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ON THE EXCEDANCE NUMBER OF COLORED PERMUTATION GROUPS

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Abstract. We generalize the results of Ksavrelof and Zeng about the multidistribution of the excedance number of S_n with some natural parameters to the colored permutation group and to the Coxeter group of type D. We define two different orders on these groups which induce two different excedance numbers. Surprisingly, in the case of the colored permutation group, we get the same generalized formulas for both orders.

1. Introduction

Let r and n be two positive integers. The *colored permutation group* $G_{r,n}$ consists of all permutations of the set

$$\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$$

satisfying $\pi(\bar{i}) = \overline{\pi(i)}$.

The symmetric group S_n is a special case of $G_{r,n}$ for $r = 1$. In S_n one can define the following well-known parameters: given $\sigma \in S_n$, $i \in [n]$ is *an excedance of* σ if and only if $\sigma(i) > i$. The number of excedances is denoted by $\mathrm{exc}(\sigma)$. Two other natural parameters on S_n are the number of fixed points and the number of cycles of σ, denoted by $\mathrm{fix}(\sigma)$ and $\mathrm{cyc}(\sigma)$ respectively.

Consider the following generating function over S_n:

$$P_n(q,t,s) = \sum_{\sigma \in S_n} q^{\mathrm{exc}(\sigma)} t^{\mathrm{fix}(\sigma)} s^{\mathrm{cyc}(\sigma)}.$$

$P_n(q,1,1)$ is the classical Eulerian polynomial, while $P_n(q,0,1)$ is the counter part for the derangements, i.e., the permutations without fixed points, see [2].

In the case $s = -1$, the two polynomials $P_n(q,1,-1)$ and $P_n(q,0,-1)$ have simple closed formulas:

$$P_n(q,1,-1) = -(q-1)^{n-1} \tag{1}$$

$$P_n(q,0,-1) = -q[n-1]_q \tag{2}$$

Recently, Ksavrelof and Zeng [1] proved some new recursive formulas which induce Equations (1) and (2).

A natural problem is to generalize the results of [1] to the colored permutation groups. The main challenge here is to choose a suitable order on the alphabet Σ of the group $G_{r,n}$ and define the parameters properly.

In this paper we cope with this challenge. We define two different orders on Σ, one of them 'forgets' the colors, while the other is much more natural, since it takes into account the color structure of $G_{r,n}$. The parameter exc will be defined according to both orders in two different ways. The interesting point is that for the group $G_{r,n}$ we get the same recursive formulas for both cases.

Define
$$P_{G_{r,n}}(q,t,s) = \sum_{\pi \in G_{r,n}} q^{\mathrm{exc}(\pi)} t^{\mathrm{fix}(\pi)} s^{\mathrm{cyc}(\pi)}.$$

Concerning $G_{r,n}$, we prove the following two main results:

Theorem 1.1.
$$P^{\mathrm{Abs}}_{G_{r,n}}(q,1,-1) = P^{\mathrm{Clr}}_{G_{r,n}}(q,1,-1) = (q^r-1)P_{G_{r,n-1}}(q,1,-1).$$
Hence,
$$P^{\mathrm{Abs}}_{G_{r,n}}(q,1,-1) = P^{\mathrm{Clr}}_{G_{r,n}}(q,1,-1) = -\frac{(q^r-1)^n}{q-1}.$$

Theorem 1.2.
$$P^{\mathrm{Abs}}_{G_{r,n}}(q,0,-1) = P^{\mathrm{Clr}}_{G_{r,n}}(q,0,-1) = [r]_q(P_{G_{r,n-1}}(q,0,-1) - q^{n-1}[r]_q^{n-1}).$$
Hence,
$$P^{\mathrm{Abs}}_{G_{r,n}}(q,0,-1) = P^{\mathrm{Clr}}_{G_{r,n}}(q,0,-1) = -q[r]_q^n[n-1]_q.$$

Even though $P_{G_{r,n}}(q,0,-1)$ is a special case of $P_{G_{r,n}}(q,s,t)$, we can deduce a formula for the exponential generating function of the general case as presented in the following corollary:

Corollary 1.3.
$$\sum_{n\geq 0} P_{G_{r,n}}(q,s,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0}\left(\frac{[r]_q^n}{1-q}((q-s)^n - q(1-s)^n)\right)\frac{x^n}{n!}\right)^{-t}.$$

One can easily check that the formulas appeared in Theorem 1.1, Theorem 1.2 and Corollary 1.3 indeed generalize the formulas of Ksavrelof and Zeng (for $r=1$).

As noted above, in the symmetric group case, the polynomials $P_n(q,1,1)$ are equal to the Eulerian polynomials, which are usually defined using descent numbers. It is possible to define descent numbers for the colored permutation groups, which are equidistributed with our parameters. A work in this direction is in progress.

We apply our techniques also to obtain permutations statistics on the group of even signed permutations, D_n, also known as the *Coxeter group of type D*. We get the following results:

Theorem 1.4.
$$P^{\mathrm{Clr}}_{D_n}(q,1,-1) = (q^2-1)P^{\mathrm{Clr}}_{D_{n-1}}(q,1,-1).$$
Hence,
$$P^{\mathrm{Clr}}_{D_n}(q,1,-1) = (1-q^2)^{n-1}.$$

Theorem 1.5.
$$P^{\mathrm{Abs}}_{D_n}(q,1,-1) = -\frac{1}{2}(q-1)^{n-1}((1+q)^n+(1-q)^n).$$

This paper is organized as follows. In Section 2, we recall some properties of $G_{r,n}$. In Section 3 we define the new statistics on $G_{r,n}$. Sections 4 and 5 deal with the proofs of Theorems 1.1 and 1.2 respectively. Section 6 deals with the proof of Corollary 1.3. Section 7 includes the proofs of Theorems 1.4 and 1.5.

