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EQUIDISTRIBUTION AND SIGN-BALANCE ON 132-AVOIDING PERMUTATIONS

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Abstract. Let R_n be the set of all permutations of length n which avoid 132. In this paper we study the statistics *last descent* ("ldes"), *first descent* ("fdes"), *last rise* ("lris"), and *first rise* ("fris") on the set R_n. In particular, we prove that the bistatistic ("fris","lris") on the set of all permutations $R_n \backslash \{n \dots 21\}$ and the bistatistic ("$n-$ldes","$n-$fdes") on the set of all permutations of $R_n \backslash \{12 \dots n\}$ are equidistributed. Furthermore, we consider the case of sign balance for these statistics on the set of all permutations R_n, and we give a combinatorial interpretation for some of these statistics.

1. Introduction

Let S_n denote the set of permutations of $\{1, \dots, n\}$, written in one-line notation, and suppose that $\pi \in S_n$ and $\sigma \in S_k$. We say that a subsequence of π has *type* σ whenever it has the same pairwise comparisons as σ. For example, the subsequence 24869 of the permutation 214538769 has type 12435. We say that π *avoids* σ (or π *is* σ*-avoiding*) whenever π contains no subsequence of type σ. For example, the permutation 214538769 avoids 4321 and 2413, but it has 2589 as a subsequence, so it does not avoid 1234. We denote the set of σ-avoiding permutations in S_n by $S_n(\sigma)$. We define $R_n = S_n(132)$. For $\pi \in R_n$, we define the following statistics:

(1) $\mathrm{ldes}(\pi)$ = last descent of $\pi = \max\{1 \leq i \leq n-1 | \pi_i > \pi_{i+1}\}$ where $\mathrm{ldes}(12\dots n) = 0$,
(2) $\mathrm{fdes}(\pi)$ = first descent of $\pi = \min\{1 \leq i \leq n-1 | \pi_i > \pi_{i+1}\}$ where $\mathrm{fdes}(12\dots n) = 0$,
(3) $\mathrm{lris}(\pi)$ = last rise of $\pi = \max\{1 \leq i \leq n-1 | \pi_i < \pi_{i+1}\}$ where $\mathrm{lris}(n\dots 21) = 0$,
(4) $\mathrm{fris}(\pi)$ = first rise of $\pi = \min\{1 \leq i \leq n-1 | \pi_i < \pi_{i+1}\}$ where $\mathrm{fris}(n\dots 21) = 0$,
(5) $\mathrm{lind}(\pi) = \pi^{-1}(n)$ = the index of the letter "n" in π,
(6) $\mathrm{find}(\pi) = \pi^{-1}(1)$ = the index of the letter "1" in π.

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Foata and Schützenberger [FS, Theorem 1] proved that the major index and the inversion number are equidistributed on S_n whose inverse has a prescribed descent set. Recently, Adin and Roichman [AR, Theorem 1.1] gave an analogue for this result for $S_n(321)$. They proved that the statistics "ldes" and "lind $-$ 1" are equidistributed on the set $S_n(321)$. Note that these statistics are identical on the set R_n. (To prove that, let $s = \pi_1^{-1}$. We may assume that $s > 1$, otherwise π equals the identity of S_n, which has no descents. Clearly, $\pi_{s-1} > \pi_s$. Furthermore, we have $1 = \pi_s < \pi_{s+1} < \cdots < \pi_n$ since π avoids 132. Therefore, $s-1$ is exactly the last descent of π.)

The *Catalan sequence* is the sequence $(C_n)_{n\geq 0} = (1, 1, 2, 5, 14, 42, 132, 429, 1430, \dots)$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is called the nth *Catalan number*. The generating function for the Catalan numbers is denoted by $C(t) = \frac{1-\sqrt{1-4t}}{2t}$. The Catalan numbers provide a complete answer to the problem of counting certain properties of more than 100 different combinatorial structures (see [S, page 219 and Exercise 6.19]). The structures that are useful for us in the present paper are Dyck paths (see [S]) and 132-avoiding permutations (see [Kn]).

