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EQUIDISTRIBUTION AND SIGN-BALANCE ON 321-AVOIDING PERMUTATIONS

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ABSTRACT. Let T_n be the set of 321-avoiding permutations of order n. Two properties of T_n are proved: (1) The *last descent* and *last index minus one* statistics are equidistributed over T_n, and also over subsets of permutations whose inverse has an (almost) prescribed descent set. An analogous result holds for Dyck paths. (2) The sign-and-last-descent enumerators for T_{2n} and T_{2n+1} are essentially equal to the last-descent enumerator for T_n. The proofs use a recursion formula for an appropriate multivariate generating function.

1. INTRODUCTION

1.1. **Equidistribution.** One of the frequent themes in combinatorics is identifying two distinct parameters on the same set which are *equidistributed*, i.e., share the same generating function. The first substantial result of this kind on permutations, by MacMahon [15], received a remarkable refinement by Foata and Schützenberger [9] (see also [12, 10]). They proved the equidistribution of inversion number and major index, not only on the whole symmetric group, but also on distinguished subsets of permutations (those whose inverses have a prescribed descent set).

Equidistribution theorems on descent classes were shown to be closely related to the study of polynomial rings; see, e.g., [20, 1]. Motivated by the properties of certain quotient rings, studied by Aval and Bergeron [3, 4] (see also Section 6 below), we looked for an analogue for the set of 321-avoiding permutations of the above mentioned theorem of Foata and Schützenberger.

Let S_n be the symmetric group on n letters, and let

$$T_n := \{\pi \in S_n \mid \nexists\, i < j < k \text{ such that } \pi(i) > \pi(j) > \pi(k)\}$$

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be the set of 321-avoiding (or two-row shaped) permutations in S_n. For $\pi \in T_n$ define the following statistics:

$$\begin{aligned}
\mathrm{inv}(\pi) &:= \text{inversion number of } \pi \\
&(= \text{length of } \pi \text{ w.r.t. the usual generators of } S_n) \\
\mathrm{ldes}(\pi) &:= \text{last descent of } \pi = \max\{1 \le i \le n-1 \mid \pi(i) > \pi(i+1)\} \\
& \text{(where } \mathrm{ldes}(\pi) := 0 \text{ for } \pi = id) \\
\mathrm{lind}(\pi) &:= \pi^{-1}(n), \text{ the index of the digit ``}n\text{'' in } \pi \\
\mathrm{Des}(\pi^{-1}) &:= \text{descent set of } \pi^{-1} = \{1 \le i \le n-1 \mid \pi^{-1}(i) > \pi^{-1}(i+1)\}
\end{aligned}$$

The following theorem is a T_n-analogue of [9, Theorem 1].

Theorem 1.1. *The statistics "* ldes *" and "* lind -1 *" are equidistributed over T_n. Moreover, for any $B \subseteq [n-2]$ they are equidistributed over the set*

$$T_n(B) := \{\pi \in T_n \mid \mathrm{Des}(\pi^{-1}) \cap [n-2] = B\},$$

namely:

$$\sum_{\pi \in T_n(B)} q^{\mathrm{ldes}(\pi)} = \sum_{\pi \in T_n(B)} q^{\mathrm{lind}(\pi)-1}.$$

Let $\mathcal{P}_n$ be the set of Dyck paths of length $2n$. For $p = (p_1, \ldots, p_{2n}) \in \mathcal{P}_n$, let

$$\begin{aligned}
\mathrm{ldes}(p) &:= \max\{1 \le i \le n-1 \mid p_i = +1, p_{i+1} = -1\} \\
& \text{(where } \mathrm{ldes}(p) := 0 \text{ for } p = (+\ldots+-\ldots-)) \\
\mathrm{lind}(p) &:= \max\{1 \le i \le n \mid p_i = +1, p_{i+1} = -1\} \\
\mathrm{Des}(p^{-1}) &:= \{1 \le i \le n-1 \mid p_{2n-i} = +1, p_{2n-i+1} = -1\}
\end{aligned}$$

Theorem 1.2.

$$\sum_{\pi \in T_n} q^{\mathrm{ldes}(\pi)} = \sum_{p \in \mathcal{P}_n} q^{\mathrm{ldes}(p)} = \sum_{p \in \mathcal{P}_n} q^{\mathrm{lind}(p)-1}.$$