2. Preliminaries

2.1. **Notations.** For $n \in \mathbb{N}$, let $[n] := \{1,2,\dots,n\}$ (where $[0] := \emptyset$). Also, let
$$[n]_q := \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}$$
(so $[0]_q = 0$), and
$$[n]_q! := [n]_q \cdot [n-1]_q \cdots [1]_q.$$

2.2. **The group of colored permutations.**

Definition 2.1. Let r and n be positive integers. *The group of colored permutations of n digits with r colors* is the wreath product
$$G_{r,n} = \mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n,$$
consisting of all the pairs (z,τ) where z is an n-tuple of integers between 0 and $r-1$ and $\tau \in S_n$. The multiplication is defined by the following rule: for $z = (z_1,\dots,z_n)$ and $z' = (z'_1,\dots,z'_n)$
$$(z,\tau)\cdot(z',\tau') = ((z_1+z'_{\tau(1)},\dots,z_n+z'_{\tau(n)}),\tau\circ\tau')$$
(here $+$ is taken modulo r).

We use some conventions along this paper. For an element $\pi = (z, \tau) \in G_{r,n}$ with $z = (z_1, ..., z_n)$ we write $z_i(\pi) = z_i$. For $\pi = (z, \tau)$, we denote $|\pi| = (0, \tau), (0 \in \mathbb{Z}_r^n)$. An element

$$(z, \tau) = ((1, 0, 3, 2), (2, 1, 4, 3)) \in G_{3,4}$$

will be written as $(\bar{2}1\bar{\bar{\bar{4}}}\bar{\bar{3}})$.

A much more natural way to present $G_{r,n}$ is the following. Consider the alphabet $\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$ as the set $[n]$ colored by the colors $0, \dots, r-1$. Then, an element of $G_{r,n}$ is a *colored permutation*, i.e., a bijection $\pi : \Sigma \to \Sigma$ such that $\pi(\bar{i}) = \overline{\pi(i)}$.

In particular, $G_{1,n} = C_1 \wr S_n$ is the symmetric group S_n, while $G_{2,n} = C_2 \wr S_n$ is the group of signed permutations B_n, also known as the *hyperoctahedral group*, or the *classical Coxeter group of type B*. We also define here the following normal subgroup of B_n of index 2, called the *even signed permutation group* or the *Coxeter group of type D*:

$$D_n = \{\pi \in B_n \mid \sum_{i=1}^{n} z_i(\pi) \equiv 0 \pmod 2\}.$$

3. Statistics on $G_{r,n}$

Given any ordered alphabet Σ', we recall the definition of the *excedance set* of a permutation π on Σ':

$$\mathrm{Exc}(\pi) = \{i \in \Sigma' \mid \pi(i) > i\}$$

and the *excedance number* is defined to be $\mathrm{exc}(\pi) = |\mathrm{Exc}(\pi)|$.

We start by defining two orders on the set

$$\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}.$$

Definition 3.1. The *absolute order* on Σ is defined to be

$$1^{[r-1]} < \cdots < \bar{1} < 1 < 2^{[r-1]} < \cdots < \bar{2} < 2 < \cdots < n^{[r-1]} < \cdots < \bar{n} < n.$$

The *color order* on Σ is defined to be

$$1^{[r-1]} < \cdots < n^{[r-1]} < 1^{[r-2]} < 2^{[r-2]} < \cdots < n^{[r-2]} < \cdots < 1 < \cdots < n.$$

Example 3.2. *Given the color order*

$$\bar{\bar{1}} < \bar{\bar{2}} < \bar{\bar{3}} < \bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3,$$

we write $\sigma = (3\bar{1}\bar{\bar{2}}) \in G_{3,3}$ *in an extended form,*

$$\begin{pmatrix} \bar{\bar{1}} & \bar{\bar{2}} & \bar{\bar{3}} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 \\ \bar{\bar{3}} & 1 & \bar{2} & \bar{3} & \bar{\bar{1}} & 2 & 3 & \bar{1} & \bar{\bar{2}} \end{pmatrix}$$

and calculate $\mathrm{Exc}(\sigma) = \{\bar{\bar{1}}, \bar{\bar{2}}, \bar{\bar{3}}, \bar{1}, \bar{3}, 1\}$ *and* $\mathrm{exc}(\sigma) = 6$.

Before defining the excedance numbers with respect to both orders, we have to introduce some notions.

Let $\sigma \in G_{r,n}$. We define

$$\mathrm{csum}(\sigma) = \sum_{i=1}^{n} z_i(\sigma),$$
$$\mathrm{Exc}_A(\sigma) = \{i \in [n-1] \mid \sigma(i) > i\},$$

where the comparison is with respect to the color order, and

$$\mathrm{exc}_A(\sigma) = |\mathrm{Exc}_A(\sigma)|.$$

Example 3.3. *Take* $\sigma = (\bar{1}\bar{\bar{3}}4\bar{2}) \in G_{3,4}$. *Then* $\mathrm{csum}(\sigma) = 4$, $\mathrm{Exc_A}(\sigma) = \{3\}$ *and hence* $\mathrm{exc}_A(\sigma) = 1$.

Let $\sigma \in G_{r,n}$. Recall that for $\sigma = (z, \tau) \in G_{r,n}$, $|\sigma|$ is the permutation of $[n]$ satisfying $|\sigma|(i) = \tau(i)$. For example, if $\sigma = (\bar{2}\bar{\bar{3}}1\bar{4})$ then $|\sigma| = (2314)$.

Now we can define the excedance numbers for $G_{r,n}$.

Definition 3.4. Define

$$\mathrm{exc}^{\mathrm{Abs}}(\sigma) = \mathrm{exc}(|\sigma|) + \mathrm{csum}(\sigma),$$
$$\mathrm{exc}^{\mathrm{Clr}}(\sigma) = r \cdot \mathrm{exc}_A(\sigma) + \mathrm{csum}(\sigma).$$

Note that the computation of the parameter $\mathrm{exc}^{\mathrm{Abs}}$ uses the absolute order.

The parameters $\mathrm{exc}^{\mathrm{Abs}}$ and $\mathrm{exc}^{\mathrm{Clr}}$ are indeed different: for $\sigma = (21) \in G_{r,2}$, $(r > 1)$ one has $\mathrm{exc}^{\mathrm{Abs}}(\sigma) = 1$ but $\mathrm{exc}^{\mathrm{Clr}}(\sigma) = r$.

