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# EULER-MAHONIAN PARAMETERS ON COLORED PERMUTATION GROUPS

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ABSTRACT. New combinatorial statistics on colored permutation groups are introduced here. We present two different generalizations of major index and descent number, one of them is combinatorial in nature and the other is algebraic. We also present Euler-Mahonian type joint distributions of our parameters.

## 1. INTRODUCTION

One of the most active branches in Enumerative Combinatorics is the study of permutation statistics. Let $S_n$ be the symmetric group on $n$ letters and let $f_i : S_n \longrightarrow \boldsymbol{Z}_+, (1 \leq i \leq t)$ be (non-negative, integer valued) combinatorial parameters. Then one is interested in the refined enumeration of permutations according to these parameters:

$$\sum_{\pi \in S_n} q_1^{f_1(\pi)} \cdots q_t^{f_t(\pi)},$$

where $q_i$ are variables. Some of the most important parameters are the following: A *descent* in a permutation $\pi = \pi_1, \ldots, \pi_n$ is a position $i$ such that $\pi(i+1) < \pi(i)$, an *inversion* is a pair $i < j$ such that $\pi(i) > \pi(j)$, and the *major index* of $\pi$ is the sum of its descents. The last parameter was introduced by MacMahon in [9] and [10]. He called it the "greater index" . He proved algebraically that the major index and the inversion number are equi-distributed over the symmetric group. In other words:

$$\sum_{\pi \in S_n} q^{inv(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} = [n]_q!,$$

where $[n]_q = \frac{1-q^n}{1-q}$. (In fact, MacMahon proved the same result for the more general case where $S_n$ is replaced by the set of all rearrangements of a given word - see [11]. The first combinatorial proof of this result was given by Foata [5]).

A permutation statistic that is equi-distributed with the number of descents is called *Eulerian*, while a permutation statistic that is equi-distributed with the inversion number is called *Mahonian.*

A natural extension of the study of permutation statistics is the study of pairs of permutation statistics and their joint distributions. Of particular interest is the joint distribution of the descent number and the major index. The generating function for this joint distribution is given by Carlitz's q-Eulerian polynomial:

$$A_n(t,q) = \sum_{\pi \in S_n} t^{des(\pi)} q^{maj(\pi)} = \prod_{i=0}^{n} (1-tq^i) \sum_{k \geq 0} [k+1]_q^n t^k.$$

(See [4], [7]).

The following problem was first suggested by Foata:

**Problem 1.1.** *(Foata), Extend the Euler-Mahonian distribution of descent number and major index to the hyperoctahedral group $B_n$.*

A solution of this problem was given by Adin, Brenti and Roichman [1] meanwhile generalizing the concept of major index in two different ways, one of them algebraic in nature ($fmaj$) and the other combinatorial in nature ($nmaj$). Let $G_{r,n} = \mathbb{Z}_r \wr S_n$ be the group of colored permutations (see Section 2.2 below). In this paper we further extend the major index to $G_{r,n}$ in two different ways:

• The parameter $lmaj$ is equi-distributed with the length function of $G_{r,n}$. We prove the following (See Theorem 5.2):

**Theorem.**

$$\sum_{\pi \in G_{r,n}} q^{lmaj(\pi)} = \sum_{\pi \in G_{r,n}} q^{\ell(\pi)}$$

*where $\ell$ is the length function with respect to the standard generators of $G_{r,n}$ (see Section 2.2 below).*

We define also the parameter $ldes$ which is a length-oriented generalization of the descent number. The parameters $ldes$ and $lmaj$ have Euler-Mahonian type joint distribution. We prove (see Theorem 5.3):

**Theorem.**

$$\frac{\sum\limits_{\pi \in G_{r,n}} q^{lmaj(\pi)} t^{ldes(\pi)}}{\prod\limits_{i=0}^{n} (1-tq^i) \Pi_{k=1}^{n} (1+q^k t[r-1]_{qt})} = \sum_{k \geq 0} [k+1]_q^n t^k.$$

• The second direction is to generalize the negative descent number, *ndes*, and the negative major index, *nmaj*, defined for $B_n$ in [1]. The pair $(ndes, nmaj)$ has a simpler Euler-Mahonian type joint distribution. We prove (see Theorem 6.8):

**Theorem.**

$$\sum_{\pi \in G_{r,n}} q^{nmaj(\pi)} t^{ndes(\pi)} = \frac{(t^r; q^r)_{n+1}}{[r]_t} \sum_{k \geq 0} [k+1]_q^n t^k$$

*The notation* $(t^r; q^r)_{n+1}$ *will be explained in Section 2.1.*

The rest of this paper is organized as follows: In the next section we present some needed notations to be used in the sequel, including the colored permutation groups. In Section 3 we present some basic statistics on them. In Section 4 we present a formula for the length function of $G_{r,n}$ (see also [13]). In Section 5 we introduce the parameters *ldes* and *lmaj* and find their joint distributions. The parameters *nmaj* and *ndes* will be presented in Section 6 along with their joint distribution. It should be noted that, unlike *lmaj*, *nmaj* is not equi-distributed with the standard length function.