A *Dyck path* is a path in the plane integer lattice $\mathbb{Z}^2$ consisting of up-steps $U = (1, 1)$ and down-steps $D = (1, -1)$ which never passes below the x-axis. We denote the set of Dyck paths of length $2n$ by $\mathcal{P}_n$. A point on the Dyck path is called a *peak* if it is immediately preceded by an up-step and immediately followed by a down-step. For $p \in \mathcal{P}_n$, we define

$$\text{lpeak}(p) = \text{the height (the } y\text{-coordinate) of the last peak of } p.$$

Recently, Adin and Roichman [AR, Theorem 1.2] proved that the statistics "ldes" on the set $S_n(321)$, "ldes" on the set $\mathcal{P}_n$, and "lind $-$ 1" are equidistributed on the set $\mathcal{P}_n$. In this paper we prove the following analogue of this result.

Theorem 1.1. *For all $n \geq 1$, we have*

$$\sum_{p\in\mathcal{P}_n} q^{n-\text{lpeak}(p)} = \sum_{\pi\in R_n} q^{\text{ldes}(\pi)}.$$

In addition, we study the sign-and-"find" and sign-and-"lind" enumerators for R_n (see [AR, Theorem 1.3] for $S_n(321)$), and we prove the following result.

Theorem 1.2. *For all $n \geq 1$,*

$$\sum_{\pi\in R_{2n}} \text{sign}(\pi) q^{\text{ldes}(\pi)} = (1-q) \sum_{\pi\in R_{2n+1}} \text{sign}(\pi) q^{\text{find}(\pi)-1},$$

and

$$\sum_{\pi\in R_{2n-1}} \text{sign}(\pi) q^{\text{ldes}(\pi)} = \sum_{\pi\in R_{n-1}} q^{2(\text{find}(\pi)-1)}.$$

The paper is organized as follows. In Section 2, we study the statistics "ldes", "fdes", "lris", and "fris" on the set R_n. In particular, we prove that the bistatistic ("fris","lris") on the set $R_n \backslash \{n \dots 21\}$ and the bistatistic ("n $-$ ldes","n $-$ fdes") on the set $R_n \backslash \{12 \dots n\}$ are equidistributed. In Section 3, we consider the case of sign balance of some statistics on the set R_n. Finally, in Section 4, we give a combinatorial interpretation (using Dyck paths) for some of these statistics.

2. Equidistribution of statistics

In this section, we study the statistics "ldes", "fdes", "lris", and "fris" on the set R_n by using the block decomposition approach of 132-avoiding permutations in S_n (see [MV]).

First of all, let us describe the block decomposition of a permutation $\pi \in R_n$. Let $n \geq 1$ and $\pi = (\pi', n, \pi'') \in R_n$ such that $\pi_j = n$. π avoids 132 if and only if π' is a permutation of the numbers $n-j+1, n-j+2, \dots, n-1$, π'' is a permutation of the numbers $1, 2, \dots, n-j$, and both π' and π'' avoid 132. This representation is called the *block decomposition* of π.

Theorem 2.1. *The statistic "fris" on the set $R_n \backslash \{n \dots 21\}$ and the statistic "$n-$ldes" on the set $R_n \backslash \{12 \dots n\}$ are equidistributed, that is, for $n \geq 1$, we have*

$$\sum_{\pi \in R_n \backslash \{n \dots 21\}} q^{\mathrm{fris}(\pi)} = \sum_{\pi \in R_n \backslash \{12 \dots n\}} q^{n-\mathrm{ldes}(\pi)}.$$

Moreover,

$$\sum_{n \geq 0} \sum_{\pi \in R_n} q^{\mathrm{fris}(\pi)} x^n = \frac{1}{1-x} - \frac{1}{1-qx} + \frac{1}{1-qxC(x)}$$

and

$$\sum_{n \geq 0} \sum_{\pi \in R_n} q^{\mathrm{ldes}(\pi)} = \frac{1}{1-xC(qx)}.$$

Proof. Let $F_n(q) = \sum_{\pi \in R_n} q^{\mathrm{ldes}(\pi)}$ for $n \geq 0$. Using the block decomposition of $\pi \in R_n$, we may derive a recurrence for $F_n(q)$. Namely, by making use of the fact that $|S_m(132)| = C_m = \frac{1}{m+1}\binom{2m}{m}$ (see [Kn]), we get for $n \geq 1$,

$$F_n(q) = F_{n-1}(q) + q \sum_{j=0}^{n-2} q^j C_j F_{n-1-j}(q). \tag{2.1}$$

Multiplying both sides by x^n, summing over all $n \geq 1$, and finally using $F_0(q) = 1$, we obtain that

$$F(x;q) = 1 + \sum_{n \geq 1} \sum_{\pi \in R_n} q^{\mathrm{ldes}(\pi)} x^n = \frac{1}{1-xC(xq)}.$$

