$). Let

$$r = r^1 r^2 \ldots r^p$$

be the unique decomposition of r into subwords $r^i = r_1^i r_2^i \dots r_{m_i}^i$, $1 \le i \le p$, such that $r_{m_i}^i \le x$ (respectively $r_{m_i}^i > x$) and $r_j^i > x$ (respectively $r_j^i \le x$) for all $1 \le j < m_i$. Define $\gamma_x(r)$ by

$$\gamma_x(r) = r_{m_1}^1 r_1^1 r_2^1 \dots r_{m_1-1}^1 r_{m_2}^2 r_1^2 \dots r_{m_2-1}^2 \dots r_{m_p}^p r_1^p \dots r_{m_p-1}^p.$$

For example, with the usual order on the integers, $r = 1\,2\,6\,7\,8\,3\,4$ and $x = 5$, r decomposes into $r^1 = 1$, $r^2 = 2$, $r^3 = 6\,7\,8\,3$ and $r^4 = 4$, so

$$\gamma_5(1\,2\,6\,7\,8\,3\,4) = 1\,2\,3\,6\,7\,8\,4.$$

Definition 3.2. Define Φ recursively as follows. First, $\Phi(r) := r$ if r is of length 1. If x is a letter and r is a nonempty word, define $\Phi(rx) = \gamma_x(\Phi(r))\,x$.

For example,

$$\begin{aligned}\Phi(6\,5\,3\,1\,4\,2) &= \gamma_2(\gamma_4(\gamma_1(\gamma_3(\gamma_5(6)\,5)\,3)\,1)\,4)\,2\\ &= \gamma_2(\gamma_4(\gamma_1(\gamma_3(6\,5)\,3)\,1)\,4)\,2\\ &= \gamma_2(\gamma_4(\gamma_1(6\,5\,3)\,1)\,4)\,2\\ &= \gamma_2(\gamma_4(6\,5\,3\,1)\,4)\,2\\ &= \gamma_2(3\,6\,5\,1\,4)\,2\\ &= 3\,6\,5\,4\,1\,2.\end{aligned}$$

The following algorithmic description of Φ from [FS78] is more useful in calculations.

Algorithm 3.3 (Φ)**.** Let $r = x_1 x_2 \dots x_m$;

1. Let $i := 1$, $r_i' := x_1$;
2. If $i = m$, let $\Phi(r) := r_i'$ and stop; else continue;
3. If the last letter of r_i' is less than or equal to (respectively greater than) x_{i+1}, cut r_i' after every letter less than or equal to (respectively greater than) x_{i+1} ;
4. In each compartment of r_i' determined by the previous cuts, move the last letter in the compartment to the beginning of it; let t_i' be the word obtained after all those moves; put $r_{i+1}' := t_i'\,x_{i+1}$; replace i by $i+1$ and go to step 2.

Example 3.4. Calculating $\Phi(r)$, where $r = 6\,5\,3\,1\,4\,2$, using the algorithm:

$$\begin{aligned} r_1' &= 6\mid\\ r_2' &= 6\mid 5\mid\\ r_3' &= 6\mid 5\mid 3\mid\\ r_4' &= 6\quad 5\quad 3\mid 1\mid\\ r_5' &= 3\mid 6\mid 5\mid 1\quad 4\mid\\ \Phi(r) = r_6' &= 3\quad 6\quad 5\quad 4\quad 1\quad 2\quad .\end{aligned}$$

The main property of Φ is the following theorem.

Theorem 3.5 (see [Foa68])**.** (1) *Φ is a bijection of S_n onto itself.*
(2) *For every $\sigma \in S_n$, $\mathrm{maj}_S(\sigma) = \ell_S(\Phi(\sigma))$.*

Some further properties of Φ are given in Theorem 5.1

Let $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_n] \in S_n$. Denote the *reverse* and the *complement* of σ by

$$\mathbf{r}(\sigma) := [\sigma_n, \sigma_{n-1}, \dots, \sigma_1] \in S_n$$

and

$$\mathbf{c}(\sigma) := [n+1-\sigma_1, n+1-\sigma_2, \dots, n+1-\sigma_n] \in S_n$$

respectively.

Remark 3.6. Let $\rho = [n, n-1, \dots, 1] \in S_n$. Then for $\sigma \in S_n$, $\mathbf{r}(\sigma) = \sigma\rho$ and $\mathbf{c}(\sigma) = \rho\sigma$. Thus it is obvious that $\mathbf{r}$ and $\mathbf{c}$ are involutions and that $\mathbf{rc} = \mathbf{cr}$. Moreover, $(\mathbf{r}(\sigma))^{-1} = \mathbf{c}(\sigma^{-1})$.

Definition 3.7. Let $\overleftarrow{\Phi} := \mathbf{r}\Phi\mathbf{r}$, the *right-to-left Foata transformation.*

While $\overleftarrow{\Phi}(w)$ is easy enough to calculate by reversing w, applying Algorithm 3.3 and reversing the result, it is easy to see that it can be calculated "directly" by applying a "right-to-left" version of the Algorithm, namely:

Algorithm 3.8 ($\overleftarrow{\Phi}$)**.** Let $r = x_1 x_2 \dots x_m$;
1. Let $i := m$, $r'_i := x_m$;
2. If $i = 1$, let $\Phi(r) := r'_i$ and stop; else continue;
3. If the first letter of r'_i is less than or equal to (respectively greater than) x_{m-i}, cut r'_i before every letter less than or equal to (respectively greater than) x_{m-i} ;
4. In each compartment of r'_i determined by the previous cuts, move the first letter in the compartment to the end of it; let t'_i be the word obtained after all those moves; put $r'_{i-1} := x_{m-i}\, t'_i$; replace i by $i-1$ and go to step 2.

For an example of applying Algorithm 3.8, see the calculation of $\overleftarrow{\Phi}(w)$ in Section 6.

The key property of $\overleftarrow{\Phi}$ used in this paper is the following.

Theorem 3.9. *For every $\sigma \in S_n$, $\mathrm{rmaj}_{S_n}(\sigma) = \ell_S(\overleftarrow{\Phi}(\sigma))$.*

The proof requires the following lemmas.

Lemma 3.10. *The bijections $\Phi : S_n \to S_n$ and $\mathbf{c} : S_n \to S_n$ commute with each other.*

Proof. We prove a slightly stronger claim, namely that Φ and $\mathbf{c}_k$ commute as maps on $\mathbb{Z}^n$, where $\mathbf{c}_k(a_1, a_2, \dots, a_n) := (k+1-a_1, k+1-a_2, \dots, k+1-a_n)$. Let $\sigma = [\sigma_1\sigma_2 \dots \sigma_n] \in \mathbb{Z}^n$. We need to show that $\Phi\mathbf{c}_k(\sigma) = \mathbf{c}_k\Phi(\sigma)$. The proof is by induction on n. For $n = 1$, everything is trivial. For $n \geq 2$, write $\Phi(\sigma_1 \dots \sigma_{n-1}) = b_1 \dots b_{n-1}$ and $\gamma_{\sigma_n}(b_1 \dots b_{n-1}) = c_1 \dots c_{n-1}$. Using the notation $\overline{a} = k+1-a$, we have

$$\begin{aligned}\Phi\mathbf{c}_k(\sigma) &= \gamma_{\overline{\sigma_n}}(\Phi(\overline{\sigma_1}\,\overline{\sigma_2} \dots \overline{\sigma_{n-1}}))\,\overline{\sigma_n} \\ &= \gamma_{\overline{\sigma_n}}(\overline{b_1}\,\overline{b_2} \dots \overline{b_{n-1}})\,\overline{\sigma_n}\end{aligned}$$

by the induction hypothesis, and

$$\mathbf{c}_k\Phi(\sigma) = \mathbf{c}_k(c_1 \dots c_{n-1}\,\sigma_n) = \overline{c_1} \dots \overline{c_{n-1}}\,\overline{\sigma_n},$$

so it remains to show that $\gamma_{\overline{\sigma_n}}(\overline{b_1}\,\overline{b_2} \dots \overline{b_{n-1}}) = \overline{c_1} \dots \overline{c_{n-1}}$.