Moreover, for any $B \subseteq [n-2]$,

$$\sum_{\pi \in T_n(B)} q^{\mathrm{ldes}(\pi)} = \sum_{p \in \mathcal{P}_n(B)} q^{\mathrm{ldes}(p)} = \sum_{p \in \mathcal{P}_n(B)} q^{\mathrm{lind}(p)-1},$$

where $T_n(B)$ is as in Theorem 1.1 and

$$\mathcal{P}_n(B) := \{p \in \mathcal{P}_n \mid \mathrm{Des}(p^{-1}) \cap [n-2] = B\}.$$

1.2. **Signed Enumeration.** Sign balance for linear extensions of posets was studied in [21, 27, 25]. The study of sign balance for pattern-avoiding permutations started with Simion and Schmidt [23], who proved that the numbers of even and odd 123-avoiding permutations in S_n are equal if n is even, and differ (up to a sign) by a Catalan number if n is odd. Refined sign balance (i.e., a generating function taking into account the signs of permutations) was investigated by Désarménien and Foata [6] and by Wachs [26]. A beautiful formula for the signed Mahonian (i.e., the sign-and-major-index enumerator on S_n) was found by Simion and Gessel [26, Cor. 2]; see also [2].

The last descent statistic is a T_n-analogue of the major index on S_n, as demonstrated by Theorem 1.1 above; see also the discussion in Section 6. Thus, a T_n-analogue of the signed Mahonian on S_n is the sum

$$\sum_{\pi \in T_n} \mathrm{sign}(\pi) q^{\mathrm{ldes}(\pi)}.$$

For this sum we prove

Theorem 1.3.

$$\begin{aligned}
\sum_{\pi \in T_{2n+1}} \mathrm{sign}(\pi) \cdot q^{\mathrm{ldes}(\pi)} &= \sum_{\pi \in T_n} q^{2 \cdot \mathrm{ldes}(\pi)} \qquad (n \geq 0), \\
\sum_{\pi \in T_{2n}} \mathrm{sign}(\pi) \cdot q^{\mathrm{ldes}(\pi)} &= (1-q) \sum_{\pi \in T_n} q^{2 \cdot \mathrm{ldes}(\pi)} \qquad (n \geq 1).
\end{aligned}$$

This result refines (and is actually implicit in) some results in [23]. Our approach is different from that of [23]; see Subsection 1.3 below. After the archive posting of the current paper, Theorem 1.3 has been followed by various extensions, analogues and refinements:

(1) A. Reifegerste gave a bivariate refinement [18, Cor. 4.4].
(2) T. Mansour proved an analogue for 132-avoiding permutations [16].
(3) M. Shynar found analogues for the signed enumeration of tableaux with respect to certain natural statistics [22].

The following phenomenon appears, with small variations, in all these results: the standard enumerator of objects of size n is essentially equal to the signed enumerator of objects of size $2n$. This phenomenon deserves further study; it seems to be related to the cyclic sieving phenomenon of [19].

1.3. **Recursion.** Our main tool in proving the above results is a new recursive formula for the multivariate generating function, on 321-avoiding permutations, involving the inversion number, the last descent, the last index, and the descent set of the inverse (Theorem 2.1 below).

It should be noted that, shortly after the archive posting of this paper, bijective proofs of Theorems 1.1 and 1.3 were presented by A. Reifegerste [18].

2. A Recursion Formula

Define the multivariate generating function

$$f_n(\bar{t}, x, y, z) := \sum_{\pi \in T_n} t_{\mathrm{Des}(\pi^{-1})} x^{\mathrm{inv}(\pi)} y^{\mathrm{ldes}(\pi)} z^{\mathrm{lind}(\pi)}, \tag{1}$$

where $\bar{t} = (t_1, t_2, \ldots)$ and $t_D := \prod_{i \in D} t_i$ for $D \subseteq [n-1]$.