Let now $H_n(q) = \sum_{\pi \in R_n} q^{\mathrm{fris}(\pi)}$ for $n \geq 0$. Again, using the block decomposition of π, we may derive a recurrence for $H_n(q)$. First of all, the contribution of the permutations π with $\pi_1 = n$ gives $1 + q(H_{n-1}(q) - 1)$. Secondly, the contribution of the permutations π with $\pi_2 = n$ gives $C_{n-2}q$. Finally, for $\pi_j = n$ with $j \geq 3$, we get as contribution $(H_{j-1}(q) - 1)C_{n-j}$ whenever $\pi' \neq (n-1)\dots(n+1-j)$ in the decomposition $\pi = (\pi', n, \pi'')$, and $q^{j-1}C_{n-j}$ for $\pi' = (n-1)\dots(n+1-j)$. Hence, for $n \geq 1$, we have

$$H_n(q) = 1 + q(H_{n-1}(q) - 1) + C_{n-2}q + \sum_{j=2}^{n-1} C_{n-1-j}(H_j(q) - 1) + \sum_{j=2}^{n-1} q^j C_{n-1-j},$$

or, equivalently,

$$H_n(q) = 1 + q(H_{n-1}(q) - 1) + \sum_{j=2}^{n-1} C_{n-1-j}(H_j(q) - 1) + \sum_{j=1}^{n-1} q^j C_{n-1-j}. \tag{2.2}$$

If we write $K_n(q)$ for $q^n(H_n(q^{-1}) - 1) + 1$, then it is easy to see that

$$K_n(q) = K_{n-1}(q) + q\sum_{j=0}^{n-2} q^j C_j K_{n-1-j}(q).$$

Since $K_0(q) = K_1(q) = 1$, an induction on n together with Equation (2.1) gives for $n \geq 0$,

$$q^n(H_n(1/q) - 1) + 1 = F_n(q).$$

The rest follows now easily. $\square$

Theorem 2.2. *The statistic "*lris*" on the set $R_n \backslash \{n \dots 21\}$ and the statistic "$n-$*fdes*" on the set $R_n \backslash \{12 \dots n\}$ are equidistributed, that is, for $n \geq 1$, we have*

$$\sum_{\pi \in R_n \backslash \{n \dots 21\}} q^{\mathrm{lris}(\pi)} = \sum_{\pi \in R_n \backslash \{12 \dots n\}} q^{n-\mathrm{fdes}(\pi)}.$$

Moreover,

$$\sum_{n \geq 0} \sum_{\pi \in R_n} q^{\mathrm{fdes}(\pi)} x^n = \frac{1 - q + q(1-x)^2 C(x)}{(1-x)(1-xq)}$$

and

$$\sum_{n \geq 0} \sum_{\pi \in R_n} q^{\mathrm{lris}(\pi)} x^n = \frac{1 + x(C(xq) - 1)C(xq)}{1-x}.$$

Proof. Let $L_n(q) = \sum_{\pi \in R_n} q^{\mathrm{fdes}(\pi)}$ for $n \geq 0$. Let $\pi = (\pi', n, \pi'') \in R_n$ such that $\pi_j = n$. Using this block decomposition of π, we may derive a recurrence for $L_n(q)$. Namely, the contribution of the permutations with $j = 1$ gives $C_{n-1}q$, the contribution for $j = 2$ gives $C_{n-2}q^2$, the contribution for $n \geq j \geq 3$ and $\pi' \neq (n+1-j)\dots(n-1)$ gives $C_{n-j}(L_{j-1}(q) - 1)$, the contribution for $n - 1 \geq j \geq 3$ and $\pi' = (n+1-j)\dots(n-1)$ gives $q^j C_{n-j}$, and the contribution for $j = n$ and $\pi' = (n+1-j)\dots(n-1)n$ gives 1. Hence, for $n \geq 1$, we have

$$L_n(q) = C_{n-1}q + C_{n-2}q^2 + \sum_{j=3}^{n} C_{n-j}(L_{j-1}(q) - 1) + \sum_{j=3}^{n-1} q^j C_{n-j} + 1,$$

or, equivalently,

$$L_n(q) = 1 + q\sum_{j=0}^{n-2} q^j C_{n-1-j} + \sum_{j=0}^{n-1} C_{n-1-j}(L_j(q) - 1). \tag{2.3}$$