Assume for now that $b_{n-1} < \sigma_n$ (the case $b_{n-1} > \sigma_n$ is entirely symmetric and will be left to the reader). Let $M = \{\, 1 \leq m \leq n-1 \mid b_m < \sigma_n \,\} = \{m_1, \dots, m_p\}$, $m_1 < \cdots < m_p$. Note that $\overline{b_{n-1}} > \overline{\sigma_n}$ and $M = \{\, 1 \leq m \leq n-1 \mid \overline{\sigma_m} > \overline{b_n} \}$. Therefore, using the notation from Definition 3.1, we have the decompositions

$$b_1 \dots b_{n-1} = r^1_1 \dots r^1_{m_1}\, r^2_1 \dots r^2_{m_2} \dots r^p_1 \dots r^p_{m_p}$$

and

$$\overline{b_1}\ \dots\ \overline{b_{n-1}} = \overline{r_1^1}\dots\overline{r_{m_1}^1}\,\overline{r_1^2}\dots\overline{r_{m_2}^2}\dots\overline{r_1^p}\dots\overline{r_{m_p}^p},$$

so

$$c_1\dots c_{n-1} = \gamma_{\sigma_n}(b_1\dots b_{n-1}) = r_{m_1}^1\, r_1^1\dots r_{m_1-1}^1\, r_{m_2}^2\, r_1^2\dots r_{m_2-1}^2\dots r_{m_p}^p\, r_1^p\dots r_{m_p-1}^p$$

and

$$\gamma_{\overline{\sigma_n}}(\overline{b_1}\dots\overline{b_{n-1}}) = \overline{r_{m_1}^1}\,\overline{r_1^1}\dots\overline{r_{m_1-1}^1}\,\overline{r_{m_2}^2}\,\overline{r_1^2}\dots\overline{r_{m_2-1}^2}\dots\overline{r_{m_p}^p}\,\overline{r_1^p}\dots\overline{r_{m_p-1}^p}.$$

Thus $\gamma_{\overline{\sigma_n}}(\overline{b_1}\,\overline{b_2}\ \dots\ \overline{b_{n-1}}) = \overline{c_1}\ \dots\ \overline{c_{n-1}}$ as desired. □

Lemma 3.11. *For every $w \in S_n$, $\ell_S(\mathbf{rc}(w)) = \ell_S(w)$.*

Proof. The lemma follows from the definitions of $\mathbf{r}$ and $\mathbf{c}$ and from the fact that for all $\sigma \in S_n$, $\ell_S(\sigma) = \operatorname{inv}(\sigma) = \#\{\,(i,j) \mid 1 \le i < j \le n,\ \sigma(i) > \sigma(j)\,\}$:

$$\begin{aligned}
\ell_S(w) &= \#\{\,(i,j) \mid 1 \le i < j \le n,\ w(i) > w(j)\,\}\\
&= \#\{\,(i,j) \mid 1 \le i < j \le n,\ \mathbf{c}(w)(i) < \mathbf{c}(w)(j)\,\}\\
&= \#\{\,(i,j) \mid 1 \le i < j \le n,\ \mathbf{rc}(w)(n+1-i) < \mathbf{rc}(w)(n+1-j)\,\}\\
&= \#\{\,(n+1-s, n+1-r) \mid 1 \le r < s \le n,\ \mathbf{rc}(w)(s) < \mathbf{rc}(w)(r)\,\}\\
&= \#\{\,(r,s) \mid 1 \le r < s \le n,\ \mathbf{rc}(w)(r) > \mathbf{rc}(w)(s)\,\}\\
&= \ell_S(\mathbf{rc}(w))
\end{aligned}$$

□

Lemma 3.12. *For every $w \in S_n$, $\operatorname{maj}_S(\mathbf{rc}(w)) = \operatorname{rmaj}_{S_n}(w)$.*

Proof. By the definitions of $\mathbf{r}$, $\mathbf{c}$ and Des_S,

$$\begin{aligned}
i \in \operatorname{Des}_S(\mathbf{rc}(w)) &\iff \mathbf{rc}(w)(i) > \mathbf{rc}(w)(i+1)\\
&\iff \mathbf{c}(w)(n-i+1) > \mathbf{c}(w)(n-i)\\
&\iff n+1-w(n-i+1) > n+1-(w)(n-i)\\
&\iff w(n-i+1) < (w)(n-i)\\
&\iff n-i \in \operatorname{Des}_S(w).
\end{aligned}$$

Therefore

$$\operatorname{maj}_S(\mathbf{rc}(w)) = \sum_{i\in \operatorname{Des}_S(\mathbf{rc}(w))} i = \sum_{i\in\operatorname{Des}_S(w)} n-i = \operatorname{rmaj}_{S_n}(w).$$

□

Proof of Theorem 3.9.

$$\begin{aligned}
\operatorname{rmaj}_{S_n}(\sigma) &= \operatorname{maj}_S(\mathbf{rc}(\sigma)) && \text{(by Lemma 3.12)}\\
&= \ell_S(\Phi\mathbf{rc}(\sigma)) && \text{(by Theorem 3.5)}\\
&= \ell_S(\mathbf{rc}\Phi\mathbf{rc}(\sigma)) && \text{(by Lemma 3.11)}\\
&= \ell_S(\overleftarrow{\Phi}(\sigma)) && \text{(by Lemma 3.10 and Remark 3.6)}
\end{aligned}$$

□

4. The delent statistics

Definition 4.1 (see [RR04, Definition 7.1])**.** Let $\sigma \in S_n$. Define $\mathrm{Del}_S(\sigma)$ as

$$\mathrm{Del}_S(\sigma) = \{\, 1 < j \le n \mid \forall i < j \;\; \sigma(i) > \sigma(j) \,\}.$$

These are the positions of the l.t.r.min, excluding the first position.

Definition 4.2. Let $\sigma \in S_n$. Define the *left-to-right minima set* of σ as

$$\overrightarrow{\min}(\sigma) = \sigma\left(\mathrm{Del}_S(\sigma) \cup \{1\}\right) = \{\, \sigma(j) \mid 1 \le j \le n,\ \forall i < j \;\; \sigma(i) > \sigma(j) \,\}.$$

These are the actual (letters) l.t.r.min, including the first letter.

Example 4.3. Let $\sigma = [5,2,3,1,4]$. Then $\mathrm{Del}_S(\sigma) = \{2,4\}$ and $\overrightarrow{\min}(\sigma) = \{5,2,1\}$.

Proposition 4.4. *For every $\sigma \in S_n$, $\overrightarrow{\min}(\sigma) = \mathrm{Del}_S(\sigma^{-1}) \cup \{1\}$.*

Proof. Let $k \in \overrightarrow{\min}(\sigma)$. Then $j = \sigma^{-1}(k) \in \mathrm{Del}_S(\sigma) \cup \{1\}$. Therefore, by negation, for all $1 \le i \le n$, if $\sigma(i) < \sigma(j) = k$ then $i > j = \sigma^{-1}(k)$. By the change of variables $i' = \sigma(i)$, we get that for all $1 \le i' \le n$, $i' < k$ implies $\sigma^{-1}(i') > \sigma^{-1}(k)$, so by definition, $k \in \mathrm{Del}_S(\sigma^{-1}) \cup \{1\}$. This proves that $\overrightarrow{\min}(\sigma) \subseteq \mathrm{Del}_S(\sigma^{-1}) \cup \{1\}$.