## 2. Preliminaries

2.1. **Notations.** Let $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ and let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ be the field of rational numbers and $\mathbb{C}$ the field of complex numbers.

For $a \in \mathbb{N}$ let $[a] := \{1, 2, \dots, a\}$ (where $[0] := \emptyset$). The cardinality of a set $A$ will be denoted by $|A|$. More generally, given a multiset $M = \{1^{a_1}, 2^{a_2}, \dots, r^{a_r}\}$, denote by $|M|$ its cardinality, so $|M| = \sum_{i=1}^{r} a_i$.

Given a variable $q$ and a commutative ring $R$, denote by $R[q]$ (respectively, $R[[q]]$) the ring of polynomials (respectively, formal power series) in $q$ with coefficients in $R$.

Define:

$$(a; q)_n := \begin{cases} 1 & \text{if } n = 0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{otherwise.} \end{cases}$$

Also, let:

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$$

(so $[0]_q = 0$) and

$$[n]_q! := \frac{(q; q)_n}{(1-q)^n} = [n]_q \cdot [n-1]_q \cdots [1]_q.$$

Let $n$ be a non-negative integer. A *partition* of $n$ is an infinite sequence of non-negative integers with finitely many non-zero terms $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum\limits_{i=1}^{\infty} \lambda_i = n$.

The sum $\sum \lambda_i = n$ is called the *size* of $\lambda$, denoted $|\lambda|$; write also $\lambda \vdash n$. The number of parts of $\lambda$, $\ell(\lambda)$, is the maximal $j$ for which $\lambda_j > 0$. The unique partition of $n = 0$ is the *empty partition* $\emptyset = (0, 0, \dots)$, which has length $\ell(\emptyset) := 0$.

For any partition $\lambda$ with at most $n$ positive parts let

$$m_j(\lambda) := |\ \{1 \leq i \leq n \mid \lambda_i = j\}\ | \qquad \text{(for all } j \geq 0),$$

and let $\binom{n}{\bar{m}(\lambda)}$ denote the multinomial coefficient $\binom{n}{m_0(\lambda), m_1(\lambda), \ldots}$.

## 2.2. **The Group of Colored Permutations.**

**Definition 2.1.** Let $r$ and $n$ be positive integers. The group of colored permutations of $n$ digits with $r$ colors is the wreath product $G_{r,n} = \mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n$, consisting of all the pairs $(z, \tau)$ where $z$ is an $n$-tuple of integers between 0 and $r-1$ and $\tau \in S_n$. The multiplication is defined by the following rule: For $z = (z_1, \ldots, z_n)$ and $z' = (z'_1, \ldots, z'_n)$

$$(z, \tau) \cdot (z', \tau') = ((z_1 + z'_{\tau^{-1}(1)}, \ldots, z_n + z'_{\tau^{-1}(n)}), \tau \circ \tau')$$

(here $+$ is taken mod $r$).

Here are some conventions we use along this paper:

For an element $\pi = (z, \tau) \in G_{r,n}$ with $z = (z_1, \ldots, z_n)$ we write $z_i(\pi) = z_i$. For $\pi = (z, \tau)$, we write $|\pi| = (0, \tau), (0 \in \mathbb{Z}_r^n)$.

A much more natural way to present $G_{r,n}$ is the following: Consider the alphabet $\Sigma = \{1, \ldots, n, \bar{1}, \ldots, \bar{n}, \ldots, 1^{[r-1]}, \ldots, n^{[r-1]}\}$ as the set $[n]$ colored by the colors $0, \ldots, r-1$. Then an element of $G_{r,n}$ is a *colored permutation*, i.e., a bijection $\pi : \Sigma \to \Sigma$ such that $\pi(\bar{i}) = \overline{\pi(i)}$. In this manner we write, for example, the colored permutation $(z, \tau) = ((1, 0, 3, 2), (2, 1, 4, 3)) \in G_{3,4}$ as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ \bar{2} & 1 & \bar{\bar{\bar{4}}} & \bar{\bar{3}} \end{pmatrix}$ or even just as: $\bar{2}1\bar{\bar{\bar{4}}}\bar{\bar{3}}$.

The group $G_{r,n}$ is generated by the set $S_{G_{r,n}} = \{s_0, s_1, \ldots, s_{n-1}\}$ defined by their action on the set $[n]$ as follows:

$$s_i(j) := \begin{cases} i+1, & \text{if} \quad j = i; \\ i, & \text{if} \quad j = i+1; \\ j, & \text{otherwise.} \end{cases}$$

whereas the exceptional generator $s_0$ is defined by

$$s_0(j) := \begin{cases} \bar{1}, & \text{if} \quad j = 1; \\ j, & \text{otherwise.} \end{cases}$$

Note that, unlike the case of Coxeter groups, here the set of generators is not symmetric. Note also that $G_{1,n} = C_1 \wr S_n$ is the symmetric group $S_n$ while $G_{2,n} = C_2 \wr S_n$ is the group of signed permutations, also known as the hyperoctahedral group, or the classical Weyl group of type B.