Theorem 2.1. (Recursion Formula)

$$f_1(\bar{t}, x, y, z) = z,$$

and, for $n \geq 2$,

$$\begin{aligned}(x - yz) f_n(\bar{t}, x, y, z) &= t_{n-1} x^n yz \cdot f_{n-1}(\bar{t}, x, yz/x, 1) \\ &\quad + (1 - t_{n-1}) x^n yz \cdot f_{n-1}(\bar{t}, x, 1, yz/x) \\ &\quad + (x - yz) z^n \cdot f_{n-1}(\bar{t}, x, y, 1) \\ &\quad - x y^n z^n \cdot f_{n-1}(\bar{t}, x, 1, 1).\end{aligned}$$

Proof. The case $n = 1$ is clear. Assume $n \geq 2$.
Given a permutation

$$\pi = (\pi(1), \ldots, \pi(n-1)) \in T_{n-1},$$

insert the digit n between the kth and $(k+1)$st places in π $(0 \leq k \leq n-1)$ to get a permutation

$$\bar{\pi} = (\pi(1), \ldots, \pi(k), n, \pi(k+1), \ldots, \pi(n-1)) \in S_n.$$

Clearly, a necessary and sufficient condition for $\bar{\pi} \in T_n$ is that π has no descent after the kth place, i.e.,

$$\mathrm{ldes}(\pi) \leq k \leq n-1. \tag{2}$$

If π has no descents, i.e. $\pi = id$, this still holds with $\mathrm{ldes}(\pi) := 0$. The new statistics for $\bar{\pi}$ are:

$$\begin{aligned}\mathrm{inv}(\bar{\pi}) &= \mathrm{inv}(\pi) + (n - 1 - k) \\ \mathrm{ldes}(\bar{\pi}) &= \begin{cases} k+1, & \text{if } k < n-1; \\ \mathrm{ldes}(\pi), & \text{if } k = n-1. \end{cases} \\ \mathrm{lind}(\bar{\pi}) &= k+1 \\ \mathrm{Des}(\bar{\pi}^{-1}) &= \begin{cases} \mathrm{Des}(\pi^{-1}), & \text{if } \pi^{-1}(n-1) \leq k; \\ \mathrm{Des}(\pi^{-1}) \cup \{n-1\}, & \text{if } k < \pi^{-1}(n-1). \end{cases}\end{aligned}$$

Note that, for any $\pi \in T_{n-1}$,

$$\mathrm{ldes}(\pi) \leq \pi^{-1}(n-1) \leq n-1;$$

and, actually, exactly one of the following two cases holds: either

$$\mathrm{ldes}(\pi) = \pi^{-1}(n-1) < n-1 \qquad (\text{if } n-1 \text{ is not the last digit in } \pi)$$

or

$$\mathrm{ldes}(\pi) < \pi^{-1}(n-1) = n-1 \qquad (\text{if } n-1 \text{ is the last digit in } \pi).$$

We can therefore compute

$$\begin{aligned}
f_n &= f_n(\bar{t}, x, y, z) \\
&= \sum_{\pi \in T_{n-1}} \sum_{k=\mathrm{ldes}(\pi)}^{n-1} t_{\mathrm{Des}(\bar{\pi}^{-1})} x^{\mathrm{inv}(\bar{\pi})} y^{\mathrm{ldes}(\bar{\pi})} z^{\mathrm{lind}(\bar{\pi})} \\
&= \sum_{\pi \in T_{n-1}} t_{\mathrm{Des}(\pi^{-1})} x^{\mathrm{inv}(\pi)} \cdot \left[\sum_{k=\mathrm{ldes}(\pi)}^{\pi^{-1}(n-1)-1} t_{n-1} x^{n-1-k} y^{k+1} z^{k+1} \right. \\
&\qquad \left. + \sum_{k=\pi^{-1}(n-1)}^{n-2} x^{n-1-k} y^{k+1} z^{k+1} + y^{\mathrm{ldes}(\pi)} z^n \right] \\
&= \sum_{\pi \in T_{n-1}} t_{\mathrm{Des}(\pi^{-1})} x^{\mathrm{inv}(\pi)} \cdot \left[t_{n-1} x^{n-1} yz \sum_{k=\mathrm{ldes}(\pi)}^{\pi^{-1}(n-1)-1} (yz/x)^k \right. \\
&\qquad \left. + x^{n-1} yz \sum_{k=\pi^{-1}(n-1)}^{n-2} (yz/x)^k + y^{\mathrm{ldes}(\pi)} z^n \right] \\
&= (1 - yz/x)^{-1} \sum_{\pi \in T_{n-1}} t_{\mathrm{Des}(\pi^{-1})} x^{\mathrm{inv}(\pi)} \cdot \\
&\quad \cdot \Big[t_{n-1} x^{n-1} yz \left((yz/x)^{\mathrm{ldes}(\pi)} - (yz/x)^{\pi^{-1}(n-1)} \right) \\
&\qquad + x^{n-1} yz \left((yz/x)^{\pi^{-1}(n-1)} - (yz/x)^{n-1} \right) + y^{\mathrm{ldes}(\pi)} z^n (1 - yz/x) \Big] \\
&= (1 - yz/x)^{-1} [t_{n-1} x^{n-1} yz f_{n-1}(\bar{t}, x, yz/x, 1) \\
&\qquad + (1 - t_{n-1}) x^{n-1} yz f_{n-1}(\bar{t}, x, 1, yz/x) - y^n z^n f_{n-1}(\bar{t}, x, 1, 1) \\
&\qquad + (1 - yz/x) z^n f_{n-1}(\bar{t}, x, y, 1)] .
\end{aligned}$$