One can view $\mathrm{exc}^{\mathrm{Clr}}(\sigma)$ in a different way:

Lemma 3.5. *Let* $\sigma \in G_{r,n}$. *Consider the set* Σ *ordered by the color order. Then*

$$\mathrm{exc}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma).$$

Proof. Let $i \in [n]$. We divide our proof into two cases: $z_i(\sigma) = 0$ and $z_i(\sigma) \neq 0$.

If $z_i(\sigma) = 0$, then $i \in \mathrm{Exc}_A(\sigma)$ if and only if $\sigma(i) > i$. In this case, we have $\sigma(i^{[j]}) > i^{[j]}$ for every color $1 \leq j \leq r-1$. Hence, we have $\{i, i^{[1]}, \cdots, i^{[r-1]}\} \subseteq \mathrm{Exc}(\sigma)$. Hence, each $i \in \mathrm{Exc}_A(\sigma)$ contributes r excedances to $\mathrm{exc}(\sigma)$.

On the other hand, if $z_i(\sigma) = k \neq 0$, we have that $i \notin \mathrm{Exc}(\sigma)$. By definition, we have $\sigma(i^{[j]}) = |\sigma(i)|^{[(j+k) \pmod r)]}$ for all j. Thus, for $0 \leq j \leq r-k-1$, $i^{[j]} \notin \mathrm{Exc}(\sigma)$, and for the k indices $r-k \leq j \leq r-1$, $i^{[j]} \in \mathrm{Exc}(\sigma)$. Consequently, we have

$$\mathrm{exc}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma).$$

□

Recall that any permutation of S_n can be decomposed into a product of disjoint cycles. This notion can be easily generalized to the group $G_{r,n}$ as follows. Given any $\pi \in G_{r,n}$ we define the *cycle number* of $\pi = (z, \tau)$ to be the number of cycles in τ.

We say that $i \in [n]$ is an *absolute fixed point* of $\sigma \in G_{r,n}$ if $|\sigma(i)| = i$.

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. The idea of proving this type of identities is constructing a subset S of $G_{r,n}$ whose contribution to the generating function is exactly the right side of the identity and a killing involution on $G_{r,n} - S$, i.e., an involution on $G_{r,n} - S$ which preserves the number of excedances but changes the sign of every element of $G_{r,n} - S$ and hence shows that $G_{r,n} - S$ contributes nothing to the generating function.

4.1. **Proof for the absolute order.** We divide $G_{r,n}$ into $2r+1$ disjoint subsets as follows:

$$K_{r,n} = \{\sigma \in G_{r,n} \mid |\sigma(n)| \neq n, |\sigma(n-1)| \neq n\},$$
$$T^i_{r,n} = \{\sigma \in G_{r,n} \mid \sigma(n) = n^{[i]}\}, \qquad (0 \leq i \leq r-1),$$
$$R^i_{r,n} = \{\sigma \in G_{r,n} \mid \sigma(n-1) = n^{[i]}\}, \qquad (0 \leq i \leq r-1).$$

We first construct a killing involution on the set $K_{r,n}$. Let $\sigma \in K_{r,n}$. Define $\varphi : K_{r,n} \to K_{r,n}$ by

$$\sigma' = \varphi(\sigma) = (\sigma(n-1), \sigma(n))\sigma.$$

Note that φ exchanges $\sigma(n-1)$ with $\sigma(n)$. It is obvious that φ is indeed an involution.

We will show that $\mathrm{exc}^{\mathrm{Abs}}(\sigma) = \mathrm{exc}^{\mathrm{Abs}}(\sigma')$. First, for $i < n-1$, it is clear that $i \in \mathrm{Exc}(|\sigma|)$ if and only if $i \in \mathrm{Exc}(|\sigma'|)$. Now, as $\sigma(n-1) \neq n$, $n-1 \notin \mathrm{Exc}(|\sigma|)$ and thus $n \notin \mathrm{Exc}(|\sigma'|)$. Finally, $|\sigma(n)| \neq n$ implies that $n-1 \notin \mathrm{Exc}(|\sigma'|)$ and thus $\mathrm{exc}^{\mathrm{Abs}}(\sigma) = \mathrm{exc}^{\mathrm{Abs}}(\sigma')$.

On the other hand, $\mathrm{cyc}(\sigma)$ and $\mathrm{cyc}(\sigma')$ have different parities due to a multiplication by a transposition. Hence, φ is indeed a killing involution on $K_{r,n}$.

We turn now to the sets $T^i_{r,n}$ $(0 \leq i \leq r-1)$. Note that there is a natural bijection between $T^i_{r,n}$ and $G_{r,n-1}$ defined by ignoring the last digit. Let $\sigma \in T^i_{r,n}$. Denote the image of $\sigma \in T^i_{r,n}$ under this bijection by σ'. Since $n \notin \mathrm{Exc}(|\sigma|)$, we have $\mathrm{exc}(|\sigma|) = \mathrm{exc}(|\sigma'|)$. Now, $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - \mathrm{i}$, since $z_n(\sigma) = i$ and hence we have

$$\mathrm{exc}^{\mathrm{Abs}}(\sigma) - \mathrm{i} = \mathrm{exc}^{\mathrm{Abs}}(\sigma').$$

Finally, since n is an absolute fixed point of σ, $\mathrm{cyc}(\sigma') = \mathrm{cyc}(\sigma) - 1$ and we get that the total contribution of $T^i_{r,n}$ is

$$P^{\mathrm{Abs}}_{T^i_{r,n}} = -q^i P^{\mathrm{Abs}}_{G_{r,n-1}}(q, 1, -1)$$

for $0 \leq i \leq r-1$.

Now, we treat the sets $R^i_{r,n} \quad (0 \leq i \leq r-1)$. There is a bijection between $R^i_{r,n}$ and $T^i_{r,n}$ using the same function φ we used above. Let $\sigma \in R^i_{r,n}$. Define $\varphi : R^i_{r,n} \to T^i_{r,n}$ by

$$\sigma' = \varphi(\sigma) = (\sigma(n-1), \sigma(n))\sigma.$$

In σ, we have that $n-1 \in \mathrm{Exc}(|\sigma|)$ (since $|\sigma(n-1)| = n$) and $n \notin \mathrm{Exc}(|\sigma|)$, but in σ', $n-1, n \notin \mathrm{Exc}(|\sigma'|)$. Hence, $\mathrm{exc}(|\sigma|) - 1 = \mathrm{exc}(|\sigma'|)$. We also have that $\mathrm{csum}(\sigma) = \mathrm{csum}(\sigma')$ (since $z_{n-1}(\sigma) + z_n(\sigma) = z_{n-1}(\sigma') + z_n(\sigma')$). Hence, we have that

$$\mathrm{exc}^{\mathrm{Abs}}(\sigma) - 1 = \mathrm{exc}^{\mathrm{Abs}}(\sigma').$$

As before, the number of cycles changes its parity due to the multiplication by a transposition, and thus $(-1)^{\mathrm{cyc}(\sigma)} = -(-1)^{\mathrm{cyc}(\sigma')}$.