Returning now to the view of $G_{r,n}$ as a semidirect product: $\mathbb{Z}_r^n \rtimes S_n$, we note that it would be worthwhile to consider $\mathbb{Z}_r^n$ in another way: Given any order on the alphabet $\Sigma$, $\mathbb{Z}_r^n$ can be identified with the subgroup $T$ of $G_{r,n}$ consisting of the ordered permutations, i.e. those satisfying: $i < j \Longrightarrow \pi(i) < \pi(j)$ with respect to the given order. (This is done by sending a vector $z = (z_1, \ldots, z_n) \in \mathbb{Z}_r^n$ to the unique ordered permutation $\pi \in G_{r,n}$ satisfying $z_i(\pi) = z_i$. For example, given the order $\bar{3} < \bar{2} < \bar{1} < 1 < 2 < 3$ on the set $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$, the vector $(0, 1, 1) \in \mathbb{Z}_2^3$ corresponds to the colored permutation $(\bar{3}\bar{2}1) \in G_{2,3}$). Note that if we denote by $\mathbb{S}$ the group, isomorphic to $S_n$, generated by $S = S_{G_{r,n}} - \{s_0\}$, then $T$ is just a set of coset representatives of $\mathbb{S}$. (Indeed, any colored permutation $\pi \in G_{r,n}$ can be written uniquely in the following way: $\pi = \sigma \cdot u$ where $\sigma$ is an ordered permutation and $u \in \mathbb{S}$).

## 3. Basic Statistics on $G_{r,n}$

For $\pi \in G_{r,n}$ we define the *negative set* of $\pi$ by :

$$Neg(\pi) = \{i | z_i(\pi) \neq 0\}. \tag{1}$$

The size of the set $Neg(\pi)$ will be denoted by $neg(\pi)$. The following parameters we define on $G_{r,n}$ depend on the assumption that we have some order on the alphabet $\Sigma$.

Let $\pi \in G_{r,n}$. We say that the pair $i < j$ is an inversion of $\pi$ if $\pi(i) > \pi(j)$. The number of inversions in $\pi$ is denoted by $inv(\pi)$.

$i \in [n-1]$ is a *descent* of $\pi$ if $\pi(i) > \pi(i+1)$. We define:

$$Des(\pi) := \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$$

to be the *descent set* of $\pi$ and we denote by $des(\pi)$ the size of $Des(\pi)$. We also let

$$maj(\pi) := \sum_{i \in Des(\pi)} i$$

and call it the *major index* of $\pi$.

For example, consider the order $\bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3$ defined on the set $\{1,2,3,\bar{1},\bar{2},\bar{3}\}$. Also consider the colored permutation:

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ \bar{3} & 2 & \bar{1} \end{pmatrix}.$$

Then one has $inv(\pi) = 2, des(\pi) = 1$ and $maj(\pi) = 2$.

Now, given any order on the alphabet $\Sigma$, we write $\pi = \sigma \cdot u$ where $\sigma \in T$ and $u \in \mathbb{S}$ ($T$ and $\mathbb{S}$ were defined at the end of Section 2.2). We present a few simple facts concerning this coset decomposition which will be used later.

$$des(\pi) = des(u). \tag{2}$$

$$neg(\pi) = neg(\sigma). \tag{3}$$

$$\sum_{i=1}^{n} z_i(\pi) = \sum_{i=1}^{n} z_i(\sigma). \tag{4}$$

$$maj(\pi) = maj(u). \tag{5}$$

$$\sum_{z_i(\pi)\neq 0} |\pi(i)| = \sum_{z_i(\sigma)\neq 0} |\sigma(i)|. \tag{6}$$

## 4. A Length function for $G_{r,n}$

In this section we present a formula for the length of $G_{r,n}$ with respect to $S_{G_{r,n}}$. A similar expression appears in [13]. The proof of our formula depends on the length order we define next although the length function itself is independent of order. We start with the definition of a length function for $G_{r,n}$:

**Definition 4.1.** For every $\pi \in G_{r,n}$ define the *length* of $\pi$ with respect to the set of the generators $S_{G_{r,n}} = \{s_0, s_1, \dots, s_{n-1}\}$ to be the minimal number of generators satisfying that their product is $\pi$. Formally:

$$\ell(\pi) = min\{r \in N : \pi = s_{i_1} \cdots s_{i_r}, \text{for some } i_1, \dots, i_r \in [0, n-1]\}.$$

**Definition 4.2.** The *length order* on the alphabet

$$\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$$

is defined as follows:

$$n^{[r-1]} < \cdots < \bar{n} < \cdots < 1^{[r-1]} < \cdots < \bar{1} < 1 < \cdots < n.$$