Multiply both sides by $(x - yz)$ to get the claimed recursion. □

Corollary 2.2. *The first few values of f_n are:*

$$\begin{aligned}
f_1(\bar{t}, x, y, z) &= z, \\
f_2(\bar{t}, x, y, z) &= z^2 + t_1 xyz, \\
f_3(\bar{t}, x, y, z) &= z^3 + t_1(x^2y^2z^2 + xyz^3) + t_2(x^2yz + xy^2z^2), \\
f_4(\bar{t}, x, y, z) &= z^4 + t_1(x^3y^3z^3 + x^2y^2z^4 + xyz^4) \\
&\quad + t_2(x^4y^2z^2 + x^3y^3z^3 + x^2y^3z^3 + x^2yz^4 + xy^2z^4), \\
&\quad + t_3(x^3yz + x^2y^2z^2 + xy^3z^3) + t_1t_3(x^3y^2z^2 + x^2y^3z^3).
\end{aligned}$$

3. Equidistribution of ldes and lind -1

In this section we prove Theorem 1.1.

Note. Most of the permutations $\pi \in T_n$, namely those with $\pi(n) \neq n$, satisfy $\mathrm{lind}(\pi) = \mathrm{ldes}(\pi)$. Nevertheless, the equidistributed parameters are not ldes and lind but rather ldes and lind -1.

Note. $T_n(B)$ "forgets" whether or not $n-1$ belongs to $\mathrm{Des}(\pi^{-1})$. The corresponding claim, without this "forgetfulness", is false!

Proof of Theorem 1.1. We have to show that, letting $x = 1$:

$$q\hat{f}_n(\bar{t}, 1, q, 1) = \hat{f}_n(\bar{t}, 1, 1, q),$$

where $\hat{f}$ denotes f under the additional substitution $t_{n-1} = 1$. This clearly holds for $n = 1$ (as well as for $n = 2, 3, 4$, by Corollary 2.2). By Theorem 2.1, for $n \geq 2$:

$$\begin{aligned}
(1 - yz)\hat{f}_n(\bar{t}, 1, y, z) &= yzf_{n-1}(\bar{t}, 1, yz, 1) + (1 - yz)z^n f_{n-1}(\bar{t}, 1, y, 1) \\
&\quad - y^n z^n f_{n-1}(\bar{t}, 1, 1, 1).
\end{aligned}$$

Letting $y = q$ and $z = 1$ gives

$$\begin{aligned}
(1 - q)\hat{f}_n(\bar{t}, 1, q, 1) &= qf_{n-1}(\bar{t}, 1, q, 1) + (1 - q)f_{n-1}(\bar{t}, 1, q, 1) \\
&\quad - q^n f_{n-1}(\bar{t}, 1, 1, 1) \\
&= f_{n-1}(\bar{t}, 1, q, 1) - q^n f_{n-1}(\bar{t}, 1, 1, 1),
\end{aligned}$$

whereas letting $y = 1$ and $z = q$ yields

$$\begin{aligned}
(1 - q)\hat{f}_n(\bar{t}, 1, 1, q) &= qf_{n-1}(\bar{t}, 1, q, 1) + (1 - q)q^n f_{n-1}(\bar{t}, 1, 1, 1) \\
&\quad - q^n f_{n-1}(\bar{t}, 1, 1, 1) \\
&= qf_{n-1}(\bar{t}, 1, q, 1) - q^{n+1} f_{n-1}(\bar{t}, 1, 1, 1).
\end{aligned}$$

This completes the proof. ∎

4. Sign Balance of T_n

In this section we prove Theorem 1.3.