The reverse containment is obtained by substituting σ^{-1} for σ and applying σ to both sides. $\square$

Definition 4.5 (see [RR04, Definition 7.4])**.** Let $\pi \in A_{n+1}$. Define $\mathrm{Del}_A(\pi)$ as

$$\mathrm{Del}_A(\pi) = \{\, 2 < j \le n+1 \mid \text{there is at most one } i < j \text{ such that } \pi(i) < \pi(j) \,\}.$$

Definition 4.6. Let $\pi \in A_{n+1}$. Define the *left-to-right almost-minima set* of π as

$$\begin{aligned}\overrightarrow{\mathrm{amin}}(\pi) &= \pi\left(\mathrm{Del}_A(\pi) \cup \{1,2\}\right) \\ &= \{\, \pi(j) \mid 1 \le j \le n+1 \text{ and there is at most one } i < j \text{ such that } \pi(i) < \pi(j) \,\}.\end{aligned}$$

Example 4.7. Let $\pi = [4,2,6,3,1,5]$. Then $\mathrm{Del}_A(\pi) = \{4,5\}$ and $\overrightarrow{\mathrm{amin}}(\pi) = \{4,2,3,1\}$.

Proposition 4.8. *For every $\pi \in A_{n+1}$, $\overrightarrow{\mathrm{amin}}(\pi) = \mathrm{Del}_A(\pi^{-1}) \cup \{1,2\}$.*

Proof. Let $k \in \overrightarrow{\mathrm{amin}}(\pi)$. Then $j = \pi^{-1}(k) \in \mathrm{Del}_A(\pi) \cup \{1,2\}$. Therefore for all $1 \le i \le n+1$ except at most one, if $\pi(i) < \pi(j) = k$ then $i > j = \pi^{-1}(k)$. By the change of variables $i' = \pi(i)$, we get that for all $1 \le i' \le n+1$ except at most one, $i' < k$ implies $\pi^{-1}(i') > \pi^{-1}(k)$, so by definition, $k \in \mathrm{Del}_A(\pi^{-1}) \cup \{1,2\}$. This proves that $\overrightarrow{\min}(\pi) \subseteq \mathrm{Del}_A(\pi^{-1}) \cup \{1,2\}$.

The reverse containment is obtained by substituting π^{-1} for π and applying π to both sides. $\square$

We now quote the following theorem. The bijection Ψ of Theorem 5.8 bellow provides a (short) bijective proof of that theorem.

Theorem 4.9 (see [RR04, Theroem 9.1(2)])**.** *For every subsets $D_1 \subseteq \{1,\dots,n-1\}$ and $D_2 \subseteq \{3,\dots,n+1\}$,*

$$\sum_{\{\sigma \in A_{n+1} \mid \mathrm{Des}_A(\sigma^{-1}) \subseteq D_1,\ \mathrm{Del}_A(\sigma^{-1}) \subseteq D_2\}} q^{\mathrm{rmaj}_{A_{n+1}}(\sigma)} = \sum_{\{\sigma \in A_{n+1} \mid \mathrm{Des}_A(\sigma^{-1}) \subseteq D_1,\ \mathrm{Del}_A(\sigma^{-1}) \subseteq D_2\}} q^{\ell_A(\sigma)}.$$

5. The bijection Ψ

Recall the notations for the reverse and the complement of $\sigma = [\sigma_1\sigma_2\ldots\sigma_n] \in S_n$, which are $\mathbf{r}(\sigma) = [\sigma_n\sigma_{n-1}\ldots\sigma_1]$ and $\mathbf{c}(\sigma) = [n+1-\sigma_1, n+1-\sigma_2, \ldots, n+1-\sigma_n]$, respectively, and the notations Φ and $\overleftarrow{\Phi} = \mathbf{r}\Phi\mathbf{r}$ for Foata's *second fundamental transformation* and the *right-to-left Foata transformation* (both described in detail in Section 3), respectively.

We shall need the following properties of Φ and $\overleftarrow{\Phi}$, see also Theorem 3.5.

Theorem 5.1. (1) *Φ is a bijection of S_n onto itself.*
(2) *For every $\sigma \in S_n$, $\mathrm{maj}_S(\sigma) = \ell_S(\Phi(\sigma))$.*
(3) (see [BW91, Example 5.3]) *For every $\sigma \in S_n$, $\overleftarrow{\min}(\sigma) = \overleftarrow{\min}(\Phi(\sigma))$, where $\overleftarrow{\min}(\sigma) = \{\,\sigma(j) \mid 1 \le j \le n,\ \forall i > j\ \ \sigma(i) > \sigma(j)\,\}$ is the set of right-to-left minima of σ.*
(4) (see [FS78, Theorem 1]) *For every $\sigma \in S_n$, $\mathrm{Des}_S(\sigma^{-1}) = \mathrm{Des}_S([\Phi(\sigma)]^{-1})$.*
(5) *By Theorem 3.9, for every $\sigma \in S_n$, $\mathrm{rmaj}_{S_n}(\sigma) = \ell_S(\overleftarrow{\Phi}(\sigma))$.*

The S- and the A-canonical presentations and the map f were discussed in Section 2. A key property of f is the way it relates between certain pairs of statistics on A_{n+1} and on S_n.

Definition 5.2 (see [RR04, Definition 5.2]). Let m_S be a statistic on the symmetric groups and m_A a statistic on the alternating groups. We say that (m_S, m_A) is an *f-pair (of statistics)* if for any n and $v \in A_{n+1}$, $m_A(v) = m_S(f(v))$.

Proposition 5.3 (see [RR04, Propositions 5.3 and 5.4]). *The following pairs are f-pairs: (ℓ_S, ℓ_A), $(\mathrm{rmaj}_{S_n}, \mathrm{rmaj}_{A_{n+1}})$, $(\mathrm{del}_S, \mathrm{del}_A)$ and $(\mathrm{Des}_A, \mathrm{Des}_S)$.*

We also have

Proposition 5.4 (see [RR05, Propositions 8.4 and 8.5]). *For every $v \in A_{n+1}$, $f(v)^{-1} = f(v^{-1})$.*

The covering map f is obviously not injective. The family of maps g_u defined next serve as local inverses of f (see Remark 5.6).

Definition 5.5. For $u \in A_{n+1}$ with A-canonical presentation $u = u_1u_2\cdots u_{n-1}$, define $g_u : R_j^S \to R_j^A$ by

$$g_u(s_js_{j-1}\cdots s_\ell) = a_ja_{j-1}\cdots a_\ell \quad \text{if} \ \ \ell \ge 2, \ \text{ and } \quad g_u(s_js_{j-1}\cdots s_1) = u_j.$$

Now extend $g_u : S_n \to A_{n+1}$ as follows: let $w \in S_n$, $w = w_1\cdots w_{n-1}$ its S-canonical presentation, then

$$g_u(w) := g_u(w_1)\cdots g_u(w_{n-1}),$$

which is clearly the A-canonical presentation of $g_u(w)$.

Remark 5.6. Let $w \in S_n$ and $u \in A_{n+1}$. Then $f(g_u(w)) = w$ if for all $1 \le j \le n-1$,

$$u_j = a_j\cdots a_2a_1^{\pm1} \iff w_j = s_j\cdots s_1,$$

where $w = w_1\cdots w_{n-1}$ and $u = u_1\cdots u_{n-1}$ are the S- and A-canonical presentations of w and u respectively.