**Theorem 4.3.** *For every $\pi \in G_{r,n}$:*

$$\ell(\pi) = inv(\pi) + \sum_{z_i(\pi)\neq 0} (|\pi(i)| + z_i(\pi) - 1)$$

*where $inv(\pi)$ is calculated with respect to the length order defined above.*

*Proof.* Denote

$$L(\pi) = inv(\pi) + \sum_{z_i(\pi)\neq 0} (|\pi(i)| + z_i(\pi) - 1).$$

We prove first that $\ell(\pi) \leq L(\pi)$ by presenting an algorithm which expresses $\pi$ as a product of $L(\pi)$ generators. For $\pi \in G_{r,n}$ write $\pi = \sigma u$ where $\sigma \in T, u \in \mathbb{S}$. Our algorithm sends the identity permutation first to $\sigma$ and then from $\sigma$ to $\sigma u = \pi$. This is done by multiplying on the right by Coxeter generators.

Start with the identity permutation.

- For every $j = |\pi(i)|$ such that $i \in Neg(\pi)$, in increasing order of $j$:
  - Move $j$ to place 1 by multiplying on the right by the $j-1$ successive decreasing generators: $s_{j-1}, \ldots, s_1$.
  - Equip $j$ with $z_i(\pi)$ bars, to get $\pi(i)$. This will be done by multiplying on the right by $s_0^{z_i(\pi)}$ and thus will cost exactly $z_i(\pi)$ steps.

  After doing this process once for every 'colored digit' we get $\sigma \in T$, i.e., $\sigma$ is increasing according to the length order.
- Mix the permutation $\sigma$ in order to get $\pi$ out of it. This will cost $inv(\pi)$ steps.

**Example:** $\pi = \bar{\bar{3}}\bar{2}4\bar{1}$. Here $\sigma = \bar{\bar{3}}\bar{2}\bar{1}4$, and $u = 1243$.

The process is (Multiplication is always on the right):

$$1234 \xrightarrow{s_0} \bar{1}234 \xrightarrow{s_1} 2\bar{1}34 \xrightarrow{s_0} \bar{2}\bar{1}34 \xrightarrow{s_2} \bar{2}3\bar{1}4 \xrightarrow{s_1} 3\bar{2}\bar{1}4 \xrightarrow{s_0} \bar{3}\bar{2}\bar{1}4 \xrightarrow{s_0} \bar{\bar{3}}\bar{2}\bar{1}4 = \sigma.$$

Now, we are left with an increasing ordered permutation which we have to mix. This will be done by $inv(\pi)$ elements of the generating set $S = S_{G_{r,n}} - \{s_0\}$ which form $u$ and thus we have $\pi = \sigma u$. In summary, we used $inv(\pi) + \sum\limits_{z_i(\pi)\neq 0} (|\pi(i)| - 1) + \sum\limits_{i=1}^{n} z_i$ steps. This proves $\ell(\pi) \leq L(\pi)$.

We prove now the other direction. For $r = 2$, $\mathbb{Z}_r \wr S_n$ is the Coxeter group of type $B$. In this case it is known that $L(\pi) = \ell(\pi)$ (See for example [3]).

Take $r > 2$ and let $\pi = ((z_1, \ldots, z_n), \tau) \in G_{r,n}$. We construct $\pi' = ((z'_1, \ldots, z'_n), \tau) \in G_{2,n}$ by defining:

$$z'_i = \begin{cases} 0 & z_i = 0; \\ 1 & z_i > 0. \end{cases}$$

For example, if $\pi = \bar{\bar{4}}\bar{3}2\bar{1} \in G_{3,4}$ then $\pi' = \bar{4}\bar{3}2\bar{1} \in G_{2,4}$.

Denoting $k = \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} z'_i$ we have, by the construction of $\pi'$, that $L(\pi) = L(\pi') + k$. Now, assume to the contrary that $L(\pi) > \ell(\pi)$. We have $\ell(\pi) \geq \ell(\pi') + k$. (Indeed, take any reduced word representing $\pi$ and delete from it the $k$ occurrences of the generator $s_0$, which are responsible for coloring the digits in $\pi$ that are reducing colors in the passage to $\pi'$. This will give us a word representing $\pi'$).

We have now:

$$L(\pi') + k = L(\pi) > \ell(\pi) \geq \ell(\pi') + k$$

which contradicts the fact that $L(\pi') = \ell(\pi')$ in $G_{2,n}$. $\square$

We proceed to the calculation of the generating function of the length function.

**Theorem 4.4.**

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = [n]_q! \cdot \prod_{i=1}^{n} (1 + q^i [r-1]_q).$$

*Proof.* If $\pi \in G_{r,n}$ then using the coset decomposition we can write $\pi = \sigma \cdot u$ where $\sigma \in T, u \in \mathbb{S}$. As can be easily deduced by the proof of Theorem 4.3, the length function can also be written as:

$$\ell(\pi) = inv(u) + \ell(\sigma)$$

where $\ell(\sigma) = \sum_{z_i(\sigma) \neq 0} (|\sigma(i)| - 1 + z_i(\sigma))$ is the length function of $\sigma$.