Proof of Theorem 1.3. Substituting $\bar{t} = (1, 1, \ldots)$, $z = 1$, and $x = -1$ in the generating function (1), denote

$$g_n(-1, y) := f_n(\bar{1}, -1, y, 1) = \sum_{\pi \in T_n} (-1)^{\mathrm{inv}(\pi)} y^{\mathrm{ldes}(\pi)} \qquad (n \geq 1).$$

This function records the sign and the last descent of 321-avoiding permutations. Recall also the generating function for last descent, from Theorem 1.1:

$$g_n(1, y) := f_n(\bar{1}, 1, y, 1) = \sum_{\pi \in T_n} y^{\mathrm{ldes}(\pi)} \qquad (n \geq 1).$$

Let $g_0(1, y) := 1$. We have to prove that

$$g_{2n+1}(-1, y) = g_n(1, y^2) \qquad (n \geq 0),$$

and

$$g_{2n}(-1, y) = (1 - y) g_n(1, y^2) \qquad (n \geq 1).$$

By Theorem 2.1, the polynomial $g_n(-1, y)$ satisfies the recursion formula

$$\begin{aligned} (-1 - y) g_n(-1, y) &= (-1)^n y g_{n-1}(-1, -y) + (-1 - y) g_{n-1}(-1, y) \\ &\quad + y^n g_{n-1}(-1, 1). \end{aligned}$$

Clearly,

$$g_1(-1, y) = 1 = g_0(1, y^2),$$

and also

$$g_2(-1, y) = 1 - y = (1 - y) g_1(1, y^2).$$

We can proceed by induction. If

$$g_{2n}(-1, y) = (1 - y) g_n(1, y^2),$$

then

$$\begin{aligned} (-1 - y) g_{2n+1}(-1, y) &= -y g_{2n}(-1, -y) + (-1 - y) g_{2n}(-1, y) \\ &\quad + y^{2n+1} g_{2n}(-1, 1) \\ &= -y(1 + y) g_n(1, y^2) + (-1 - y)(1 - y) g_n(1, y^2) \\ &\quad + 0 \\ &= (-1 - y) g_n(1, y^2), \end{aligned}$$

so that

$$g_{2n+1}(-1, y) = g_n(1, y^2).$$

This, in turn, implies

$$\begin{aligned}
(-1-y)g_{2n+2}(-1,y) &= yg_{2n+1}(-1,-y)+(-1-y)g_{2n+1}(-1,y) \\
&\qquad + y^{2n+2}g_{2n+1}(-1,1) \\
&= yg_n(1,y^2)+(-1-y)g_n(1,y^2)+y^{2n+2}g_n(1,1) \\
&= -g_n(1,y^2)+y^{2n+2}g_n(1,1),
\end{aligned}$$

and therefore

$$g_{2n+2}(-1,y)=\frac{g_n(1,y^2)-y^{2n+2}g_n(1,1)}{1+y}=(1-y)h_n(y^2),$$

where

$$h_n(q)=\frac{g_n(1,q)-q^{n+1}g_n(1,1)}{1-q}.$$

We need to show that

$$h_n(q)=g_{n+1}(1,q).$$

Indeed, the equation

$$(1-q)g_{n+1}(1,q)=g_n(1,q)-q^{n+1}g_n(1,1)$$

follows immediately from Theorem 2.1. □

Corollary 4.1. (equivalent to [23, Prop. 2])
The sign-balance enumerator for T_n

$$g_n(-1,1):=f_n(\bar{1},-1,1,1)=\sum_{\pi\in T_n}(-1)^{\mathrm{inv}(\pi)}$$

is either a Catalan number or zero:

$$\begin{aligned}
g_{2n+1}(-1,1) &= |T_n|=\frac{1}{n+1}\genfrac{}{}{0pt}{}{2n}{n} \qquad (n\geq 0) \\
g_{2n}(-1,1) &= 0 \qquad (n\geq 1).
\end{aligned}$$

The amazing connection between T_n, T_{2n}, and T_{2n+1} calls for further study.

5. Dyck Paths

Let $\mathcal{P}_n$ be the set of Dyck paths of length $2n$. Thus, each $p\in\mathcal{P}_n$ is a sequence $(p_1,\ldots,p_{2n})$ of n "+1"s and n "−1"s, with nonnegative initial partial sums:

$$p_1+\cdots+p_i\geq 0 \qquad (\forall i).$$

Of course, $p_1+\cdots+p_{2n}=0$ by definition.

