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RESTRICTED PERMUTATIONS FROM CATALAN TO FINE AND BACK

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Abstract

We give some history and recent results in the area of pattern restricted permutations. We also present a new bijection between certain pattern restricted permutations.

Introduction

It was a great pleasure to have been given the opportunity to speak at the landmark 50^{th} meeting of the Séminaire Lotharingien de Combinatoire, held in March 2003, at Domaine Saint Jacques, near Ottrott, France. The meeting was a wonderful celebration, replete with wonderful company, wonderful food, and, of course, wonderful mathematics.

This article is an expanded version of the talk that I presented at the meeting. We give some history and recent results in the area of pattern restricted permutations, as well as present a pertinent new bijection.

1. Preliminaries

We first get the definitions and notation out of the way.

Let $\pi \in S_n$ be a permutation of $\{1, 2, \ldots, n\}$ written in one-line notation. Let $\alpha \in S_m$. We say that π *contains the pattern* α if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_m}$ is order-equivalent to α. By order-equivalent we mean that the usual order on the integers is the same for both sequences. For example, 475 is order-equivalent to 132 since both have smallest element first, largest element second, and middle element last. If π does not contain a pattern β, we say that π is β-avoiding. We write $S_n(\beta)$ for the set of permutations of S_n that are β-avoiding.

We next extend this notation. Let $\mathcal{S} = \bigcup_n S_n$. Let $T \subseteq \mathcal{S}$ and let R be a multisubset of $\mathcal{S}$. Then $S_n(T)$ is the set of permutations of S_n that avoid all patterns in T while $S_n(T;R)$ is the set of permutations of $S_n(T)$ that contain each pattern (including multiplicities) in R exactly once. If $|T| = 1$ or $|R| = 1$ then we drop the set notation.

So, for example, $S_n(132, \{123, 123\})$ is the set of 132-avoiding permutations in S_n that contain exactly two 123 patterns. As a concrete example, $124635 \in S_6(321; 213)$.

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One last bit of notation: we let $s_n(R) = |S_n(R)|$ and $s_n(R;T) = |S_n(R;T)|$.

2. Some History

In 1838, Catalan [3] defined what we now call the Catalan numbers:

$$\frac{1}{n+1}\binom{2n}{n} = C_n.$$

In this paper he addressed the question

De combien de manières peut-on effectuer le produit de n facteurs différents?

and used essentially an argument that shows that $s_n(132) = C_n$ by proving that $C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}$. This argument that shows $s_n(132) = C_n$ goes as follows. Let $\pi = \pi_1\pi_2\ldots\pi_n$ and let $\pi_i = n$. Then $\pi_1\ldots\pi_{i-1}$ must consist of the elements $n-i+1, n-i+2, \ldots, n-1$ and $\pi_{i+1}\ldots\pi_n$ must consist of the elements $1, 2, \ldots, n-i$, for otherwise $\pi_j\, n\, \pi_k$ would be a 132 pattern for some j, k, $j < i < k$. Next, $\pi_1\ldots\pi_{i-1}$ and $\pi_{i+1}\ldots\pi_n$ must each be 132-avoiding permutations themselves. Hence, we have

$$s_n(132) = \sum_{i=1}^{n} s_{i-1}(132)s_{n-i}(132),$$

which gives (using initial conditions) $s_n(132) = C_n$.

Of course, Catalan did not consider $s_n(132)$. It was not until 1915 that $s_n(\alpha)$ was determined for the first time for some $\alpha \in S_3$. MacMahon [see 14] proved that $s_n(123) = C_n$. This result also gives us $s_n(321) = C_n$ by reading permutations from right-to-left instead of left-to-right. This re-reading is called the reversal bijection, one of the three standard bijections.

2.1 Three Standard Bijections

There are three standard bijections that allow us to look at a smaller number of patterns when considering pattern-avoiding permutations. We define the following bijections.

The reversal bijection: $r : S_n \to S_n$, $\pi_1\pi_2\ldots\pi_n \mapsto \pi_n\ldots\pi_2\pi_1$.

The complement bijection: $c : S_n \to S_n$, $\pi_1\pi_2\ldots\pi_n \mapsto (n+1-\pi_1)(n+1-\pi_2)\ldots(n+1-\pi_n)$.

The inverse bijection $i : S_n \to S_n$, $\pi \mapsto \pi^{-1}$ (group theoretic).

Using these bijections we see that $\pi \in S_n(\alpha)$ if and only if $x(\pi) \in S_n(x(\alpha))$ where x is one of r, c, i.

2.2 More Recent History

It was not until 1973 that the first non-monotonic pattern restricted permutations were considered. Knuth [12] showed that $s_n(231) = C_n$. Now, since $r(231) = 132$, $c(231) = 213$, and $i(231) = 312$, Knuth's and MacMahon's results show that $s_n(\alpha) = C_n$ for any $\alpha \in S_3$.

This result leads us to the following definition.