Now, it is well known that $\sum_{u \in \mathbb{S}} q^{inv(u)} = [n]_q!$. Thus,

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = \sum_{u \in \mathbb{S}} q^{inv(u)} \cdot \sum_{\sigma \in T} q^{\ell(\pi)} = [n]_q! A_n$$

where by induction

$$A_n = (1 + q^{n-1+1} + q^{n-1+2} + \cdots + q^{n-1+r-1}) A_{n-1}.$$

We have in summary:

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = [n]_q! \prod_{i=1}^{n} (1 + q^i [r-1]_q).$$

$\square$

## 5. The Parameter $lmaj$

In this section we introduce the first generalization of the parameter $maj$ defined for the symmetric groups and prove its equi-distribution with the length function of $G_{r,n}$. We introduce also the parameter $ldes$ which is a generalization of the parameter $des$ defined for the symmetric groups. We start with the following definitions:

**Definition 5.1.** For every $\pi \in G_{r,n}$ we define:

$$ldes(\pi) = des(\pi) + \sum_{i=1}^{n} z_i(\pi)$$

and

$$lmaj(\pi) = maj(\pi) + \sum_{z_i(\pi) \neq 0} (|\pi(i)| - 1) + \sum_{i=1}^{n} z_i(\pi)$$

where the descents are computed with respect to the length order.

**Theorem 5.2.** *The parameter $lmaj$ equi distributes with the length function over $G_{r,n}$, i.e.,*

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = \sum_{\pi \in G_{r,n}} q^{lmaj(\pi)}.$$

*Proof.* We use the semidirect decomposition of $G_{r,n}$ and the equations (3), (4), (5) and (6) to get:

$$\begin{aligned}
\sum_{\pi \in G_{r,n}} q^{lmaj(\pi)} &= \sum_{\sigma u \in G_{r,n}} q^{maj(\sigma u) + \sum\limits_{z_i(\sigma u) \neq 0} (|\sigma u(i)| - 1 + z_i(\sigma u))} \\
&= \sum_{u \in \mathbb{S}} q^{maj(u)} \sum_{\sigma \in T} q^{\sum\limits_{z_i(\sigma) \neq 0} (|\sigma(i)| - 1 + z_i(\sigma))} \\
&= \sum_{u \in \mathbb{S}} q^{inv(u)} \sum_{\sigma \in T} q^{\sum\limits_{z_i(\sigma) \neq 0} (|\sigma(i)| - 1 + z_i(\sigma))} \\
&= \sum_{\pi \in G_{r,n}} q^{\ell(\pi)}.
\end{aligned}$$

□

5.1. **Euler-Mahonian Type Distribution.** In this section we present an Euler-Mahonian type bi-distribution for the parameters $lmaj$ and $ldes$ over $G_{r,n}$.

**Theorem 5.3.**
$$\frac{\sum\limits_{\pi\in G_{r,n}} t^{ldes(\pi)}q^{lmaj(\pi)}}{(t;q)_{n+1}(1+q^k t[r-1]_{qt})} = \sum_{k\geq 0}[k+1]_q^n t^k.$$

*Proof.* We use the semidirect decomposition of $G_{r,n}$ and the equations (2), (3), (4), (5) and (6) to get:

$$\begin{aligned}\sum_{\pi\in G_{r,n}} t^{ldes(\pi)}q^{lmaj(\pi)} &= \sum_{\pi\in G_{r,n}} t^{des(\pi)+\sum\limits_{i=1}^{n} z_i(\pi)} q^{maj(\pi)+\sum\limits_{z_i(\pi)\neq 0}(|\pi(i)|-1+z_i(\pi))} \\ &= \sum_{\sigma\in T}\sum_{u\in\mathbb{S}} t^{des(u)+\sum\limits_{i=1}^{n} z_i(\sigma)} q^{maj(u)+\sum\limits_{z_i(\sigma)\neq 0}(|\sigma(i)|-1+z_i(\sigma))} \\ &= \sum_{\sigma\in T} t^{\sum\limits_{i=1}^{n} z_i(\sigma)} q^{\sum\limits_{z_i(\sigma)\neq 0}(|\sigma(i)|-1+z_i(\sigma))} \sum_{u\in\mathbb{S}} t^{des(u)}q^{maj(u)}.\end{aligned}$$

By [1, Theorem 2.2], we have:

$$\sum_{u\in\mathbb{S}} t^{des(u)}q^{maj(u)} = \prod_{i=0}^{n}(1-tq^i)\sum_{k\geq 0}[k+1]_q^n t^k, \tag{7}$$

so we are left with the sum :

$$\sum_{\sigma\in T} t^{\sum\limits_{i=1}^{n} z_i(\sigma)} q^{\sum\limits_{z_i(\sigma)\neq 0}(|\sigma(i)|-1+z_i(\sigma))}.$$