A *peak* of a Dyck path $p = (p_1, \ldots, p_{2n}) \in \mathcal{P}_n$ is an index $1 \le i \le 2n-1$ such that $p_i = +1$ and $p_{i+1} = -1$. Let $\mathrm{Peak}(p)$ be the set of all peaks of p. Define the *Descent set* of p to be

$$\mathrm{Des}(p) := \{1 \le i \le n-1 \,|\, i \in \mathrm{Peak}(p)\}.$$

Define the *inverse path* $p^{-1} \in \mathcal{P}_n$ to be

$$p^{-1} := (-p_{2n}, \ldots, -p_1),$$

i.e., p^{-1} is obtained by reversing the order of steps as well as the sign of each step in p. Clearly,

$$i \in \mathrm{Peak}(p^{-1}) \iff 2n - i \in \mathrm{Peak}(p),$$

so that

$$\begin{aligned} \mathrm{Des}(p^{-1}) &= \{1 \le i \le n-1 \,|\, i \in \mathrm{Peak}(p^{-1})\} \\ &= \{1 \le i \le n-1 \,|\, 2n - i \in \mathrm{Peak}(p)\}. \end{aligned}$$

Thus the peaks strictly *before* the midpoint of p record the descents of p, whereas those strictly *after* the midpoint of p record the descents of p^{-1}, in reverse order. The reason for this terminology is the following result.

Lemma 5.1. *There exists a bijection $\phi : T_n \to \mathcal{P}_n$ such that, if $\pi \in T_n$ and $p := \phi(\pi) \in \mathcal{P}_n$, then*

$$\mathrm{Des}(p) = \mathrm{Des}(\pi)$$

and

$$\mathrm{Des}(p^{-1}) = \mathrm{Des}(\pi^{-1}).$$

Proof. We shall define the bijection $\phi : T_n \to \mathcal{P}_n$ in three steps.

First, let $\phi_1 : T_n \to \mathcal{SYT}_2(n)$ be the Robinson-Schensted correspondence, where $\mathcal{SYT}_2(n)$ is the set of all pairs (P, Q) of standard Young tableaux of size n with the same two-rowed shape.

Example.

$$\phi_1(25134) = \begin{pmatrix} 1 & 3 & 4 & & 1 & 2 & 5 \\ 2 & 5 & & , & 3 & 4 & \end{pmatrix}.$$

Next, define $\phi_2 : \mathcal{SYT}_2(n) \to \mathcal{SYT}(n,n)$, where $\mathcal{SYT}(n,n)$ is the set of standard Young tableaux of (size $2n$ and) shape (n,n). $T = \phi_2(P,Q)$ is obtained by gluing Q along its "eastern frontier" to the skew tableau obtained by rotating P by 180^0 and replacing each entry $1 \le j \le n$ of P by $2n+1-j$.

Example.

$$\phi_2 \begin{pmatrix} 1 & 3 & 4 & & 1 & 2 & 5 \\ 2 & 5 & & , & 3 & 4 & \end{pmatrix} = \begin{matrix} 1 & 2 & 5 & 6 & 9 \\ 3 & 4 & 7 & 8 & 10 \end{matrix}$$

Finally, define $\phi_3 : \mathcal{S}YT(n,n) \to \mathcal{P}_n$ as follows: the increasing $(+1)$ steps in $\phi_3(T)$ are at places indexed by the entries in row 1 of T, whereas the decreasing (-1) steps are at places indexed by row 2 of T.

Example.

$$\phi_3 \begin{pmatrix} 1 & 2 & 5 & 6 & 9 \\ 3 & 4 & 7 & 8 & 10 \end{pmatrix} = (++--++--+-).$$

The bijection we want is the composition $\phi = \phi_3\phi_2\phi_1$.

Let us examine what happens to the descent sets of π and of π^{-1} under this sequence of transformations.

By a well-known property of the Robinson-Schensted correspondence ϕ_1, if

$$\pi \overset{\phi_1}{\longmapsto} (P,Q) \overset{\phi_2}{\longmapsto} T \overset{\phi_3}{\longmapsto} p,$$

then

$$\pi^{-1} \overset{\phi_1}{\longmapsto} (Q,P) \overset{\phi_2}{\longmapsto} T^{-1} \overset{\phi_3}{\longmapsto} p^{-1}.$$

Here T^{-1} is the tableau obtained from $T \in \mathcal{S}YT(n,n)$ by a 180^0 rotation and replacement of each entry $1 \le j \le 2n$ by $2n+1-j$.