Hence, the total contribution of the elements in $R^i_{r,n}$ is

$$P^{\mathrm{Abs}}_{R^i_{r,n}} = q^{i+1} \cdot P^{\mathrm{Abs}}_{G_{r,n-1}}(q, 1, -1)$$

for $0 \leq i \leq r-1$.

Now, if we sum up all the parts, we get

$$\begin{aligned}
P^{\mathrm{Abs}}_{G_{r,n}}(q,1,-1) &= P^{\mathrm{Abs}}_{K_{r,n}}(q,1,-1) \\
&\qquad + \sum_{i=0}^{r-1} P^{\mathrm{Abs}}_{T^i_{r,n}}(q,1,-1) + \sum_{i=0}^{r-1} P_{R^i_{r,n}}(q,1,-1) \\
&= \sum_{i=0}^{r-1}(-q^i P^{\mathrm{Abs}}_{G_{r,n-1}}(q,1,-1)) + \sum_{i=0}^{r-1} q^{i+1} P^{\mathrm{Abs}}_{G_{r,n-1}}(q,1,-1) \\
&= (q^r - 1) P^{\mathrm{Abs}}_{G_{r,n-1}}(q,1,-1)
\end{aligned}$$

as claimed.

Now, for $n = 1$, $G_{r,1}$ is the cyclic group of order r and thus

$$P^{\mathrm{Abs}}_{G_{r,1}}(q,1,-1) = -(1 + q + \cdots + q^{r-1}) = -\frac{q^r - 1}{q - 1},$$

so we have

$$P^{\mathrm{Abs}}_{G_{r,n}}(q,1,-1) = -\frac{(q^r-1)^n}{q-1}.$$

4.2. **Proof for the color order.** As in the previous proof, we divide $G_{r,n}$ into the same $2r+1$ disjoint subsets $K_{r,n}$, $T^i_{r,n}\quad(0 \le i \le r-1)$ and $R^i_{r,n}\quad(0 \le i \le r-1)$ used there.

As before, we first construct a killing involution on the set $K_{r,n}$. Let $\sigma \in K_{r,n}$. As before, define $\varphi : K_{r,n} \to K_{r,n}$ by

$$\sigma' = \varphi(\sigma) = (\sigma(n-1), \sigma(n))\sigma.$$

The proof that φ is a killing involution is similar to the one we presented in Section 4.1.

We turn now to the sets $T^i_{r,n}$. We use again the bijection between $T^i_{r,n}$ and $G_{r,n-1}$ defined by ignoring the last digit. Let $\sigma \in T_{r,n}$. As in the previous proof, we have

$$\mathrm{exc}^{\mathrm{Clr}}(\sigma) - \mathrm{i} = \mathrm{exc}^{\mathrm{Clr}}(\sigma').$$

Now, since n is an absolute fixed point of σ, $\mathrm{cyc}(\sigma') = \mathrm{cyc}(\sigma) - 1$.

To summarize, we get that the total contribution of elements in $T^i_{r,n}$ is

$$P^{\mathrm{Clr}}_{T^i_{r,n}} = -q^i P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1)$$

for $0 \le i \le r-1$.

Finally, we treat the sets $R^i_{r,n}$. Let $\sigma \in R^i_{r,n}$. Recall the bijection $\varphi : R^i_{r,n} \to T^i_{r,n}$ defined in Section 4.1 by

$$\sigma' = \varphi(\sigma) = (\sigma(n-1), \sigma(n))\sigma.$$

When we compute the change in the excedance, we split our treatment into two cases: $i = 0$ and $i > 0$. For the case $i = 0$, we get $\mathrm{exc}^{\mathrm{Clr}}(\sigma) - \mathrm{r} = \mathrm{exc}^{\mathrm{Clr}}(\sigma')$. For the case $i > 0$, we show that $\mathrm{exc}^{\mathrm{Clr}}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma')$.

We start with case $i = 0$. Note that $n-1 \in \mathrm{Exc_A}(\sigma)$ (since $\sigma(n-1) = n$) and $n \notin \mathrm{Exc_A}(\sigma)$. On the other hand, in σ', $n-1, n \notin \mathrm{Exc_A}(\sigma')$. Hence, $\mathrm{exc_A}(\sigma) - 1 = \mathrm{exc_A}(\sigma')$.

Now, for the case $i > 0 : n-1, n \notin \mathrm{Exc_A}(\sigma)$ (since $\sigma(n-1) = n^{[i]}$ is not an excedance with respect to the color order). We also have $n-1, n \notin \mathrm{Exc_A}(\sigma')$, and thus $\mathrm{Exc_A}(\sigma) = \mathrm{Exc_A}(\sigma')$ for $\sigma \in R^i_{r,n}$ where $i > 0$.

In both cases, we have that $\mathrm{csum}(\sigma) = \mathrm{csum}(\sigma')$. Hence, we have that $\mathrm{exc}^{\mathrm{Clr}}(\sigma) - \mathrm{r} = \mathrm{exc}^{\mathrm{Clr}}(\sigma')$ for $i = 0$ and $\mathrm{exc}^{\mathrm{Clr}}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma')$ for $i > 0$.

As before, the number of cycles changes its parity due to the multiplication by a transposition, and hence $(-1)^{\mathrm{cyc}(\sigma)} = -(-1)^{\mathrm{cyc}(\sigma')}$. Hence, the total contribution of elements in $R^i_{r,n}$ is

$$q^r P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1)$$

for $i = 0$, and

$$q^i P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1)$$

for $i > 0$.