We are now ready to define the bijection Ψ.

Definition 5.7. Define $\Psi : A_{n+1} \to A_{n+1}$ by $\Psi(v) = g_v(\overleftarrow{\Phi}(f(v)))$.

That is, the image of v under Ψ is obtained by applying $\overleftarrow{\Phi}$ to $f(v)$ in S_n, then using g_v as an "inverse" of f in order to "lift" the result back to A_{n+1}.

The following is our main theorem, which can be seen as an A_{n+1}-analogue of Theorem 5.1.

Theorem 5.8. (1) *The mapping Ψ is a bijection of A_{n+1} onto itself.*
(2) *For every $v \in A_{n+1}$, $\mathrm{rmaj}_{A_{n+1}}(v) = \ell_A(\Psi(v))$.*
(3) *For every $v \in A_{n+1}$, $\mathrm{del}_A(v) = \mathrm{del}_A(\Psi(v))$.*
(4) *For every $v \in A_{n+1}$, $\mathrm{Del}_A(v^{-1}) = \mathrm{Del}_A([\Psi(v)]^{-1})$.*
(5) *For every $v \in A_{n+1}$, $\mathrm{Des}_A(v^{-1}) = \mathrm{Des}_A([\Psi(v)]^{-1})$.*

In order to prove the theorem we need the following lemmas.

Lemma 5.9. (1) *Let $w \in S_n$, $w = w_1 \cdots w_{n-1}$ its S-canonical presentation. Then for every $1 < j \le n$, $j \in \overrightarrow{\min}(w)$ if and only if $w_{j-1} = s_{j-1}s_{j-2}\cdots s_1$.*
(2) *Let $v \in A_{n+1}$, $v = v_1 \cdots v_{n-1}$ its A-canonical presentation. Then for every $2 < j \le n+1$, $j \in \overrightarrow{\mathrm{amin}}(v)$ if and only if $v_{j-2} = a_{j-2}a_{j-3}\cdots a_1^{\pm 1}$.*

Proof. (1) By induction on n. Let $\sigma = w_1 \cdots w_{n-2} \in S_{n-1} \subseteq S_n$ and assume that the assertion is true for σ. If $w_{n-1} = 1$, then the claim is correct by the induction hypothesis. Otherwise, $w_{n-1} = s_{n-1}s_{n-2}\cdots s_\ell$ for some $1 \le \ell \le n-1$. Writing $\sigma = [b_1, \dots, b_{n-1}]$, we have that $w = \sigma w_{n-1} = [b_1, \dots, b_{\ell-1}, n, b_\ell, \dots, b_{n-1}]$. For every $1 < j \le n-1$, $j = b_k$ for some k, so j is a left-to-right minimum of w if and only if it is a left-to-right minimum of σ, which, by the induction hypothesis, is true if and only if $w_{j-1} = s_{j-1}\cdots s_1$. Finally, n is an additional left-to-right minimum of w if and only if $\ell = 1$, that is if and only if $w_{n-1} = s_{n-1}s_{n-2}\cdots s_1$.

(2) By induction on n. Let $\pi = v_1 \cdots v_{n-2} \in A_n \subseteq A_{n+1}$ and assume that the assertion is true for π. If $v_{n-1} = 1$, then the claim is correct by the induction hypothesis. Otherwise, $v_{n-1} = a_{n-1}a_{n-2}\cdots a_\ell^\epsilon$ for some $1 \le \ell \le n-1$ and $\epsilon = \pm 1$. Writing $\pi = [c_1, c_2, \dots, c_n]$, we have that

$$v = \pi v_{n-1} = \begin{cases} [c_1 c_2, \dots, c_\ell, n+1, c_{\ell+1}, \dots, c_n], & \text{if } \ell > 1 \text{ and } n-\ell \text{ is even;} \\ [c_2 c_1, \dots, c_\ell, n+1, c_{\ell+1}, \dots, c_n], & \text{if } \ell > 1 \text{ and } n-\ell \text{ is odd;} \\ [c_1, n+1, c_2, \dots, c_n], & \text{if } \ell = 1,\ n \text{ is odd and } \epsilon = 1; \\ [c_2, n+1, c_1, \dots, c_n], & \text{if } \ell = 1,\ n \text{ is even and } \epsilon = 1; \\ [n+1, c_1, c_2, \dots, c_n], & \text{if } \ell = 1,\ n \text{ is even and } \epsilon = -1; \\ [n+1, c_2, c_1, c_3, \dots, c_n], & \text{if } \ell = 1,\ n \text{ is odd and } \epsilon = -1. \end{cases}$$

For every $2 < j \le n$, $j = c_k$ for some k, so j is a left-to-right almost-minimum of v if and only if it is a left-to-right almost-minimum of π, which, by the induction hypothesis, is true if and only if $v_{j-2} = a_{j-2}\cdots a_1^{\pm 1}$. Finally, $n+1$ is an additional left-to-right almost-minimum of v if and only if $\ell = 1$, that is if and only if $v_{n-1} = a_{n-1}a_{n-2}\cdots a_1^{\pm 1}$. □

Corollary 5.10. *For every $v \in A_{n+1}$, $\overrightarrow{\mathrm{amin}}(v) = \overrightarrow{\min}(f(v)) - 1$, where $X - 1 = \{x - 1 \mid x \in X\}$.*

Lemma 5.11. *For every $w \in S_n$, $\overrightarrow{\min}(w) = \overrightarrow{\min}(\overleftarrow{\Phi}(w))$, hence $\mathrm{del}_S(w) = \mathrm{del}_S(\overleftarrow{\Phi}(w))$.*

Proof. This follows immediately from the definitions and from Theorem 5.1(3):

$$\begin{aligned} j \in \overrightarrow{\min}(w) &\iff j \in \overleftarrow{\min}(\mathbf{r}(w)) \\ &\iff j \in \overleftarrow{\min}(\Phi(\mathbf{r}(w)) \\ &\iff j \in \overrightarrow{\min}(\mathbf{r}(\Phi(\mathbf{r}(w))) = \overrightarrow{\min}(\overleftarrow{\Phi}(w)). \end{aligned}$$ □

The following is an easy corollary of Lemmas 5.9 and 5.11.

Corollary 5.12. *Let $w \in S_n$, $w = w_1 \cdots w_{n-1}$ its S-canonical presentation, and let $\sigma = \overleftarrow{\Phi}(w)$, $\sigma = \sigma_1 \cdots \sigma_{n-1}$ its S-canonical presentation. Then $\sigma_j = s_j \cdots s_1$ if and only if $w_j = s_j \cdots s_1$.*

Lemma 5.13. *Let $v \in A_{n+1}$. Then $f(\Psi(v)) = \overleftarrow{\Phi}(f(v))$.*

Proof. Let $v = v_1 \cdots v_{n-1}$ and $w = \overleftarrow{\Phi}(f(v)) = w_1 \cdots w_{n-1}$ be the A- and S-canonical presentations of v and $\overleftarrow{\Phi}(f(v))$ respectively. By definition of f and Corollary 5.12, for every $1 \le j \le n-1$, $w_j = s_j s_{j-1} \cdots s_1$ if and only if $v_j = a_j \cdots a_2 a_1^{\pm 1}$. Therefore, by Remark 5.6,

$$f(\Psi(v)) = f(g_v(\overleftarrow{\Phi}(f(v)))) = f(g_v(w)) = w = \overleftarrow{\Phi}(f(v)).$$ □