For a standard two-rowed Young tableau P of size n define:

$$\mathrm{Des}(P) := \{1 \le i \le n-1 \mid i \text{ is in row 1 and } i+1 \text{ is in row 2 of } P\}.$$

For $T \in \mathcal{S}YT(n,n)$ (of size $2n$) define

$$\begin{aligned} \mathrm{Des}_1(T) &:= \mathrm{Des}(T) \cap [n-1] \\ &= \{1 \le i \le n-1 \mid i \text{ is in row 1 and } i+1 \text{ is in row 2 of } T\}. \end{aligned}$$

Using all this notation we now have, for $\pi \in T_n$,

$$\mathrm{Des}(\pi) = \mathrm{Des}(Q) = \mathrm{Des}_1(T) = \mathrm{Des}(p),$$

where the equality $\mathrm{Des}(\pi) = \mathrm{Des}(Q)$ is again a well-known property of the Robinson-Schensted correspondence. Similarly,

$$\mathrm{Des}(\pi^{-1}) = \mathrm{Des}(P) = \mathrm{Des}_1(T^{-1}) = \mathrm{Des}(p^{-1}).$$

This completes the proof. $\square$

Note. The bijection $\phi : T_n \to \mathcal{P}_n$ in the proof of Lemma 5.1, so well suited for our purposes, is essentially due to Knuth [13, p. 64] (up to an extra rotation of T). Its ingredient ϕ_2 was already used by MacMahon [15, p. 131]. For other bijections between these two sets see [5, 14, 7, 8].

For a Dyck path $p = (p_1, \ldots, p_{2n}) \in \mathcal{P}_n$, let

$$\mathrm{ldes}(p) := \max \mathrm{Des}(p) = \max\{1 \le i \le n-1 \mid p_i = +1, p_{i+1} = -1\}.$$

If $\mathrm{Des}(p) = \emptyset$, i.e., if $p = (+\ldots+-\ldots-)$ is the "unimodal path", we define $\mathrm{ldes}(p) := 0$.

Define also

$$\mathrm{lind}(p) := \max\{1 \le i \le n \mid p_i = +1, p_{i+1} = -1\}.$$

In other words,

$$\mathrm{lind}(p) = \begin{cases} n, & \text{if } n \in \mathrm{Peak}(p); \\ \mathrm{ldes}(p), & \text{otherwise.} \end{cases}$$

Lemma 5.2. *Let*

$$\pi \xmapsto{\phi_1} (P, Q) \xmapsto{\phi_2} T \xmapsto{\phi_3} p,$$

as in the proof of Lemma 5.1. The following conditions are equivalent:

(1) $\pi(n) = n$.
(2) n *is in row 1 of both* P *and* Q.
(3) n *is in row 1 of* T, *and* $n + 1$ *is in row 2 of* T.
(4) $n \in \mathrm{Peak}(p)$.

Proof. If $\pi(n) = n$ then n is the last element inserted into P, and it stays in row 1 due to its size. Thus n is in row 1 of both P and Q. Conversely, if n is in row 1 of both P and Q then it is necessarily the *last* element in each of these rows. Thus it is the last element inserted into P, i.e., $\pi(n) = n$. This proves the equivalence of conditions 1 and 2.

Conditions 2 and 3 are clearly equivalent, by the definition of ϕ_2.

Similarly for conditions 3 and 4, by the definitions of ϕ_3 and of $\mathrm{Peak}(p)$. □

Lemma 5.3. *If* $\pi \in T_n$ *and* $p := \phi(\pi) \in \mathcal{P}_n$, *then*

$$\mathrm{ldes}(p) = \mathrm{ldes}(\pi)$$

and

$$\mathrm{lind}(p) = \mathrm{lind}(\pi).$$

Proof. The first claim follows immediately from Lemma 5.1, since

$$\mathrm{ldes}(p) = \max \mathrm{Des}(p) = \max \mathrm{Des}(\pi) = \mathrm{ldes}(\pi).$$

(If $\mathrm{Des}(p) = \mathrm{Des}(\pi) = \emptyset$ then $\mathrm{ldes}(p) = \mathrm{ldes}(\pi) = 0$ by definition.)