Multiplying by x^n, summing over $n \geq 1$, and finally using $L_0(q) = L_1(q) = 1$, we obtain that

$$\sum_{n \geq 0} L_n(q) x^n = \frac{1 - q + q(1-x)^2 C(x)}{(1-x)(1-xq)}.$$

Now, let $P_n(q) = \sum_{\pi\in R_n} q^{\mathrm{lris}(\pi)}$ for $n \geq 0$. Again, we may derive a recurrence for $P_n(q)$ by using the above block decomposition of π. Namely, the contribution for $j = n$ gives $C_{n-1}q^{n-1}$, the contribution for $j = n-1$ gives $C_{n-2}q^{n-2}$, the contribution for $1 \leq j \leq n-2$ and $\pi'' \neq (n-j)\dots 21$ gives $C_{j-1}q^j(P_{n-j}(q)-1)$, and the contribution for $n-2 \geq j \geq 1$ and $\pi'' = (n-j)\dots 21$ gives $q^{j-1}C_{j-1}$. Hence, for $n \geq 1$, we have

$$P_n(q) = C_{n-1}q^{n-1} + C_{n-2}q^{n-2} + \sum_{j=1}^{n-2} C_{j-1}q^j(P_{n-j}(q)-1) + \sum_{j=1}^{n-2} q^{j-1}C_{j-1},$$

or, equivalently,

$$P_n(q) = \sum_{j=0}^{n-1} q^jC_j + q\sum_{j=0}^{n-1} C_jq^j(P_{n-1-j}(q)-1). \tag{2.4}$$

Multiplying by x^n, summing over $n \geq 1$, and using finally $P_0(q) = P_1(q) = 1$, we obtain that

$$\sum_{n\geq 0} P_n(q)x^n = \frac{1 + x(C(xq)-1)C(xq)}{1-x}.$$

Using Equations (2.3) and (2.4) together with an induction on n, we conclude that $q^n(P_n(q^{-1})-1)+1 = L_n(q)$, and this completes the proof. □

More generally, we prove that the bistatistics ("fris","lris") on the set $R_n\backslash\{n\dots 21\}$ and ("$n-\mathrm{ldes}$","$n-\mathrm{fdes}$") on the set $R_n\backslash\{12\dots n\}$ are equidistributed.

Theorem 2.3. *The bistatistic ("fris", "lris") on the set $R_n\backslash\{n\dots 21\}$ and the bistatistic ("$n-\mathrm{ldes}$", "$n-\mathrm{fdes}$") on the set $R_n\backslash\{12\dots n\}$ are equidistributed, that is, for $n \geq 1$, we have*

$$\sum_{\pi\in R_n\backslash\{12\dots n\}} p^{n-\mathrm{ldes}(\pi)}q^{n-\mathrm{fdes}(\pi)} = \sum_{\pi\in R_n\backslash\{n\dots 21\}} p^{\mathrm{fris}(\pi)}q^{\mathrm{lris}(\pi)}.$$

Moreover,

$$\sum_{n\geq 0}\sum_{\pi\in R_n} p^{\mathrm{fris}(\pi)}q^{\mathrm{lris}(\pi)}x^n = 1 + \frac{x(1-p+p(1-2qx+pq^2x^2)C(qx))}{(1-x)(1-pqx)(1-pqxC(qx))}.$$

Proof. Let $A_n(p,q) = \sum_{\pi\in R_n} p^{\mathrm{fris}(\pi)}q^{\mathrm{lris}(\pi)}$ for $n \geq 1$ and $A_0(p,q) = 1$. Using the block decomposition $\pi = (\pi', n, \pi'') \in R_n$ where $\pi_j = n$ once more, we derive a recurrence for $A_n(p,q)$ if $n \geq 1$. Namely, the contribution of the permutations with $\pi' = (n-1)\dots(n+1-j)$ and $\pi'' = (n-j)\dots 1$ gives $\sum_{j=0}^{n-1}(pq)^j$. The contribution of the permutations where $\pi' = (n-1)\dots(n+1-j)$ and $\pi'' \neq (n-j)\dots 1$ gives