By the same technique we adopted in the proof of Theorem 4.4 we can prove that

$$\sum_{\sigma\in T} t^{\sum\limits_{i=1}^{n} z_i(\sigma)} q^{\sum\limits_{z_i(\sigma)\neq 0}(|\sigma(i)|-1+z_i(\sigma))} = \prod_{i=1}^{n}(1+q^i t[r-1]_{qt}).$$

Combining this with equation (7) we get:

$$\sum_{\pi\in G_{r,n}} t^{ldes(\pi)}q^{lmaj(\pi)} =$$
$$\prod_{k=1}^{n}(1+q^k t[r-1]_{qt})\prod_{i=0}^{n}(1-tq^i)\sum_{k\geq 0}[k+1]_q^n t^k,$$

and thus

$$\frac{\sum\limits_{\pi\in\mathbb{Z}_r\wr S_n} t^{ldes(\pi)}q^{lmaj(\pi)}}{\Pi_{i=0}^{n}(1-tq^i)\Pi_{k=1}^{n}(1+q^k t[r-1]_{qt})} = \sum_{k\geq 0}[k+1]_q^n t^k.$$

$\square$

## 6. The Parameter *nmaj*

In this section we define another parameter on $\mathbb{Z}_r \wr S_n$. This parameter is a generalization of the parameter $nmaj$ defined in [2]. The results of this section do not depend depend on the order one chooses on the alphabet $\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$.

**Definition 6.1.** For $\pi \in \mathbb{Z}_r \wr S_n$ define the multiset:

$$NNeg(\pi) = \{i^{z_i(\pi)}\}.$$

Note that each $i$ with $z_i(\pi) > 0$ appears $z_i(\pi)$ times.

**Definition 6.2.** For $\pi \in G_{r,n}$ define the multiset:

$$NDes(\pi) = Des(\pi) \uplus NNeg(\pi^{-1}).$$

We define also $ndes(\pi) = |NDes(\pi)|$.

**Definition 6.3.** For $\pi \in G_{r,n}$ define

$$nmaj(\pi) = \sum_{i \in NDes(\pi)} i,$$

and also

$$ndes(\pi) = |NDes(\pi)|.$$

**Example 6.4.** *Consider the order* $\bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3$ *and take* $\pi = \bar{3}\bar{\bar{1}}2$. *Here* $Des(\pi) = \{1\}$, $\pi^{-1} = \bar{2}3\bar{\bar{1}}$, $NNeg(\pi^{-1}) = \{1, 3, 3\}$, $NDes(\pi) = \{1, 1, 3, 3\}$, $nmaj(\pi) = 1 + 1 + 3 + 3 = 8$ *and* $ndes(\pi) = 4$.

We define also some refinements of the parameters $ndes$ and $nmaj$. They will be used in the proof of the main result of this section.

**Definition 6.5.** For every $\pi \in G_{r,n}$ define

$$d_i(\pi) := |\{j \in Des(\pi) : j \geq i\}| \quad (1 \leq i \leq n).$$

This is the number of descents in $\pi$ from position $i$ on.

Define also for every $\pi \in G_{r,n}$

$$n_i(\pi) = |\{j \in NNeg(\pi) : j \geq i\}|.$$

Note that $n_i(\pi)$ counts the number of colors from position $i$ on.

**Observation 6.6.** *Let* $\pi \in G_{r,n}$. *Then*

$$ndes(\pi) = d_1(\pi) + n_1(\pi^{-1}),$$

$$nmaj(\pi) = \sum_{i=1}^{n} [d_i(\pi) + n_i(\pi^{-1})].$$

6.1. **Euler-Mahonian Distribution.** In what follows we present an Euler-Mahonian type distribution of the parameters $ndes$ and $nmaj$ defined earlier. We use here the Hilbert series of the algebra $\mathbb{C}[q_1,\ldots,q_n]$ with respect to multi-degree rearranged into a weakly decreasing sequence, i.e., a partition. The right-hand side in the following result is the Hilbert series of the above algebra.

**Theorem 6.7.**

$$\sum_{\ell(\lambda)\leq n}\binom{n}{\bar{m}(\lambda)}\prod_{i=1}^{n}q_i^{\lambda_i}=\frac{\sum\limits_{\pi\in G_{r,n}}\prod\limits_{i=1}^{n}q_i^{d_i(\pi)+n_i(\pi^{-1})}}{\prod\limits_{i=1}^{n}(1-q_1^r\cdots q_i^r)}$$

*in* $\mathbb{C}[[q_1,\ldots,q_n]]$.

*Proof.* Recall from Section 3 the definition of $T\subseteq G_{r,n}$ as the set of ordered permutations. As already shown in Section 3, $T$ can be seen as a copy of $\mathbb{Z}_r^n$ and thus the fact that $G_{r,n}=\mathbb{Z}_r^n\rtimes S_n$ implies that every $\pi\in G_{r,n}$ can be uniquely written as $\pi=\sigma u$ where $\sigma\in T$ and $u\in\mathbb{S}$, where $\mathbb{S}$ is the subgroup of $G_{r,n}$ generated by $S_{G_{r,n}}-\{s_0\}$.