Proof of Theorem 5.8. (1) To prove that Ψ is a bijection, it suffices to find its inverse. Let $v \in A_{n+1}$, and let $v = v_1 \cdots v_{n-1}$, $w = \overleftarrow{\Phi}(f(v)) = w_1 \cdots w_{n-1}$ and $u = \Psi(v) = g_v(w) = u_1 \cdots u_{n-1}$ be the A-, S- and A-canonical presentations of v, $\overleftarrow{\Phi}(f(v))$ and $\Psi(v)$ respectively. By Lemma 5.13,

$$\overleftarrow{\Phi}^{-1}(f(\Psi(v))) = \overleftarrow{\Phi}^{-1}(\overleftarrow{\Phi}(f(v))) = f(v),$$

so

$$(*) \qquad g_{\Psi(v)}(\overleftarrow{\Phi}^{-1}(f(\Psi(v)))) = g_{\Psi(v)}(f(v)) = g_u(f(v)) = g_u(f(v_1)) \cdots g_u(f(v_{n-1})).$$

We claim that $\pi \mapsto g_\pi(\overleftarrow{\Phi}^{-1}(f(\pi)))$ is the inverse of Ψ, or in other words, that the right hand side of $(*)$ equals $v_1 v_2 \cdots v_{n-1}$. Let $1 \le j \le n-1$. If $v_j = a_j a_{j-1} \cdots a_\ell$, $\ell > 1$, then $g_u(f(v_j)) = g_u(s_j s_{j-1} \cdots s_\ell) = a_j a_{j-1} \cdots a_\ell = v_j$. If $v_j = a_j \cdots a_2 a_1^{\pm 1}$, then $f(v_j) = s_j s_{j-1} \cdots s_1$, so by Corollary 5.12, $w_j = s_j s_{j-1} \cdots s_1$, and therefore $u_j = g_v(w_j) = v_j$, so again $g_u(f(v_j)) = v_j$, and the claim is proved.

(2) By Proposition 5.3 and Lemma 5.13, $\ell_A(\Psi(v)) = \ell_S(f(\Psi(v))) = \ell_S(\overleftarrow{\Phi}(f(v)))$. By Theorem 3.9 and Proposition 5.3, $\ell_S(\overleftarrow{\Phi}(f(v))) = \mathrm{rmaj}_{S_n}(f(v)) = \mathrm{rmaj}_{A_{n+1}}(v)$. Thus $\ell_A(\Psi(v)) = \mathrm{rmaj}_{A_{n+1}}(v)$ as desired.

(3) By Proposition 5.3 and Lemma 5.13, $\mathrm{del}_A(\Psi(v)) = \mathrm{del}_S(f(\Psi(v))) = \mathrm{del}_S(\overleftarrow{\Phi}(f(v)))$, and by Lemma 5.11, the definition of del_S and Proposition 5.3, $\mathrm{del}_S(\overleftarrow{\Phi}(f(v))) = \mathrm{del}_S(f(v)) = \mathrm{del}_A(v)$. Thus $\mathrm{del}_A(\Psi(v)) = \mathrm{del}_A(v)$ as desired.

(4) By Corollary 5.10, $\overrightarrow{\mathrm{amin}}(\Psi(v)) = \overrightarrow{\min}(f(\Psi(v))) - 1$, with the notation $X - 1 = \{ x-1 \mid x \in X \}$. Therefore by Lemmas 5.13 and 5.11, $\overrightarrow{\mathrm{amin}}(\Psi(v)) = \overrightarrow{\min}(\overleftarrow{\Phi}(f(v))) - 1 = \overrightarrow{\min}(f(v)) - 1$. Again by Lemma 5.9, we get that $\overrightarrow{\mathrm{amin}}(\Psi(v)) = \overrightarrow{\mathrm{amin}}(v)$. By Proposition 4.8, this implies that $\mathrm{Del}_A([\Psi(v)]^{-1}) \cup \{1,2\} = \mathrm{Del}_A(v^{-1}) \cup \{1,2\}$, hence $\mathrm{Del}_A([\Psi(v)]^{-1}) = \mathrm{Del}_A(v^{-1})$ as desired.

(5) By Propositions 5.3 and 5.4 and Lemma 5.13,

$$\mathrm{Des}_A([\Psi(v)]^{-1}) = \mathrm{Des}_S(f([\Psi(v)]^{-1})) = \mathrm{Des}_S([f(\Psi(v))]^{-1})\,\mathrm{Des}_S([\overleftarrow{\Phi}(f(v))]^{-1}).$$

By Remark 3.6, $\overleftarrow{\Phi}(f(v))^{-1} = (\mathbf{r}\Phi\mathbf{r}f(v))^{-1} = \mathbf{c}((\Phi\mathbf{r}f(v))^{-1})$, so $\mathrm{Des}_S([\overleftarrow{\Phi}(f(v))]^{-1}) = \{1,\ldots,n-1\} \setminus \mathrm{Des}_S([\Phi\mathbf{r}f(v)]^{-1})$. By Theorem 5.1,

$$\mathrm{Des}_S([\Phi\mathbf{r}f(v)]^{-1}) = \mathrm{Des}_S([\mathbf{r}f(v)]^{-1}).$$

Hence, $\mathrm{Des}_S([\overleftarrow{\Phi}(f(v))]^{-1}) = \{1,\ldots,n-1\} \setminus \mathrm{Des}_S([\mathbf{r}f(v)]^{-1}) = \mathrm{Des}_S(\mathbf{c}([\mathbf{r}f(v)]^{-1}))$. Since $\mathbf{c}([\mathbf{r}f(v)]^{-1}) = \mathbf{c}(\mathbf{c}([f(v)]^{-1})) = f(v)^{-1}$, we get that

$$\mathrm{Des}_S([\overleftarrow{\Phi}(f(v))]^{-1}) = \mathrm{Des}_S([f(v)]^{-1}).$$

Finally, by Propositions 5.4 and 5.3, $\mathrm{Des}_S([f(v)]^{-1}) = \mathrm{Des}_S(f(v^{-1})) = \mathrm{Des}_A(v^{-1})$. $\square$

6. Example

As an example, let $v = [6,4,3,7,5,2,1] \in A_7$. We now calculate v, v^{-1}, $\Psi(v)$ and $[\Psi(v)]^{-1}$, and using the A-procedure — their A-canonical presentations. This yields the corresponding sets Del_A and Des_A, hence also the ℓ_A and the $rmaj_{A_7}$ indices, thus demonstrating Theorem 5.8 in this example. Throughout the example, when writing a canonical presentation, we will underline all factors of the form $\underline{a_j \cdots a_2 a_1^{\pm 1}}$ and $\underline{s_j \cdots s_1}$.

The A-canonical presentations of v and of v^{-1} are

$$v = \underline{v_1}\,\underline{v_2}\,v_3\,\underline{v_4}\,v_5 = (\underline{a_1})(\underline{a_2 a_1^{-1}})(a_3 a_2)(\underline{a_4 a_3 a_2 a_1})(a_5 a_4 a_3) \quad (\text{so} \quad \mathrm{del}_A(v) = 3),$$

$$v^{-1} = [7,6,3,2,5,1,4] = (\underline{a_1})(a_3 a_2)(\underline{a_4 a_3 a_2 a_1^{-1}})(\underline{a_5 a_4 a_3 a_2 a_1^{-1}}) \quad (\text{so} \quad \mathrm{del}_A(v^{-1}) = 3).$$

Thus $\mathrm{Des}_A(v) = \{1,3,4,5\}$, so $\mathrm{rmaj}_{A_7}(v) = (6-1)+(6-3)+(6-4)+(6-5) = 11$. Similarly $\mathrm{Des}_A(v^{-1}) = \{1,2,4\}$. Also, $\mathrm{Del}_A(v) = \{3,6,7\}$ and $\mathrm{Del}_A(v^{-1}) = \{3,4,6\}$.