$$pq(A_{n-1}(p,q)-1) + \sum_{j=2}^{n} q^jp^{j-1}(P_{n-j}(q)-1).$$

($P_n(q)$ is the polynomial appearing in the proof of Theorem 2.2.) The contribution of the permutations where $\pi' \neq (n-1)\dots(n+1-j)$ and $\pi'' = (n-j)\dots 1$ gives $\sum_{j=1}^{n} q^{j-1}(H_{j-1}(p)-1)$. ($H_n(p)$ is the polynomial appearing in the proof of Theorem 2.1). Finally, the contribution of the permutations where $\pi' \neq (n-1)\dots(n+1-j)$

and $\pi'' \neq (n-j)\dots 1$ equals $\sum_{j=1}^{n} q^j (H_{j-1}(p)-1)(P_{n-j}(q)-1)$. Hence, for $n \geq 1$, we have

$$\begin{aligned}A_n(p,q) &= pq(A_{n-1}(p,q)-1) + \sum_{j=0}^{n-1}(pq)^j + q\sum_{j=1}^{n-1}(pq)^j(P_{n-1-j}(q)-1)\\ &\quad + \sum_{j=0}^{n-1} q^j(H_j(p)-1) + q\sum_{j=0}^{n-1} q^j(H_j(p)-1)(P_{n-1-j}(q)-1).\end{aligned}$$

Let $A(x;p,q)$, $P(x;q)$, and $H(x;p)$ be the generating functions for the sequences $A_n(p,q)$, $P_n(q)$, and $H_n(p)$, respectively, that is,

$$A(x;p,q) = \sum_{n\geq 0} A_n(p,q)x^n, \quad P(x;q) = \sum_{n\geq 0} P_n(q)x^n, \quad \text{and} \quad H(x;p) = \sum_{n\geq 0} H_n(p)x^n.$$

Multiplying the above recurrence by x^n, summing over $n \geq 1$, and finally using $A_0(p,q) = H_0(p) = P_0(q) = 1$, we obtain that

$$\begin{aligned}(1-pqx)A(x;p,q) &= 1 + \frac{x}{(1-x)(1-pqx)} + \frac{x}{1-x}\left(H(qx;p) - \frac{1}{1-qx}\right)\\ &\quad + \frac{qx}{1-pqx}\left(P(x;q) - \frac{1}{1-x}\right) - qx\left(P(x;q) - \frac{1}{1-x}\right)\\ &\quad - \frac{pqx}{1-x} + xq\left(H(qx;p) - \frac{1}{1-qx}\right)\left(P(x;q) - \frac{1}{1-x}\right).\end{aligned}$$

From Theorem 2.1 we have

$$H(x;q) = \frac{1}{1-qxC(x)} + \frac{1}{1-x} - \frac{1}{1-qx},$$

while Theorem 2.2 yields

$$P(x;q) = \frac{1}{1-x}(1 + x(C(qx)-1)C(qx)).$$

Hence,

$$A(x;p,q) = 1 + \frac{x(1-p+p(1-2qx+pq^2x^2)C(qx))}{(1-x)(1-pqx)(1-pqxC(qx))}.$$

We now turn to the computation of the generating function

$$B(x;p,q) = \sum_{n\geq 0}\sum_{\pi\in R_n} p^{n-\mathrm{ldes}(\pi)} q^{n-\mathrm{fdes}(\pi)} x^n.$$

Using the arguments in the proof of the formula for $A(x;p,q)$, we get that

$$B(x;p,q) = A(x;p,q) - \frac{1}{1-x} + \frac{1}{1-pqx}.$$

Hence,

$$\begin{aligned}\sum_{n\geq 0}\sum_{\pi\in R_n\setminus\{12\dots n\}} p^{n-\mathrm{ldes}(\pi)}q^{n-\mathrm{fdes}(\pi)}x^n &= B(x;p,q)-\frac{1}{1-pqx}\\ &= A(x;p,q)-\frac{1}{1-x}\\ &= \sum_{n\geq 0}\sum_{\pi\in R_n\setminus\{n\dots 21\}} p^{\mathrm{fris}(\pi)}q^{\mathrm{lris}(\pi)}x^n,\end{aligned}$$

as requested. □

3. Sign Balance on R_n

We denote the set of all *even* (respectively *odd*) permutations $\pi\in R_n$ by R_n^+ (respectively R_n^-). We define $e_n=|R_n^+|$, $o_n=|R_n^-|$, and $m_n=e_n-o_n$ for all $n\geq 0$. (Simion and Schmidt [SimSch] proved that $m_n=C_{(n-1)/2}$ if n is odd and $m_n=0$ otherwise.) In this section we study the sign-balance of R_n with respect to certain statistics.