It is clear from the definitions that $d_i(\sigma u)=d_i(u)$ and $n_i(u^{-1}\sigma^{-1})=n_i(\sigma^{-1})$ for all $\sigma\in T$, $u\in S_n$ and $1\leq i\leq n$. Therefore

$$\begin{aligned}\sum_{\pi\in G_{r,n}}\prod_{i=1}^{n}q_i^{d_i(\pi)+n_i(\pi^{-1})} &= \sum_{u\in\mathbb{S}}\sum_{\sigma\in T}\prod_{i=1}^{n}q_i^{d_i(\sigma u)+n_i((\sigma u)^{-1})}\\ &= \sum_{u\in\mathbb{S}}\sum_{\sigma\in T}\prod_{i=1}^{n}q_i^{d_i(u)+n_i(\sigma^{-1})}\\ &= \sum_{u\in\mathbb{S}}\prod_{i=1}^{n}q_i^{d_i(u)}\cdot\sum_{\sigma\in T}\prod_{i=1}^{n}q_i^{n_i(\sigma^{-1})}.\end{aligned}$$

In [2, Theorem 6.2], it is proven that the Hilbert series with respect to multi-degree of $\mathbb{C}[q_1,\ldots,q_n]$ can be written as a product of the generating function of the descent basis for type A and the generating function of the symmetric functions. Explicitly:

$$\sum_{\ell(\lambda)\leq n}\binom{n}{\bar{m}(\lambda)}\prod_{i=1}^{n}q_i^{\lambda_i}=\frac{\sum\limits_{\pi\in S_n}\prod\limits_{i=1}^{n}q_i^{d_i(\pi)}}{\prod\limits_{i=1}^{n}(1-q_1\cdots q_i)}\tag{8}$$

in $\mathbb{C}[[q_1,\dots,q_n]]$, where the sum on the left-hand side is taken over all partitions with at most $n$ parts.

Thus, in order to complete the proof we have to prove that:

$$\sum_{\sigma\in T}\prod_{i=1}^{n} q_i^{n_i(\sigma^{-1})} = \frac{\prod\limits_{i=1}^{n}(1-q_1^r\cdots q_i^r)}{\prod\limits_{i=1}^{n}(1-q_1\cdots q_i)}. \tag{9}$$

Define a function $\phi:\{0,\dots,r-1\}\to\{0,..,r-1\}$ by

$$\phi(i)=\begin{cases} 0 & i=0\\ r-i & i\neq 0\end{cases}.$$

Note that there is a bijection between the elements $\sigma\in T$ and the multisets of the form $\{1^{j_1},\dots,n^{j_n}\}$ where $0\le j_i\le r-1$, given by

$$\sigma\mapsto NNeg(\sigma^{-1})=\{|\sigma(i)|^{\phi(z_i)(\sigma)}\}.$$

(Indeed, given any multiset $A=\{1^{j_1},\dots,n^{j_n}\}$, define $\tau$ by $\tau(i)=i_{\phi(j_i)}$ for every $1\le i\le n$ and then order $\tau$ to get $\sigma$. For example, given $A=\{1,1,2,2,3\}$, we form $\tau=\bar{1}\bar{2}\bar{\bar{3}}4$ and then $\sigma=\bar{\bar{3}}\bar{2}\bar{1}4$).

Now, in order to calculate the sum $\sum\limits_{\sigma\in T}\prod\limits_{i=1}^{n} q_i^{n_i(\sigma^{-1})}$, we can run over the multisets of the form $\{1^{j_1},\dots,n^{j_n}\}$ where $0\le j_i\le r-1$. Here, every $i$, inserted to such a multiset $j_i$ times, contributes the monomial $(q_1\cdots q_i)^{j_i}$ to our sum. This gives us:

$$\begin{aligned}
&\sum_{\sigma\in T}\prod_{i=1}^{n} q_i^{n_i(\sigma^{-1})}\\
&\quad=(1+q_1+\cdots+q_1^{r-1})(1+q_1q_2+(q_1q_2)^2+\cdots+(q_1q_2)^{r-1})\\
&\qquad\qquad\cdots(1+q_1\cdots q_n+(q_1\cdots q_n)^2+\cdots+(q_1\cdots q_n)^{r-1})\\
&\quad=\frac{\prod\limits_{i=1}^{n}(1-q_1^r\cdots q_i^r)}{\prod\limits_{i=1}^{n}(1-q_1\cdots q_i)}.
\end{aligned}$$

□

This leads us to the following generalization of the Carlitz identity for the parameters $ndes$ and $nmaj$.