Now, if we sum up all the parts, we get

$$\begin{aligned} P^{\mathrm{Clr}}_{G_{r,n}}(q,1,-1) &= \sum_{i=0}^{r-1}(-q^i P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1)) \\ &\qquad + q^r P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1) + \sum_{i=1}^{r-1} q^i P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1) \\ &= (q^r - 1) P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1) \end{aligned}$$

as needed.

Now, for $n = 1$, $G_{r,1}$ is the cyclic group of order r and thus

$$P^{\mathrm{Clr}}_{G_{r,1}}(q,1,-1) = -(1 + q + \cdots + q^{r-1}) = -\frac{q^r - 1}{q - 1},$$

so we have

$$P^{\mathrm{Clr}}_{G_{r,n}}(q,1,-1) = -\frac{(q^r-1)^n}{q-1}.$$

5. Derangements in $G_{r,n}$ and the proof of Theorem 1.2

In this section, we prove Theorem 1.2. As in the previous section, we will prove Theorem 1.2 for both orders.

We start with the definition of a derangement.

Definition 5.1. An element $\sigma \in G_{r,n}$ is called a *derangement* if it has no absolute fixed points, i.e., $|\sigma(i)| \neq i$ for every $i \in [n]$. Denote by $D_{r,n}$ the set of all derangements in $G_{r,n}$

5.1. Proof for the absolute order.

We divide $D_{r,n}$ into $r+2$ disjoint subsets in the following way:

$$\begin{aligned} A^i_{r,n} &= \{\sigma \in D_{r,n} \mid \sigma(2) = 1^{[i]}, |\sigma(1)| \neq 2\}, \qquad i = 0, \ldots, r-1, \\ T_{r,n} &= \{\sigma \in D_{r,n} \mid |\sigma| = (123 \cdots n)\}, \\ \hat{D}_{r,n} &= D_{r,n} - (\bigcup_{i=0}^{r-1} A^i_{r,n} \cup T_{r,n}). \end{aligned}$$

We start by constructing a killing involution φ on $\hat{D}_{r,n}$. Given any $\sigma \in \hat{D}_{r,n}$, let i be the first number such that $|\sigma(i)| \neq i+1$. Define

$$\sigma' = \varphi(\sigma) = (\sigma(i), \sigma(i+1))\sigma.$$

For example, if $\sigma = (\bar{3}4\bar{1}\bar{5}\bar{\bar{2}})$ then $\sigma' = (4\bar{3}\bar{1}\bar{5}\bar{\bar{2}})$.

It is easy to see that φ is a well-defined involution on $\hat{D}_{r,n}$. We proceed to prove that $\mathrm{exc}^{\mathrm{Abs}}(\sigma) = \mathrm{exc}^{\mathrm{Abs}}(\sigma')$. Indeed, $\mathrm{csum}(\sigma) = \mathrm{csum}(\sigma')$.

Let i be the first number such that $|\sigma(i)| \neq i+1$ so that in the pass from σ to σ' we exchange $\sigma(i)$ with $\sigma(i+1)$. For every $j \neq i, i+1$, clearly $j \in \mathrm{Exc}(|\sigma|)$ if and only if $j \in \mathrm{Exc}(|\sigma'|)$. Since $\sigma \in D_{r,n}$, $|\sigma(i)| \neq i+1$ and $|\sigma(j)| = j+1$ for $j < i$, we have that $|\sigma(i)|, |\sigma(i+1)| \in \{1, i+2, \cdots, n\}$. Thus, exchanging $\sigma(i)$ with $\sigma(i+1)$ does not change $\mathrm{Exc}(|\sigma|)$.

Note also that the parity of $\mathrm{cyc}(\sigma')$ is opposite to the parity of $\mathrm{cyc}(\sigma)$ due to the multiplication by a transposition. Hence, we have proven that φ is a killing involution.

Now, let us calculate the contribution of each set in our decomposition to $P^{\mathrm{Abs}}_{D_{r,n}}(q, 0, -1)$. As we have shown, $\hat{D}_{r,n}$ contributes nothing. Define a bijection

$$\psi : A^i_{r,n} \to D_{r,n-1}$$

by $\psi(\sigma) = \sigma'$, where $\sigma'(1) = (|\sigma(1)| - 1)^{z_1(\sigma)}$ and for $i > 1$, $\sigma'(i) = (|\sigma(i+1)| - 1)^{z_{i+1}(\sigma)}$. For example, if $\sigma = (3\bar{1}4\bar{2})$, then $\sigma' = (23\bar{1})$. It is easy to see that $\mathrm{exc}(|\sigma|) = \mathrm{exc}(|\sigma'|)$. On the other hand, $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - i$ and $\mathrm{cyc}(\sigma) = \mathrm{cyc}(\sigma')$ and thus the contribution of $A^i_{r,n}$ to $P^{\mathrm{Abs}}_{D_{r,n}}(q, 0, -1)$ is $q^i P^{\mathrm{Abs}}_{D_{r,n-1}}(q, 0, -1)$ for $1 \leq i \leq r-1$.

Finally, we treat the set $T_{r,n}$. For every $\sigma \in T_{r,n}$ we have $\mathrm{exc}(|\sigma|) = n-1$ and $\mathrm{cyc}(\sigma) = 1$. Concerning $\mathrm{csum}(\sigma)$ we have

$$\sum_{\sigma \in T_{r,n}} q^{\mathrm{csum}(\sigma)} = (1 + q + \cdots q^{r-1})^n.$$

To summarize, we get

$$\begin{aligned}
P^{\mathrm{Abs}}_{D_{r,n}}(q, 0, -1) &= \sum_{i=0}^{r-1} (q^i P^{\mathrm{Abs}}_{D_{r,n-1}}(q, 0, -1)) + q^{n-1}(1 + q + \cdots q^{r-1})^n \\
&= (1 + q + \cdots q^{r-1}) P^{\mathrm{Abs}}_{D_{r,n-1}}(q, 0, -1) \\
&\qquad + q^{n-1}(1 + q + \cdots q^{r-1})^n \\
&= [r]_q (P^{\mathrm{Abs}}_{D_{r,n-1}}(q, 0, -1) - q^{n-1}[r]_q^{n-1})
\end{aligned}$$

as needed.