Definition. We say that two patterns α and β are in the same *Wilf class* or are *Wilf equivalent* if and only if $s_n(\alpha) = s_n(\beta)$ for all n.

Hence, there is one Wilf class for patterns of length 3. However, historically it was not shown that $s_n(123) = s_n(132)$ directly, rather that they are both equal to C_n. This leads us into our next section.

3. Some Bijections

Using the three standard bijections we have $s_n(123) = s_n(321)$ and $s_n(132) = s_n(213) = s_n(231) = s_n(312)$. The fact that $s_n(123) = s_n(132)$ as well is surprising. Hence, we will look at some bijections from $S_n(\overline{123})$ to $S_n(\overline{132})$, where $\overline{123}$ means any element from $\{123, 321\}$ and $\overline{132}$ means any element from $\{132, 213, 231, 312\}$ since these are equivalent by the standard bijections.

The first bijection (1975) is due to Rotem [26]. He used ballot sequences and binary trees as an intermediate step from $S_n(321)$ to $S_n(231)$. The first *direct* bijection was given by Simion and Schmidt [27] in 1985. In more recent years, West [30] has used generating trees while Stanley [29] and Krattenthaler [13] have used Dyck paths.

3.1 A New Bijection

We give a new *direct* bijection $\gamma : S_n(321) \to S_n(132)$. In order to present γ we give a definition.

Definition. Let $k \geq 2$. Let $\alpha = \alpha_1 \ldots \alpha_k$ and $\beta = \beta_1 \ldots \beta_k$ be two distinct occurrences of $1k(k-1)\ldots 2$ in π. We say α is smaller than β if there exists i, $1 \leq i \leq k$, such that $\pi^{-1}(\alpha_j) = \pi^{-1}(\beta_j)$ for all $j < i$ and $\pi^{-1}(\alpha_i) < \pi^{-1}(\beta_i)$.

We can now describe the bijection $\gamma : S_n(321) \to S_n(132)$. Let $\pi_1\pi_2 \ldots \pi_n = \pi \in S_n(321)$ and let $xzy = \pi_{p_1}\pi_{p_2}\pi_{p_3}$ be the smallest 132 occurrence in π. If no such occurrence exists, $\gamma(\pi) = \pi$. Otherwise, let $\mathcal{M}$ be the operation that creates the permutation $\mathcal{M}\pi$ where $\mathcal{M}\pi_i = \pi_i$ if $i \notin \{p_1, p_2, p_3\}$ and $\mathcal{M}\pi_{p_1} = z$, $\mathcal{M}\pi_{p_2} = y$, and $\mathcal{M}\pi_{p_3} = x$. Now let xzy be the smallest 132 occurrence in $\mathcal{M}\pi$ and apply $\mathcal{M}$ again. Repeat until the resulting permutation is 132-avoiding.

Example. Let $\pi = 35612487 \in S_8(321)$, a permutation that has eight 132 occurrences, the smallest one being 354. We apply $\mathcal{M}$ to get $\mathcal{M}\pi = 54612387$. Notice that the 354 is changed to 543 in the same positions and all other elements are left untouched. Now, in 54612387, 587 is the smallest occurrence. Apply $\mathcal{M}$ again to get $\mathcal{M}^2\pi = 84612375$. Continuing, we get $\mathcal{M}^3\pi = 86512374$ and $\mathcal{M}^4\pi = 86572341$. Since $86572341 \in S_8(132)$, $\gamma(35612487) = 86572341$.

Of course, we must prove that this *is* a bijection. We start by showing that γ is well-defined. Our approach is to show that, with respect to the indices of elements, the smallest 132 occurrence in $\mathcal{M}\tau$ is larger than the smallest 132 occurrence in τ. Hence, $\mathcal{M}^j\tau$ must be 132-avoiding for some finite j.

Write $\tau = \tau_1\tau_2\ldots\tau_n$ and let $\tau_i\tau_j\tau_k$ be the smallest 132 occurrence. Clearly, since τ has no 132 occurrence $\tau_x\tau_y\tau_z$ with $x < i$, $\mathcal{M}\tau$ could only possibly have such an occurrence if $\{i,j,k\} \cap \{y,z\} \neq \emptyset$. Since $\tau_i\tau_j\tau_k$ is the smallest 132 occurrence in τ, $\tau_x > \tau_k$ and thus $i,k \notin \{y,z\}$. Hence, we must have $j \in \{y,z\}$. From the minimality of $\tau_i\tau_j\tau_k$, clearly $j \neq z$. Thus, $\tau_x\tau_j\tau_z$ with $i < z < j$ is our only possibility. But, $\tau_z > \tau_x > \tau_k$ implies that $\tau_i\tau_z\tau_k$ is a smaller 132 occurrence in τ than $\tau_i\tau_j\tau_k$, a contradiction.