We have

$$w = f(v) = \underline{w_1}\,\underline{w_2}\,w_3\,\underline{w_4}\,w_5 = (\underline{s_1})(\underline{s_2 s_1})(s_3 s_2)(\underline{s_4 s_3 s_2 s_1})(s_5 s_4 s_3) = [5,3,6,4,2,1].$$

Note that $\mathrm{Des}_S(w) = \mathrm{Des}_S(f(v)) = \{1,3,4,5\} = \mathrm{Des}_A(v)$, and also, $\mathrm{rmaj}_{S_6}(w) = 11 = \mathrm{rmaj}_{A_7}(v)$ and $\mathrm{del}_S(w) = 3 = \mathrm{del}_A(v)$, in accordance with Proposition 5.3.

Let us calculate $\Psi(v)$ and $[\Psi(v)]^{-1}$. Using Algorithm 3.8 we obtain $\overleftarrow{\Phi}(w)$:

$$\begin{aligned}
w'_1 &= & &\mid 1\\
w'_2 &= & &\mid 2 \mid 1\\
w'_3 &= & &\mid 4 \mid 2 \mid 1\\
w'_4 &= & &\mid 6 \mid 4 \;\; 2 \;\; 1\\
w'_5 &= & &\mid 3 \;\; 6 \mid 2 \mid 1 \mid 4\\
\overleftarrow{\Phi}(w) = w'_6 &= & &5,\ 6,\ 3,\ 2,\ 1,\ 4.
\end{aligned}$$

Note that $\ell_S(\overleftarrow{\Phi}(w)) = 11 = \mathrm{rmaj}_{S_6}(w)$, as asserted by Theorem 3.9.

The S-canonical presentation of $\overleftarrow{\Phi}(w)$, obtained by the S-procedure (see Example 2.2), is

$$u = \overleftarrow{\Phi}(w) = \underline{u_1}\,\underline{u_2}\,u_3\,\underline{u_4}\,u_5 = (\underline{s_1})(\underline{s_2 s_1})(1)(\underline{s_4 s_3 s_2 s_1})(s_5 s_4 s_3 s_2).$$

The underlined factors in the S-canonical presentation of w are the same as the underlined factors in the S-canonical presentation of $\overleftarrow{\Phi}(w)$, as asserted by Corollary 5.12. This is a result of the fact that $\overrightarrow{\min}(w) = \{1,2,3,5\} = \overrightarrow{\min}(\overleftarrow{\Phi}(w))$, which is a result of Lemma 5.11.

Now

$$\Psi(v) = g_v(u) = \underline{v_1 v_2}(1)\underline{v_4}(a_5a_4a_3a_2) = \\ (\underline{a_1})(\underline{a_2a_1^{-1}})(1)(\underline{a_4a_3a_2a_1})(a_5a_4a_3a_2) = [4,6,7,3,2,1,5],$$

so $[\Psi(v)]^{-1} = [6,5,4,1,7,2,3] = (1)(\underline{a_2a_1})(\underline{a_3a_2a_1})(\underline{a_4a_3a_2a_1^{-1}})(a_5a_4)$. It follows that

$$\mathrm{Des}_A(v^{-1}) = \{1,2,4\} = \mathrm{Des}_A([\Psi(v)]^{-1}) \quad \text{and} \quad \mathrm{Del}_A(v^{-1}) = \{3,4,6\} = \mathrm{Del}_A([\Psi(v)]^{-1}).$$

Also

$$\mathrm{del}_A(\Psi(v)) = 3 = \mathrm{del}_A(v)$$

and

$$\ell_A(\Psi(v)) = 11 = \mathrm{rmaj}_{A_7}(v).$$

7. q-analogues

7.1. The q statistics.

Definition 7.1 (see [RR05, Definition 4.1])**.** Let $\pi \in S_n$, and let $q < n$. Define the *q-length* of π, $\ell_q(\pi)$, as the number of Coxeter generators in the S-canonical presentation of π, where $s_1, \dots, s_{q-1}$ are not counted. For example, let $\pi = s_1s_2s_1s_4s_3s_6s_5s_4s_3s_2$, then $\ell_3(\pi) = 6$ while $\ell_4(\pi) = 4$. Clearly, $\ell_1 = \ell_S$.

Definition 7.2 (see [RR05, Definition 5.1])**.** Let $\pi \in S_n$. Define $\mathrm{Del}_{k+1}(\pi)$ as

$$\mathrm{Del}_{k+1}(\pi) = \{\, k+1 < j \le n \mid \#\{i < j \mid \pi(i) < \pi(j)\} \le k \,\}.$$

Definition 7.3. Let $\pi \in S_n$. Define the *left-to-right k-almost-minima set* of π as

$$\overrightarrow{\min}_{k+1}(\pi) = \pi\left(\mathrm{Del}_{k+1}(\pi) \cup \{1,2,\dots,k+1\}\right) \\ = \{\, \pi(j) \mid 1 \le j \le n,\ \#\{i < j \mid \pi(i) < \pi(j)\} \le k \,\}.$$

Proposition 7.4. *For every $\pi \in S_{n+q-1}$, $\overrightarrow{\min}_{k+1}(\pi) = \mathrm{Del}_{k+1}(\pi^{-1}) \cup \{1,2,\dots,k+1\}$.*

Proof. Let $r \in \overrightarrow{\min}_{k+1}(\pi)$. Then $j = \pi^{-1}(r) \in \mathrm{Del}_{k+1}(\pi) \cup \{1,\dots,k+1\}$. Therefore

$$\#\{1 \le i \le n+q-1 \mid \pi(i) < \pi(j) = r \text{ and } i < j = \pi^{-1}(r)\} \le k.$$

By the change of variables $i' = \pi(i)$, we get

$$\#\{1 \le i' \le n+q-1 \mid i' < r \text{ and } \pi^{-1}(i') < \pi^{-1}(r)\} \le k,$$

so by definition, $r \in \mathrm{Del}_{k+1}(\pi^{-1}) \cup \{1,\dots,k+1\}$. This proves that $\overrightarrow{\min}_{k+1}(\pi) \subseteq \mathrm{Del}_{k+1}(\pi^{-1}) \cup \{1,\dots,k+1\}$.

The reverse containment is obtained by substituting π^{-1} for π and applying π to both sides. $\square$

Definition 7.5 (see [RR05, Definition 5.8])**.** Let $\pi \in S_{n+q-1}$. Then i is a *q-descent* in π if $i \ge q$ and at least one of the following holds: a) $i \in \mathrm{Des}(\pi)$; b) $i+1 \in \mathrm{Del}_q(\pi)$.

Definition 7.6 (see [RR05, Definition 5.9]).

(1) The *q-descent set* of $\pi \in S_{n+q-1}$ is defined as
$$\mathrm{Des}_q(\pi) = \{\, i \mid i \text{ is a } q\text{-descent in } \pi \,\}.$$

(2) For $\pi \in S_{n+q-1}$ define the *q, m-reverse major index* of π by
$$\mathrm{rmaj}_{q,m}(\pi) = \sum_{i \in \mathrm{Des}_q(\pi)} (m - i),$$
where $m = n + q - 1$.