Now, for $n = 2$, we have

$$P^{\mathrm{Abs}}_{D_{r,2}}(q, 0, -1) = -q[r]_q^2$$

and thus

$$P^{\mathrm{Abs}}_{D_{r,n}}(q, 0, -1) = -q[r]_q^n[n-1]_q.$$

5.2. **Proof for the color order.** We use the same decomposition of $D_{r,n}$ as before. The killing involution will be also the same, due to the following observation: one can replace $\mathrm{Exc}(|\sigma|)$ in the previous proof by $\mathrm{Exc}_A(\sigma)$, and the argument still holds. Note that $i \notin \mathrm{Exc}_A(\sigma)$ if $z_i(\sigma) \neq 0$.

Now, let us calculate the contribution of each set in our decomposition to $P^{\mathrm{Clr}}_{D_{r,n}}(q, 0, -1)$. As we have shown, $\hat{D}_{r,n}$ contributes nothing.

As before, define a bijection

$$\psi : A^i_{r,n} \to D_{r,n-1}$$

by $\psi(\sigma) = \sigma'$, where $\sigma'(1) = (|\sigma(1)| - 1)^{z_1(\sigma)}$ and for $i > 1$, $\sigma'(i) = (|\sigma(i+1)| - 1)^{z_{i+1}(\sigma)}$. As before, it is easy to see that $\mathrm{exc}_A(\sigma) = \mathrm{exc}_A(\sigma')$ (since $|\sigma(2)| = 1$). On the other hand, $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - i$ and $\mathrm{cyc}(\sigma) = \mathrm{cyc}(\sigma')$, and thus the contribution of $A^i_{r,n}$ to $P^{\mathrm{Clr}}_{D_{r,n}}(q, 0, -1)$ is $q^i P^{\mathrm{Clr}}_{D_{r,n-1}}(q, 0, -1)$ for $0 \leq i \leq r-1$.

Now, we treat the set $T_{r,n}$. Let $\sigma \in T_{r,n}$. Observe that for $i < n$, $i \in \mathrm{Exc}_A(\sigma)$ if and only if $z_i(\sigma) = 0$. In this case, the place i contributes r to $\mathrm{exc}^{\mathrm{Clr}}(\sigma)$. Hence, it will be natural to construct the following bijection between $T_{r,n}$ and the following subset W of $G_{r+1,n}$:

$$\begin{aligned} W = \{\sigma \in G_{r+1,n} \mid |\sigma| = (12 \cdots n), \\ z_i(\sigma) \neq 0 \text{ for } 1 \leq i \leq n-1, z_n(\sigma) \neq r\} \end{aligned}$$

by $\psi(\sigma) = \sigma'$, where

$$\sigma'(i) = \begin{cases} \sigma(i) & i = n \text{ or } z_i(\sigma) \neq 0, \\ |\sigma(i)|^{[r]} & \text{otherwise}. \end{cases}$$

Note that $\mathrm{exc}^{\mathrm{Clr}}(\sigma) = \mathrm{csum}(\sigma')$.

Now we compute

$$\begin{aligned} \sum_{\sigma \in T_{r,n}} q^{\mathrm{exc}^{\mathrm{Clr}}(\sigma)}(-1)^{\mathrm{cyc}(\sigma)} &= \sum_{\sigma' \in W} q^{\mathrm{csum}(\sigma')}(-1)^{\mathrm{cyc}(\sigma')} \\ &= -\sum_{\sigma' \in W} q^{\mathrm{csum}(\sigma')} \\ &= -(q + q^2 + \cdots + q^r)^{n-1}(1 + q + \cdots + q^{r-1}). \end{aligned}$$

To summarize, we get

$$\begin{aligned}
P^{\mathrm{Clr}}_{D_{r,n}}(q,0,-1) &= \sum_{i=0}^{r-1}(q^i P^{\mathrm{Clr}}_{D_{r,n-1}}(q,0,-1)) \\
&\qquad - (q+q^2+\cdots+q^r)^{n-1}(1+q+\cdots+q^{r-1}) \\
&= (1+q+\cdots q^{r-1})(P^{\mathrm{Clr}}_{D_{r,n-1}}(q,0,-1)+(q+\cdots q^r)^{n-1}) \\
&= [r]_q(P^{\mathrm{Clr}}_{D_{r,n-1}}(q,0,-1)-q^{n-1}[r]_q^{n-1})
\end{aligned}$$

Now, for $n=2$, we have

$$P^{\mathrm{Clr}}_{D_{r,2}}(q,0,-1) = -q[r]_q^2$$

and thus

$$P^{\mathrm{Clr}}_{D_{r,n}}(q,0,-1) = -q[r]_q^n[n-1]_q.$$

6. Proof of Corollary 1.3

In this section, we provide a complete proof for Corollary 1.3. The proof follows the ideas of Ksavrelov and Zeng [1].

We first compute $P_{G_{r,n}}(q,s,-1)$. The polynomial counting the permutations of $G_{r,n}$ with a given number k of absolute fixed points $(0 \leq k \leq n)$ is

$$(-s)^k(1+q+\cdots q^{r-1})^k P_{G_{r,n-k}}(q,0,-1).$$

Hence,

$$\begin{aligned}
P_{G_{r,n}}(q,s,-1) &= \sum_{k=0}^{n}\binom{n}{k}[r]_q^k(-s)^k P_{G_{r,n-k}} = \\
&= \sum_{k=0}^{n}\binom{n}{k}[r]_q^k(-s)^k(-q)[r]_q^{n-k}[n-k-1]_q = \\
&= [r]_q^n\sum_{k=0}^{n}\binom{n}{k}(-s)^k(-q)\frac{1-q^{n-k-1}}{1-q} = \\
&= \frac{[r]_q^n}{1-q}\sum_{k=0}^{n}\left(\binom{n}{k}q^{n-k}(-s)^k - q\binom{n}{k}(-s)^k\right) = \\
&= \frac{[r]_q^n}{1-q}((q-s)^n - q(1-s)^n).
\end{aligned}$$