Theorem 3.1. *We have*

$$\sum_{n\geq 0}M_n(q)x^n=\sum_{n\geq 0}\sum_{\pi\in R_n}\mathrm{sign}(\pi)q^{\mathrm{ldes}\,(\pi)}=\frac{1+x-x^2qC(x^2q^2)}{1-x^2C(x^2q^2)}.$$

Proof. Let $F_n^{\pm}(q)=\sum_{\pi\in R_n^{\pm}}q^{\mathrm{ldes}(\pi)}$, and let $\pi=(\pi',n,\pi'')\in R_n$ such that $\pi_{j+1}=n$, $0\leq j\leq n-1$. Then, for this block decomposition of π, we have

$$\mathrm{sign}(\pi)=(-1)^{(j+1)(n-j-1)}\,\mathrm{sign}(\pi')\,\mathrm{sign}(\pi'')=(-1)^{(j+1)(n-1)}\,\mathrm{sign}(\pi')\,\mathrm{sign}(\pi''),$$

or, equivalently,

$$\mathrm{sign}(\pi)=\begin{cases}\mathrm{sign}(\pi')\cdot\mathrm{sign}(\pi''), & \text{if } n \text{ is odd},\\ (-1)^{j+1}\cdot\mathrm{sign}(\pi')\cdot\mathrm{sign}(\pi''), & \text{if } n \text{ is even}.\end{cases}$$

Therefore, for $n\geq 1$, we obtain

$$\begin{aligned}F_{2n+1}^{\pm}(q) &= F_{2n}^{\pm}(q)+\sum_{j=0}^{2n-1}q^{j+1}(e_jF_{2n-j}^{\pm}(q)+o_jF_{2n-j}^{\mp}(q))\\ F_{2n}^{\pm}(q) &= F_{2n-1}^{\pm}(q)+\sum_{j=0,2,4,\dots,2n-2}q^{j+1}(e_jF_{2n-1-j}^{\mp}(q)+o_jF_{2n-1-j}^{\pm}(q))\\ &\quad+\sum_{j=1,3,\dots,2n-3}q^{j+1}(e_jF_{2n-1-j}^{\pm}(q)+o_jF_{2n-1-j}^{\mp}(q).\end{aligned}$$

Hence, for all $n\geq 1$, we have

$$\begin{aligned}M_{2n+1}(q) &= M_{2n}(q)+\sum_{j=0}^{2n-1}q^{j+1}m_jM_{2n-j}(q),\\ M_{2n}(q) &= M_{2n-1}(q)-\sum_{j=0,2,4,\dots,2n-2}q^{j+1}m_jM_{2n-1-j}(q)\\ &\quad+\sum_{j=1,3,\dots,2n-3}q^{j+1}m_jM_{2n-1-j}(q).\end{aligned}$$

Let $M(x;q)=\sum_{n\geq0}M_n(q)x^n$ and $m(x)=\sum_{n\geq0}m_nx^n$. Using [M2, Lemma 2.3], we get that

$$\begin{aligned}(1-x-xqm(xq))M(x;q)-(1+x+xqm(-xq))M(-x;q)\\ =-xq(m(xq)+m(-xq)),\end{aligned}$$

and

$$\begin{aligned}(1-x+xqm(-xq))M(x;q)+(1+x-xqm(xq))M(-x;q)\\ =2-xq(m(xq)-m(-xq)).\end{aligned}$$

Hence, solving the above two equations for the variables $M(x;q)$ and $M(-x;q)$, and using the fact that $m(x)=1+xC(x^2)$ (see [SimSch]), we obtain the desired result. □

As a corollary of the above theorem, we have the following result concerning the sign-and-"ldes" statistic on R_n.

Corollary 3.2. *For all $n\geq1$, we have*

$$\sum_{\pi\in R_{2n}}\mathrm{sign}(\pi)(-1)^{\mathrm{ldes}(\pi)}=2C_n \text{ and } \sum_{\pi\in R_{2n-1}}\mathrm{sign}(\pi)(-1)^{\mathrm{ldes}(\pi)}=C_{n-1}.$$

Moreover,

$$1+\sum_{n\geq1}\sum_{\pi\in R_n}\mathrm{sign}(\pi)(-1)^{\mathrm{ldes}(\pi)}x^n=(1+x+x^2C(x^2))C(x^2).$$

More generally, Theorem 3.1 yields the following result.