We need the notion of *dashed* patterns [RR05], and we introduce it via examples: $\sigma \in S_n$ has the dashed pattern $(1-2-4,3)$ if $\sigma = [\cdots, a, \cdots, b, \cdots, d, c, \cdots]$, and it has the dashed pattern $(2-1-4,3)$ if $\sigma = [\cdots, b, \cdots, a, \cdots, d, c, \cdots]$ for some $a < b < c < d$. Given q, denote by $Pat(q)$ the following $q!$ dashed patterns:
$$Pat(q) = \{\, (\pi_1 - \pi_2 - \cdots - \pi_q - (q+2), (q+1)) \mid \pi \in S_q \,\}.$$
For example, $Pat(2) = \{(1-2-4,3),\ (2-1-4,3)\}$. If $\sigma \in S_m$ does not have any of the dashed pattern in $Pat(q)$, then σ *avoids* $Pat(q)$. We denote by $Avoid_q(n+q-1)$ the set of permutations $\sigma \in S_{n+q-1}$ avoiding all the $q!$ dashed patterns in $Pat(q)$.

The main equidistribution theorems here are the following two theorems. The bijection Ψ_q below implies bijective proofs for these theorems.

Theorem 7.7 (see [RR05, Theorem 11.5]). *For every positive integers n and q and every subsets $B_1, B_2 \subseteq \{q, \ldots, n+q-1\}$,*
$$\sum_{\{\, \pi \in S_{n+q-1} \mid \mathrm{Des}_q(\pi^{-1}) = B_1,\ \mathrm{Del}_q(\pi^{-1}) = B_2 \,\}} t^{\ell_q(\pi)} = \sum_{\{\, \pi \in S_{n+q-1} \mid \mathrm{Des}_q(\pi^{-1}) = B_1,\ \mathrm{Del}_q(\pi^{-1}) = B_2 \,\}} t^{\mathrm{rmaj}_{q,n+q-1}(\pi)}.$$

Theorem 7.8 (see [RR05, Theorem 11.7]). *For every positive integers n and q and every subsets $B \subseteq \{q, \ldots, n+q-2\}$,*
$$\sum_{\{\, \pi^{-1} \in Avoid_q(n+q-1) \mid \mathrm{Des}_q(\pi^{-1}) = B \,\}} t^{\ell_q(\pi)} = \sum_{\{\, \pi^{-1} \in Avoid_q(n+q-1) \mid \mathrm{Des}_q(\pi^{-1}) = B \,\}} t^{\mathrm{rmaj}_{q,n+q-1}(\pi)}.$$

7.2. The covering map f_q.

Definition 7.9 (see [RR05, Definition 8.1]). Let $w \in S_{n+q-1}$ and let $w = s_{i_1} \cdots s_{i_r}$ be its S-canonical presentation. Define $f_q : S_{n+q-1} \to S_n$ as follows:
$$f_q(w) = f_q(s_{i_1}) \cdots f_q(s_{i_r}),$$
where $f_q(s_1) = \cdots = f_q(s_{q-1}) = 1$, and $f_q(s_j) = s_{j-q+1}$ if $j \geq q$.

Remark 7.10. If $w = w_1 \cdots w_{n+q-2}$ is the S-canonical presentation of $w \in S_{n+q-1}$, $w_j \in R_j^S$, then $f_q(w) = f_q(w_q) \cdots f_q(w_{n+q-2})$ is the S-canonical presentation of $f_q(w)$, $f_q(w_j) \in R_{j-q+1}^S$.

Proposition 7.11 (see [RR05, Proposition 8.6 and Remark 11.1]). *For every $\pi \in S_{n+q-1}$, $\mathrm{Del}_q(\pi) - q + 1 = \mathrm{Del}_S(f_q(\pi))$, $\mathrm{Des}_q(\pi) - q + 1 = \mathrm{Des}_S(f_q(\pi))$, $\ell_q(\pi) = \ell_S(f_q(\pi))$, and $\mathrm{rmaj}_{q,n+q-1}(\pi) = \mathrm{rmaj}_{S_n}(f_q(\pi))$. Here, $X - r = \{\, x - r \mid x \in X \,\}$.*

Proposition 7.12 (see [RR05, Proposition 8.4]). *For any permutation w, $f_q(w)^{-1} = f_q(w^{-1})$.*

The map f_q is obviously not injective for $q > 1$. The family of maps $g_{q,u}$ defined next serve as local inverses of f_q (see Remark 7.14).

Definition 7.13. For $u \in S_{n+q-1}$ with S-canonical presentation $u = u_1 \cdots u_{n+q-2}$, define $g_{q,u} : R^S_j \to R^S_{j+q-1}$ by

$$g_{q,u}(s_j s_{j-1} \cdots s_\ell) = s_{j+q-1} s_{j+q-2} \cdots s_{\ell+q-1}, \quad g_u(s_j s_{j-1} \cdots s_1) = u_{j+q-1}.$$

Now extend $g_{q,u} : S_n \to S_{n+q-1}$ as follows: let $w \in S_n$, $w = w_1 \cdots w_{n-1}$ its S-canonical presentation, then

$$g_{q,u}(w) := u_1 \cdots u_{q-1} \cdot g_{q,u}(w_1) \cdots g_{q,u}(w_{n-1}),$$

which is clearly the S-canonical presentation of $g_{q,u}(w)$.

Remark 7.14. Let $w \in S_n$ and $u \in S_{n+q-1}$. Then $f_q(g_{q,u}(w)) = w$ if for all $1 \le j \le n-1$,

$$w_j = s_j \cdots s_1 \implies u_{j+q-1} = s_{j+q-1} \cdots s_\ell,\ \ell \le q,$$

where $w = w_1 \cdots w_{n-1}$ and $u = u_1 \cdots u_{n+q-2}$ are the S-canonical presentations of w and u respectively.

7.3. **The map Ψ_q.**

Definition 7.15. Define $\Psi_q : S_{n+q-1} \to S_{n+q-1}$ by $\Psi_q(v) = g_{q,v}(\overleftarrow{\Phi}(f_q(v)))$.

That is, the image of v under Ψ_q is obtained by applying $\overleftarrow{\Phi}$ to $f_q(v)$ in S_n, then using $g_{q,v}$ as an "inverse" of f_q in order to "lift" the result back to S_{n+q-1}.

Theorem 7.16. (1) *The mapping Ψ_q is a bijection of S_{n+q-1} onto itself.*
(2) *For every $v \in S_{n+q-1}$, $\mathrm{rmaj}_{q,n+q-1}(v) = \ell_q(\Psi_q(v))$.*
(3) *For every $v \in S_{n+q-1}$, $\mathrm{Del}_q(v^{-1}) = \mathrm{Del}_q(\Psi_q(v)^{-1})$.*
(4) *For every $v \in S_{n+q-1}$, $\mathrm{Des}_q(v^{-1}) = \mathrm{Des}_q(\Psi_q(v)^{-1})$.*

The proof is given below.