Thus

$$P_{G_{r,n}}(q,s,-1) = \frac{[r]_q^n}{1-q}((q-s)^n - q(1-s)^n). \tag{3}$$

Now we compute an exponential generating function for

$$\sum_{n\geq 0} P_{G_{r,n}}(q,s,t).$$

Denote by $\mathcal{C}_{r,n}$ the set of cycles of length n in $G_{r,n}$. For example, in $G_{2,2}=B_2$ we have

$$\mathcal{C}_{2,2}=\{(12),(21),(\bar{1}2),(2\bar{1}),(1\bar{2}),(2\bar{1}),(\bar{1}\bar{2})\}.$$

Note that in the following computation we use the fact that all parameters $\mathrm{exc}(\pi),\mathrm{fix}(\pi),\mathrm{cyc}(\pi)$ are additive. We have

$$\begin{aligned}
&\sum_{n\geq 0} P_{G_{r,n}}(q,s,t)\frac{x^n}{n!}\\
&\quad=\sum_{n\geq 0}\sum_{\pi\in G_{r,n}}\left(q^{\mathrm{exc}(\pi)}s^{\mathrm{fix}(\pi)}t^{\mathrm{cyc}(\pi)}\right)\frac{x^n}{n!}\\
&\quad=\sum_{n\geq 0}\sum_{\substack{(c_1,\dots,c_k)\in\mathcal{C}_{r,n_{i_1}}\times\cdots\times\mathcal{C}_{r,n_{i_k}}\\ n_{i_1}+\cdots+n_{i_k}=n}}\frac{1}{k!}\\
&\qquad\cdot\left(q^{\left(\sum_{i=1}^{k}\mathrm{exc}(c_i)\right)}s^{\left(\sum_{i=1}^{k}\mathrm{fix}(c_i)\right)}t^{\left(\sum_{i=1}^{k}\mathrm{cyc}(c_i)\right)}\right)\frac{x^{|c_1|+\cdots+|c_k|}}{|c_1|!\cdots|c_k|!}\\
&\quad=\sum_{n\geq 0}\sum_{\substack{(c_1,\dots,c_k)\in\mathcal{C}_{r,n_{i_1}}\times\cdots\times\mathcal{C}_{r,n_{i_k}}\\ n_{i_1}+\cdots+n_{i_k}=n}}\frac{1}{k!}\\
&\qquad\cdot\prod_{i=1}^{k}\left(q^{\mathrm{exc}(c_i)}s^{\mathrm{fix}(c_i)}t^{\mathrm{cyc}(c_i)}\right)\frac{x^{|c_1|+\cdots+|c_k|}}{|c_1|!\cdots|c_k|!}\\
&\quad=\sum_{k\geq 0}\frac{1}{k!}\left(\sum_{n\geq 0}\sum_{c\in\mathcal{C}_{r,n}}q^{\mathrm{exc}(c)}s^{\mathrm{fix}(c)}t^{\mathrm{cyc}(c)}\frac{x^{|c|}}{|c|!}\right)^k\\
&\quad=\exp\left(\sum_{n\geq 0}\sum_{c\in\mathcal{C}_{r,n}}q^{\mathrm{exc}(c)}s^{\mathrm{fix}(c)}t^{\mathrm{cyc}(c)}\frac{x^{|c|}}{|c|!}\right)\\
&\quad=\left(\exp\left(-\sum_{n\geq 0}\sum_{c\in\mathcal{C}_{r,n}}q^{\mathrm{exc}(c)}s^{\mathrm{fix}(c)}\frac{x^{|c|}}{|c|!}\right)\right)^{-t}.
\end{aligned}$$

Hence we have

$$\sum_{n\geq 0} P_{G_{r,n}}(q,s,t)\frac{x^n}{n!} = \left(\exp\left(-\sum_{n\geq 0}\sum_{c\in\mathcal{C}_{r,n}} q^{\mathrm{exc}(c)}s^{\mathrm{fix}(c)}\frac{x^{|c|}}{|c|!}\right)\right)^{-t}. \tag{4}$$

Substituting $t=-1$ in Equation (4), we get

$$\sum_{n\geq 0} P_{G_{r,n}}(q,s,-1)\frac{x^n}{n!} = \exp\left(-\sum_{n\geq 0}\sum_{c\in\mathcal{C}_{r,n}} q^{\mathrm{exc}(c)}s^{\mathrm{fix}(c)}\frac{x^{|c|}}{|c|!}\right). \tag{5}$$

Now, by Equation (3) we have

$$\sum_{n\geq 0}\left(\frac{[r]_q^n}{1-q}((q-s)^n - q(1-s)^n)\right)\frac{x^n}{n!}$$
$$= \exp\left(-\sum_{n\geq 0}\sum_{c\in\mathcal{C}_{r,n}} q^{\mathrm{exc}(c)}s^{\mathrm{fix}(c)}\frac{x^{|c|}}{|c|!}\right).$$

Substituting the last equation back in Equation (4) we get

$$\sum_{n\geq 0} P_{G_{r,n}}(q,s,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0}\left(\frac{[r]_q^n}{1-q}((q-s)^n - q(1-s)^n)\right)\frac{x^n}{n!}\right)^{-t}.$$

7. Statistics on the group of even signed permutations

In this section we deal with the Coxeter group of type D, namely the group of even signed permutations. We recall its definition,

$$D_n = \{\pi\in B_n \mid \sum_{i=1}^n z_i(\pi)\equiv 0 \pmod 2\}.$$

Unlike the case of the groups $G_{r,n}$, in D_n the distribution of the excedance numbers with respect to the color order is different from the distribution with respect to the absolute order. We start with the color order.

7.1. **Proof of Theorem 1.4.** We divide D_n into 5 subsets:

$$\begin{aligned}
K_n &= \{\sigma\in D_n \mid |\sigma(n)|\neq n, |\sigma(n-1)|\neq n\},\\
T_n^0 &= \{\sigma\in D_n \mid \sigma(n)=n\},\\
T_n^1 &= \{\sigma\in D_n \mid \sigma(n)=\bar{n}\},\\
R_n^0 &= \{\sigma\in D_n \mid \sigma(n-1)=n\},\\
R_n^1 &= \{\sigma\in D_n \mid \sigma(n-1)=\bar{n}\}.
\end{aligned}$$

We write

$$a_n = P^{\mathrm{Clr}}_{D_n}(q, 1, -1),$$
$$b_n = P^{\mathrm{Clr}}_{D_n^c}(q, 1, -1),$$

where D_n^c is the complement of D_n in B_n.