**Theorem 6.8.**

$$\sum_{\pi\in G_{r,n}} q^{nmaj(\pi)}t^{ndes(\pi)}=\frac{(t^r;q^r)_{n+1}}{[r]_t}\sum_{k\ge 0}[k+1]_q^n t^k.$$

*Proof.* Substitute, in Theorem 6.7, $q_1 = qt$ and $q_2 = q_3 = \cdots = q_n = q$ to get:

$$\sum_{\ell(\lambda)\le n} \binom{n}{\bar{m}(\lambda)} q^{\sum\limits_{i=1}^{n} \lambda_i} \cdot t^{\lambda_1} = \frac{\sum\limits_{\pi\in G_{r,n}} q^{nmaj(\pi)} t^{ndes(\pi)}}{\prod\limits_{i=1}^{n} (1 - t^r q^{ir})}.$$

Dividing by $(1-t)$ we have:

$$\frac{\sum\limits_{\pi\in G_{r,n}} q^{nmaj(\pi)} t^{ndes(\pi)}}{(1-t)\prod\limits_{i=1}^{n} (1 - t^r q^{ir})} = \sum_{k=0}^{\infty} \sum_{\substack{\ell(\lambda)\le n \\ \lambda_1 \le k}} \binom{n}{\bar{m}(\lambda)} q^{\sum\limits_{i=1}^{n} \lambda_i} t^k,$$

and thus the coefficient of $t^k$ is

$$\sum_{\substack{\ell(\lambda)\le n \\ \lambda_1 \le k}} \binom{n}{\bar{m}(\lambda)} q^{\sum\limits_{i=1}^{n} \lambda_i} = \sum_{(l_1,\dots,l_n)\in[0,k]^n} q^{\sum\limits_{i=1}^{n} l_i} = (\sum_{j=0}^{k} q^j)^n = [k+1]_q^n,$$

and we have proved:

$$\begin{aligned}\sum_{\pi\in G_{r,n}} q^{nmaj(\pi)} t^{ndes(\pi)} &= (1-t)\prod_{i=1}^{n} (1 - t^r q^{ir}) \sum_{k\ge 0} [k+1]_q^n t^k \\ &= \frac{(t^r;q^r)_{n+1}}{[r]_t} \sum_{k\ge 0} [k+1]_q^n t^k.\end{aligned}$$

□

## Appendix A. The distribution of *nmaj*

After the completion of this work we were told about the preprint of J. Haglund, N. Loehr and J. B. Remmel [8] : 'Statistics on wreath products, perfect matchings and signed words'. A variant of the parameter defined here as *nmaj* appears also in their work. Its distribution over $G_{r,n}$ is proven there to be $\prod\limits_{i=1}^{n} [ri]_q$. We enclose here another proof of this distribution, based on the proof of Theorem 6.8 of the last section.

**Theorem Appendix A.1.**

$$\sum_{\pi\in G_{r,n}} q^{nmaj(\pi)} = \prod_{i=1}^{n} [ri]_q.$$

*Proof.* We write:

$$\sum_{\pi\in G_{r,n}} q^{nmaj(\pi)} = \sum_{\pi\in G_{r,n}} q^{\sum_{i=1}^{n} d_i(\pi)+\sum_{i=1}^{n} n_i(\pi^{-1})}$$
$$= \sum_{u\in\mathbb{S}} q^{\sum_{i=1}^{n} d_i(u)} \sum_{\sigma\in T} q^{\sum_{i=1}^{n} n_i(\sigma^{-1})}.$$

By substituting $q = q_1 = \cdots = q_n$ in equation 8 we get:

$$\sum_{u\in\mathbb{S}} q^{\sum_{i=1}^{n} d_i(u)} = Hilb(\mathbb{C}[x_1,\ldots,x_n],q)\cdot\prod_{i=1}^{n}(1-q^i)$$

but on the other hand

$$Hilb(\mathbb{C}[x_1,\ldots,x_n],q) = \frac{1}{(1-q)^n}$$

so we have

$$\sum_{u\in\mathbb{S}} q^{\sum_{i=1}^{n} d_i(u)} = \prod_{i=1}^{n}(1-q^i)\cdot\frac{1}{(1-q)^n}.$$

We turn now to the calculation of $\sum_{\sigma\in T} q^{\sum_{i=1}^{n} n_i(\sigma^{-1})}$.

Substituting $q = q_1 = \cdots = q_n$ in equation 9 we get:

$$\sum_{\sigma\in T} q^{\sum_{i=1}^{n} n_i(\sigma^{-1})} = \frac{\prod_{i=1}^{n}(1-q^{ri})}{\prod_{i=1}^{n}(1-q^{i})}.$$

We have now:

$$\sum_{\pi\in G_{r,n}} q^{nmaj(\pi)} = \prod_{i=1}^{n}(1-q^i)\cdot\frac{1}{(1-q)^n}\cdot\frac{\prod_{i=1}^{n}(1-q^{ri})}{\prod_{i=1}^{n}(1-q^{i})} = \prod_{i=1}^{n}[ri]_q.$$

□

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