Lemma 7.17. *Let $v \in S_{n+q-1}$, $v = v_1 \cdots v_{n+q-2}$ its S-canonical presentation. Then for every $q < j \le n+q-1$, $j \in \overrightarrow{\min}_q(v)$ if and only if $v_{j-1} = s_{j-1}s_{j-2} \cdots s_\ell$ for some $\ell \le q$.*

Proof. By induction on n. Let $\pi = v_1 \cdots v_{n-1+q-2} \in S_{n+q-2} \subseteq S_{n+q-1}$ and assume that the assertion is true for π. If $v_{n+q-2} = 1$, then the claim is correct by the induction hypothesis. Otherwise, $v_{n+q-2} = s_{n+q-2}s_{n+q-3} \cdots s_\ell$ for some $1 \le \ell \le n+q-2$. Writing $\pi = [b_1, \ldots, b_{n+q-2}]$, we have that $v = \pi v_{n+q-2} = [b_1, \ldots, b_{\ell-1}, n+q-1, b_\ell, \ldots, b_{n+q-2}]$, so clearly for every $1 \le k \le n+q-2$, the set of numbers smaller than b_k and to its left in π is equal to the set of numbers smaller than b_k and to its left in v. Thus $b_k \in \overrightarrow{\min}_q(v)$ if and only if $b_k \in \overrightarrow{\min}_q(\pi)$, which, by the induction hypothesis, is true if and only if $v_{b_k-1} = s_{b_k-1} \cdots s_r$ for some $r \le q$. Finally, $n+q-1 \in \overrightarrow{\min}_q(v)$ if and only if $n+q-1$ occupies one of the q leftmost places in v, that is, if and only if $\ell \le q$. □

Lemma 7.18. *Let $v \in S_{n+q-1}$. Then $f_q(\Psi_q(v)) = \overleftarrow{\Phi}(f_q(v))$.*

Proof. Let $v = v_1 \cdots v_{n+q-2}$ and $w = \overleftarrow{\Phi}(f_q(v)) = w_1 \cdots w_{n-1}$ be the S-canonical presentations of v and $\overleftarrow{\Phi}(f_q(v))$ respectively. By definition of f_q and Corollary 5.12, for every $1 \le j \le n-1$, $w_j = s_j s_{j-1} \cdots s_1$ if and only if $v_{j+q-1} = s_{j+q-1} \cdots s_\ell$, $\ell \le q$. Therefore, by Remark 7.14,

$$f_q(\Psi_q(v)) = f_q(g_{q,v}(\overleftarrow{\Phi}(f_q(v)))) = f_q(g_{q,v}(w)) = w = \overleftarrow{\Phi}(f_q(v)).$$ □

Proof of Theorem 7.16. (1) To prove that Ψ_q is a bijection, it suffices to find its inverse. Let $v \in S_{n+q-1}$, and let $v = v_1 \cdots v_{n+q-2}$, $w = \overleftarrow{\Phi}(f_q(v)) = w_1 \cdots w_{n-1}$ and $u = \Psi_q(v) = g_{q,v}(w) = v_1 \cdots v_{q-1} u_q \cdots u_{n+q-2}$ be the S-canonical presentations of v, $\overleftarrow{\Phi}(f_q(v))$ and $\Psi_q(v)$ respectively. By Lemma 7.18,

$$\overleftarrow{\Phi}^{-1}(f_q(\Psi_q(v))) = \overleftarrow{\Phi}^{-1}(\overleftarrow{\Phi}(f_q(v))) = f_q(v),$$

so

$$(*) \quad g_{q,\Psi_q(v)}(\overleftarrow{\Phi}^{-1}(f_q(\Psi_q(v)))) = g_{q,\Psi_q(v)}(f_q(v)) \\ = g_{q,u}(f_q(v)) = v_1 \cdots v_{q-1} \cdot g_{q,u}(f_q(v_1)) \cdots g_{q,u}(f_q(v_{n-1})).$$

We claim that $\pi \mapsto g_{q,\pi}(\overleftarrow{\Phi}^{-1}(f_q(\pi)))$ is the inverse of Ψ_q, or in other words, that the right hand side of $(*)$ equals $v_1 v_2 \cdots v_{n+q-2}$. Let $q \le j \le n+q-2$, and write $v_j = s_j s_{j-1} \cdots s_\ell$. If $\ell > q$, then $g_{q,u}(f_q(v_j)) = g_{q,u}(s_{j-q+1} \cdots s_{\ell-q+1}) = s_j \cdots s_\ell = v_j$. If $\ell \le q$, then $f_q(v_j) = s_j \cdots s_1$, so by Corollary 5.12, $w_j = s_j \cdots s_1$, and therefore $u_j = g_{q,v}(w_j) = v_j$, so again $g_{q,u}(f_q(v_j)) = v_j$, and the claim is proved.

(2) By Proposition 7.11 and Lemma 7.18, $\ell_q(\Psi_q(v)) = \ell_S(f_q(\Psi_q(v))) = \ell_S(\overleftarrow{\Phi}(f_q(v)))$. By Theorem 3.9 and Proposition 7.11, $\ell_S(\overleftarrow{\Phi}(f_q(v))) = \mathrm{rmaj}_{S_n}(f_q(v)) = \mathrm{rmaj}_{q,n+q-1}(v)$. Thus $\ell_q(\Psi_q(v)) = \mathrm{rmaj}_{q,n+q-1}(v)$ as desired.

(3) By Lemma 7.17 and the definition of f_q, $\overrightarrow{\min}_q(\Psi_q(v)) = \overrightarrow{\min}(f_q(\Psi_q(v))) - q + 1$ (with the notation $X - r = \{x - r \mid x \in X\}$). Therefore by Lemmas 7.18 and 5.11, $\overrightarrow{\min}_q(\Psi_q(v)) = \overrightarrow{\min}(\overleftarrow{\Phi}(f_q(v))) - q + 1 = \overrightarrow{\min}(f_q(v)) - q + 1$. Again by Lemma 7.17, we get that $\overrightarrow{\min}_q(\Psi_q(v)) = \overrightarrow{\min}_q(v)$. By Proposition 7.4, this implies that $\mathrm{Del}_q([\Psi_q(v)]^{-1}) \cup \{1, \dots, q\} = \mathrm{Del}_q(v^{-1}) \cup \{1, \dots, q\}$, hence $\mathrm{Del}_q([\Psi_q(v)]^{-1}) = \mathrm{Del}_q(v^{-1})$ as desired.

(4) By Propositions 7.11 and 7.12 and Lemma 7.18,

$$\begin{aligned} \mathrm{Des}_q([\Psi_q(v)]^{-1}) - q + 1 &= \mathrm{Des}_S(f_q([\Psi_q(v)]^{-1})) \\ &= \mathrm{Des}_S([f_q(\Psi_q(v))]^{-1}) \\ &= \mathrm{Des}_S([\overleftarrow{\Phi}(f_q(v))]^{-1}). \end{aligned}$$

By Remark 3.6, $[\overleftarrow{\Phi}(f_q(v))]^{-1} = [\mathbf{r}\Phi\mathbf{r} f_q(v)]^{-1} = \mathbf{c}([\Phi\mathbf{r} f_q(v)]^{-1})$, so

$$\mathrm{Des}_S([\overleftarrow{\Phi}(f_q(v))]^{-1}) = \{1, \dots, n-1\} \setminus \mathrm{Des}_S([\Phi\mathbf{r} f_q(v)]^{-1}).$$

By Theorem 5.1,

$$\mathrm{Des}_S([\Phi\mathbf{r} f_q(v)]^{-1}) = \mathrm{Des}_S([\mathbf{r} f_q(v)]^{-1}).$$

Hence, $\mathrm{Des}_S([\overleftarrow{\Phi}(f_q(v))]^{-1}) = \{1, \dots, n-1\} \setminus \mathrm{Des}_S([\mathbf{r} f_q(v)]^{-1}) = \mathrm{Des}_S(\mathbf{c}([\mathbf{r} f_q(v)]^{-1}))$. Since $\mathbf{c}([\mathbf{r} f_q(v)]^{-1}) = \mathbf{c}(\mathbf{c}([f_q(v)]^{-1})) = [f_q(v)]^{-1}$, we get that

$$\mathrm{Des}_S([\overleftarrow{\Phi}(f_q(v))]^{-1}) = \mathrm{Des}_S([f_q(v)]^{-1}).$$

Finally, by Propositions 7.12 and 7.11,

$$\mathrm{Des}_S([f_q(v)]^{-1}) = \mathrm{Des}_S(f_q(v^{-1})) = \mathrm{Des}_q(v^{-1}) - q + 1.$$

□

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