Corollary 3.3. *For all $n\geq1$, we have*

$$\sum_{\pi\in R_{2n}}\mathrm{sign}(\pi)q^{\mathrm{ldes}(\pi)}=(1-q)\sum_{\pi\in R_{2n+1}}\mathrm{sign}(\pi)q^{\mathrm{ldes}(\pi)},$$

and

$$\sum_{\pi\in R_{2n-1}}\mathrm{sign}(\pi)q^{\mathrm{ldes}(\pi)}=\sum_{\pi\in R_{n-1}}q^{2(\mathrm{ldes}(\pi))}.$$

Theorem 3.4. *We have*

$$\begin{aligned}\sum_{n\geq0}N_n(q)x^n&=\sum_{n\geq0}\sum_{\pi\in R_n}\mathrm{sign}(\pi)q^{\mathrm{fdes}(\pi)}\\&=1+x\frac{(1-q)(1-xq+2q)-q(1-x^2)(1+xq-2q)C(x^2)}{(1-x)(1-x^2q^2)}.\end{aligned}$$

Proof. Let $L_n^{\pm}(q)=\sum_{\pi\in R_n^{\pm}}q^{\mathrm{fdes}(\pi)}$, and let $\pi=(\pi',n,\pi'')\in R_n$ such that $\pi_{j+1}=n$, $0\leq j\leq n-1$. Then, for this block decomposition of π, we have

$$\mathrm{sign}(\pi)=(-1)^{(j+1)(n-j-1)}\,\mathrm{sign}(\pi')\,\mathrm{sign}(\pi'')=(-1)^{(j+1)(n-1)}\,\mathrm{sign}(\pi')\,\mathrm{sign}(\pi''),$$

or, equivalently,

$$\mathrm{sign}(\pi)=\begin{cases}\mathrm{sign}(\pi')\cdot\mathrm{sign}(\pi''), & \text{if } n \text{ is odd},\\ (-1)^{j+1}\cdot\mathrm{sign}(\pi')\cdot\mathrm{sign}(\pi''), & \text{if } n \text{ is even}.\end{cases}$$

Therefore, for $n \geq 1$, we obtain

$$\begin{aligned}
L^{\pm}_{2n+1}(q) &= \sum_{j=0}^{2n-1} q^{j+1} L^{\pm}_{2n-j} + (L^{\pm}_j(q) - \epsilon^{\pm}) e_{2n-j} + (L^{\mp}_j(q) - \epsilon^{\mp}) o_{2n-j} \\
L^{\pm}_{2n}(q) &= \sum_{j=0,2,4,\ldots,2n-2} q^{j+1} L^{\mp}_{2n-1-j}(q) + (L^{\mp}_j(q) - \epsilon^{\mp}) e_{2n-1-j} + (L^{pm}_j(q) - \epsilon^{\pm}) o_{2n-1-j} \\
&\quad + \sum_{j=1,3,\ldots,2n-3} q^{j+1} L^{\pm}_{2n-1-j} \\
&\quad + \sum_{j=1,3,\ldots,2n-3} (L^{\pm}_j - \epsilon^{\pm}) e_{2n-1-j} + (L^{\mp}_j - \epsilon^{\mp}) o_{2n-1-j},
\end{aligned}$$

where $\epsilon^+ = 1$ and $\epsilon^- = 0$. Hence, for all $n \geq 1$, we have

$$\begin{aligned}
N_{2n+1}(q) &= 1 + \sum_{j=0}^{2n-1} q^{j+1} m_{2n-j} + \sum_{j=0}^{2n} (N_j(q) - 1) m_{2n-j}, \\
M_{2n}(q) &= 1 - \sum_{j=0,2,\ldots,2n-2} q^{j+1} m_{2n-1-j} + \sum_{j=1,3,\ldots,2n-3} q^{j+1} m_{2n-1-j} \\
&\quad - \sum_{j=0,2,\ldots,2n-2} (N_j(q) - 1) m_{2n-1-j} + \sum_{j=1,3,\ldots,2n-1} (N_j(q) - 1) m_{2n-1-j}.
\end{aligned}$$