If $\pi(n) = n$ then $\mathrm{lind}(\pi) = \pi^{-1}(n) = n$, and also $\mathrm{lind}(p) = n$ by Lemma 5.2.

Otherwise ($\pi(n) \ne n$) clearly $\pi^{-1}(n) \in \mathrm{Des}(\pi)$; and $\pi^{-1}(n)$ is actually the last descent of π, because of the 321-avoiding condition. Thus $\mathrm{ldes}(\pi) = \mathrm{lind}(\pi)$. On the other hand, $n \notin \mathrm{Peak}(p)$ by Lemma 5.2 so that, by definition, $\mathrm{lind}(p) = \mathrm{ldes}(p)$. Thus

$$\mathrm{lind}(p) = \mathrm{ldes}(p) = \mathrm{ldes}(\pi) = \mathrm{lind}(\pi),$$

as claimed. □

Proof of Theorem 1.2. Combine Lemmas 5.1 and 5.3 with Theorem 1.1. □

6. Final Remarks

The quotient $P_n/\langle QS_n\rangle$ of the polynomial ring $P_n = \mathbf{Q}[x_1, \ldots, x_n]$ by the ideal generated by the quasi-symmetric functions without constant term was studied by Aval, Bergeron and Bergeron [4]. They determined its Hilbert series with respect to grading by total degree. For a Dyck path $p \in \mathcal{P}_n$, let

$$\mathrm{tail}(p) := 2n - \max\{i \,|\, p_i = +1, p_{i+1} = -1\}$$

be its "tail length", i.e., the number of consecutive decreasing steps at its end. With this notation, the result of [4] can be formulated as follows.

Theorem 6.1. [4, Theorem 5.1] *The Hilbert series of the quotient $P_n/\langle QS_n\rangle$ (graded by total degree) is*

$$\sum_{p\in\mathcal{P}_n} q^{n-\mathrm{tail}(p)} = \sum_{k=0}^{n-1} \frac{n-k}{n+k}\binom{n+k}{k} q^k.$$

Corollary 6.2. *The Hilbert series of the quotient $P_n/\langle QS_n\rangle$ (graded by total degree) is equal to*

$$\sum_{\pi\in T_n} q^{\mathrm{ldes}(\pi)}.$$

Proof. By the definitions in Section 5 above, for every Dyck path $p \in \mathcal{P}_n$,

$$\mathrm{tail}(p) = 2n - \max \mathrm{Peak}(p) = \min \mathrm{Des}(p^{-1}),$$

with the convention $\min\emptyset := n$. Thus, by Lemma 5.1, if $\pi = \phi^{-1}(p) \in T_n$ then $\mathrm{tail}(p) = \min \mathrm{Des}(\pi^{-1})$. Now, define a bijection $\psi : T_n \to T_n$ by

$$\psi(\pi) := w_0\pi^{-1}w_0 \qquad (\forall \pi \in T_n),$$

where $w_0 = n \ldots 21$ is the longest element in S_n. If

$$\pi \xmapsto{\psi} \hat{\pi} \xmapsto{\phi} p$$

then clearly

$$\mathrm{tail}(p) = \min \mathrm{Des}(\hat{\pi}^{-1}) = n - \max \mathrm{Des}(\pi) = n - \mathrm{ldes}(\pi).$$

We conclude that

$$\sum_{p\in\mathcal{P}_n} q^{n-\mathrm{tail}(p)} = \sum_{\pi\in T_n} q^{\mathrm{ldes}(\pi)}.$$

□

Theorem 6.1 was proved in [4] by constructing a monomial basis for the quotient ring $P_n/\langle QS_n\rangle$. By the proof of Corollary 6.2, the elements of this basis may be indexed by permutations $\pi \in T_n$, with total degree given by

$\mathrm{ldes}(\pi)$. It is well known that the Hilbert series of the graded coinvariant algebra P_n/I_n, where I_n is the ideal generated by the symmetric functions without constant term, is equal to

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)},$$

where $\mathrm{maj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} i$. A monomial basis idexed by permutations $\pi \in S_n$, with total degree given by $\mathrm{maj}(\pi)$, was constructed in [11]. In this sense, the last descent may be considered as a T_n-analogue of the major index.

Problem 6.3. *Find algebraic interpretations and proofs for Theorems 1.1 and 1.3.*

For a different algebraic interpretation of the generating function in Theorem 6.1 see [17].

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