Define $\varphi : K_n \to K_n$ by

$$\sigma' = \varphi(\sigma) = (\sigma(n-1), \sigma(n))\sigma.$$

Note that φ exchanges $\sigma(n-1)$ with $\sigma(n)$. It is easy to see that φ is a killing involution on K_n.

We turn now to the set T_n^0. Note that there is a natural bijection between T_n^0 and D_{n-1}, defined by ignoring the last digit. Let $\sigma \in T_n^0$. Denote the image of σ under this bijection by σ'. Note that $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma)$, $\mathrm{Exc}_{\mathrm{A}}(\sigma') = \mathrm{Exc}_{\mathrm{A}}(\sigma)$ and $\mathrm{Exc}^{\mathrm{Clr}}(\sigma') = \mathrm{Exc}^{\mathrm{Clr}}(\sigma)$. On the other hand, $\mathrm{cyc}(\sigma') = \mathrm{cyc}(\sigma) - 1$ and thus the restriction of a_n to T_n^0 is just $-a_{n-1}$.

For the contribution of the set T_n^1, note that the function φ defined above gives us a bijection between T_n^1 and D_{n-1}^c. In this case, $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - 1$, $\mathrm{exc}_{\mathrm{A}}(\sigma') = \mathrm{exc}_{\mathrm{A}}(\sigma)$ and $\mathrm{exc}^{\mathrm{Clr}}(\sigma') = \mathrm{exc}^{\mathrm{Clr}}(\sigma)$. On the other hand, $\mathrm{cyc}(\sigma') = \mathrm{cyc}(\sigma) - 1$ as before. Hence, the restriction of a_n to T_n^1 is $-qb_{n-1}$.

Now, for the set R_n^0, we have the following bijection between R_n^0 and D_{n-1}: for $\sigma \in R_n^0$, exchange the last two digits, and then ignore the last digit. If we denote the image of σ by σ', we have $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma)$, $\mathrm{exc}_{\mathrm{A}}(\sigma') = \mathrm{exc}_{\mathrm{A}}(\sigma) - 1$, $\mathrm{exc}^{\mathrm{Clr}}(\sigma') = \mathrm{exc}^{\mathrm{Clr}}(\sigma) - 2$ and $\mathrm{cyc}(\sigma') \equiv \mathrm{cyc}(\sigma) \pmod 2$. Hence, the restriction of a_n to R_n^0 is $q^2 a_{n-1}$.

For the set R_n^1, we have a bijection between R_n^1 and D_{n-1}^c: for $\sigma \in R_n^1$, exchange the last two digits, and then ignore the last digit. Denoting the image of σ by σ', we have $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - 1$, $\mathrm{exc}_{\mathrm{A}}(\sigma') = \mathrm{exc}_{\mathrm{A}}(\sigma)$, and hence $\mathrm{exc}^{\mathrm{Clr}}(\sigma') = \mathrm{exc}^{\mathrm{Clr}}(\sigma) - 1$. Also, we have $\mathrm{cyc}(\sigma') \equiv \mathrm{cyc}(\sigma) \pmod 2$. Hence, the restriction of a_n to R_n^1 is qb_{n-1}.

We summarize all the contributions over all the four subsets, and we have

$$a_n = -a_{n-1} - qb_{n-1} + q^2 a_{n-1} + qb_{n-1} = (q^2 - 1)a_{n-1}$$

For computing a_1, note that $D_1 = \{1\}$ and thus $a_1 = -1$. Therefore, we have

$$P^{\mathrm{Clr}}_{D_n}(q, 1, -1) = a_n = -(q^2 - 1)^{n-1},$$

and we are done.

7.2. **Proof of Theorem 1.5.** In this subsection we present the proof of Theorem 1.5 which is based on the absolute order.

We start by dividing D_n into 5 subsets just as was shown in the proof of Theorem 1.4 and define as before

$$\begin{aligned} a_n &= P^{\mathrm{Abs}}_{D_n}(q,1,-1), \\ b_n &= P^{\mathrm{Abs}}_{D_n^c}(q,1,-1), \end{aligned}$$

where D_n^c is the complement of D_n in B_n.

It is easy to check that the sets T_n^0 and T_n^1 give the same contributions as before so we turn to the set R_n^0. By using the bijection between R_n^0 and D_{n-1} defined above which exchanges the last two digits, and then ignores the last digit, we have $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma)$, $\mathrm{exc}(|\sigma'|) = \mathrm{exc}(|\sigma|) - 1$. Thus, $\mathrm{exc}^{\mathrm{Abs}}(\sigma') = \mathrm{exc}^{\mathrm{Abs}}(\sigma) - 1$. Also, $\mathrm{cyc}(\sigma') \equiv \mathrm{cyc}(\sigma) \pmod 2$. Hence, the restriction of a_n to R_n^0 is qa_{n-1}.

For the set R_n^1, we use the same bijection, now between R_n^1 and D_{n-1}^c to get $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - 1$, $\mathrm{exc}(|\sigma'|) = \mathrm{exc}(|\sigma|) - 1$. Thus $\mathrm{exc}^{\mathrm{Abs}}(\sigma') = \mathrm{exc}^{\mathrm{Abs}}(\sigma) - 2$. Also, we have $\mathrm{cyc}(\sigma') \equiv \mathrm{cyc}(\sigma) \pmod 2$. Hence, the restriction of a_n to R_n^1 is $q^2 b_{n-1}$.

To summarize, we have

$$a_n = (q-1)(a_{n-1} + qb_{n-1}),$$

and by symmetry

$$b_n = (q-1)(b_{n-1} + qa_{n-1}).$$

Since $D_1 = \{1\}$ we get $a_1 = -1, b_1 = -q$. Solving the above system of recursive equations yields

$$\begin{aligned} P^{\mathrm{Abs}}_{D_n}(q,1,-1) &= \frac{1}{2}(q-1)^{n-1} \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} q^k \\ &= -\frac{1}{2}(q-1)^{n-1}((1+q)^n + (1-q)^n), \end{aligned}$$

as needed.

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