Let $N(x;q) = \sum_{n\geq 0} N_n(q) x^n$ and $m(x) = \sum_{n\geq 0} m_n x^n$. Using [M2, Lemma 2.3], we get that

$$\begin{aligned}
\frac{1}{2}(N(x;q) - N(-x;q)) &= \frac{x}{1-x^2} + \frac{xq}{2}\left(\frac{m(x)}{1-xq} + \frac{m(-x)}{1+xq} - \frac{2}{1-x^2q^2}\right) \\
&\quad + \frac{x}{2}\left[\left(N(x;q) - \frac{1}{1-x}\right) m(x) + \left(N(-x,q) - \frac{1}{1+x}\right) m(-x)\right],
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}(N(x;q) + N(-x,q)) &= \frac{1}{1-x^2} - \frac{xq}{2(1-x^2q^2)}(m(x) - m(-x)) \\
&\quad + \frac{x^2q^2}{2(1-x^2q^2)}(m(x) + m(-x) - 2) \\
&\quad - \frac{x}{4}\left(N(x;q) + N(-x;q) - \frac{2}{1-x^2}\right)(m(x) - m(-x)) \\
&\quad + \frac{x}{4}\left(N(x;q) - N(-x;q) - \frac{2x}{1-x^2}\right)(m(x) + m(-x)).
\end{aligned}$$

Hence, solving the above two equations for the variables $N(x;q)$ and $N(-x;q)$, and using the fact that $m(x) = 1 + xC(x^2)$ (see [SimSch]), we get the desired result. □

4. Dyck paths

Following [Kr], we define a bijection Φ between permutations in $S_n(132)$ and Dyck paths from the origin to the point $(2n, 0)$. Let $\pi = \pi_1 \ldots \pi_n$ be a 132-avoiding permutation. We read the permutation π from left to right and successively generate a Dyck path. When π_j is read, then in the path we adjoin as many up-steps as necessary, followed by a down-step from height h_j+1 to height h_j (measured from the x-axis), where h_j is the number of elements in $\pi_{j+1}, \pi_{j+2}, \dots, \pi_n$ which are larger than π_j. For example, if $\pi = 534261 \in S_6$, then the corresponding Dyck path is the one shown in Figure 1.

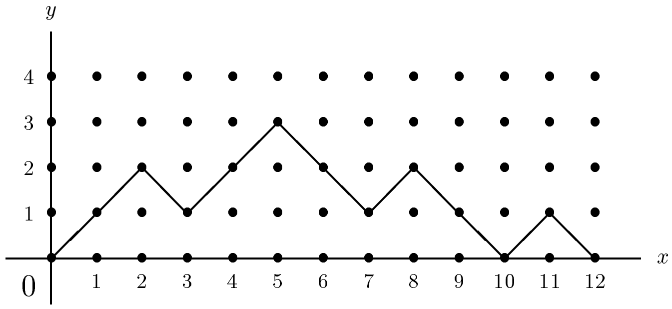

FIGURE 1. The Dyck path $\Phi(534261)$.

Namely, the first element to be read is 5. There is one element in 34261 which is larger than 5, therefore the path starts with two up-steps followed by a down-step, thus reaching height 1. Next 3 is read. There are 2 elements in 4261 which are larger than 3, therefore the path continues with two up-steps followed by a down-step, thus reaching height 2. Etc. Conversely, given a Dyck path starting at the origin and returning to the x-axis, the obvious inverse of the bijection Φ produces a 132-avoiding permutation.

Theorem 4.1. *For all $n \geq 0$, we have*
$$\sum_{p\in\mathcal{P}_n} q^{n-\mathrm{lpeak}(p)} = \sum_{\pi\in R_n} q^{\mathrm{ldes}(\pi)}.$$

Proof. We prove that $n+1-\mathrm{lpeak}(\Phi(\pi)) = fdes(\pi)$ for any $\pi \in R_n$. Let $\pi = (\pi', 1, \pi'') \in R_n$ such that $\pi_1 = j$. Since π avoids 132, the letters of π'' are increasing (i.e., $\pi''_a < \pi''_b$ for all $a < b$). Therefore, by definition of Φ, the Dyck path $p = \Phi(\pi)$ satisfies the equation $\mathrm{lpeak}(p) = |\pi''| + 1 = n + 1 - j$, where $|\pi''|$ is the number of letters in π''. $\square$

Remark 4.2. *The relation $n+1-\mathrm{peak}(\Phi(\pi)) = \mathrm{fdes}(\pi)$ for $\pi \in R_n$ can be read off immediately from the permutation diagram, see* [R].

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