Next, assume, for a contradiction, that $\tau_j\tau_x\tau_y$ in $\mathcal{M}\tau$ is a smaller 132 occurrence (with respect to the indices) than $\tau_i\tau_j\tau_k$ is in τ. This implies that $\tau_i\tau_x\tau_k$ is a smaller 132 occurrence in τ than $\tau_i\tau_j\tau_k$, contradicting the minimality of $\tau_i\tau_j\tau_k$.

We now show that γ is a bijection; it is enough to give γ^{-1}. To do so, we make the following definition.

Definition. Let $k \geq 2$. Let $\alpha = \alpha_1\ldots\alpha_k$ and $\beta = \beta_1\ldots\beta_k$ be two distinct occurrences of $k(k-1)\ldots1$ in π. We say α is larger that β if there exists i, $1 \leq i \leq k$, such that $\pi^{-1}(\alpha_j) = \pi^{-1}(\beta_j)$ for all $j > i$ and $\pi^{-1}(\alpha_i) > \pi^{-1}(\beta_i)$.

We can now describe $\gamma^{-1} : S_n(132) \to S_n(321)$. Let $\pi_1\pi_2\ldots\pi_n = \pi \in S_n(132)$ and let $zyx = \pi_{p_1}\pi_{p_2}\pi_{p_3}$ be the largest 321 occurrence in π. If no such occurrence exists, $\gamma^{-1}(\pi) = \pi$. Otherwise, let $\mathcal{N}$ be the operation that creates the permutation $\mathcal{N}\pi$ where $\mathcal{N}\pi_i = \pi_i$ if $i \notin \{p_1,p_2,p_3\}$ and $\mathcal{N}\pi_{p_1} = x$, $\mathcal{N}\pi_{p_2} = z$, and $\mathcal{N}\pi_{p_3} = y$. Now let zyx be the largest 321 occurrence in $\mathcal{N}\pi$ and apply $\mathcal{N}$ again. Repeat until the resulting permutation is 321-avoiding.

It is easy to check that $\gamma^{-1}\gamma(\pi) = \pi$ for any $\pi \in S_n(321)$.

We can extend γ to a bijection $\Gamma_k : S_n(k(k-1)\ldots1) \to S_n(1k(k-1)\cdots2)$ for any $k \geq 2$. Let $\pi_1\pi_2\ldots\pi_n = \pi \in S_n(k(k-1)\ldots1)$ and let $\alpha = \pi_{p_1}\pi_{p_2}\ldots\pi_{p_k}$ be the smallest $1k(k-1)\ldots2$ occurrence in π. If no such occurrence exists, $\Gamma_k(\pi) = \pi$. Otherwise, let $\mathcal{P}$ be the operation that creates the permutation $\mathcal{P}\pi$ where $\mathcal{P}\pi_i = \pi_i$ if $i \notin \{p_1,p_2,\ldots,p_k\}$ and $\mathcal{P}\pi_{p_i} = \pi_{p_{i+1 \pmod k}}$ for $i = 1,2,\ldots,k$.

Now let α be the smallest $1k(k-1)\ldots2$ occurrence in $\mathcal{P}\pi$ and apply $\mathcal{P}$ again. Repeat until the resulting permutation avoids the pattern $1k(k-1)\ldots2$.

To prove that Γ_k is well-defined we show that, with respect to the indices of elements, the smallest $1k\ldots2$ occurrence in $\mathcal{P}\pi$ is larger than the smallest $1k\ldots2$ occurrence in π. Hence,

$\mathcal{P}^j\tau$ has no $1k\ldots2$ occurrence for some finite j.

Let τ be our permutation and let $\gamma_1\ldots\gamma_k$ be the smallest $1k\ldots2$ occurrence in τ. Assume, for a contradiction, that $\mathcal{P}\tau$ has a $1k\ldots2$ occurrence $\sigma=\sigma_1\ldots\sigma_k$ with $\sigma_1^{-1}<\gamma_1^{-1}$. Since τ has no such occurrence, we must have

$$\{\sigma_1,\ldots,\sigma_k\}\cap\{\gamma_1,\ldots,\gamma_k\}\neq\emptyset.$$

Let γ_j be the minimal element in the above intersection such that there exists $\ell\geq1$, ℓ maximal, where we have (in $\mathcal{P}\tau$)

$$\sigma=\sigma_1\ldots\sigma_{i-1}\gamma_j\sigma_{i+1}\ldots\sigma_{i+\ell}\sigma_{i+\ell+1}\ldots\sigma_k \tag{1}$$

with

$$\{\sigma_{i+1},\ldots,\sigma_{i+\ell}\}\cap\{\gamma_1,\ldots,\gamma_k\}=\emptyset \tag{2}$$

(so that $\{\sigma_{i+\ell+1},\ldots,\sigma_k\}\subseteq\{\gamma_1,\ldots,\gamma_k\}$ where $\{\sigma_{i+\ell+1},\ldots,\sigma_k\}$ is possibly empty). We may further require that there do not exist $\gamma_{m+1},\ldots,\gamma_{m+\ell}$ such that

$$\sigma_1\ldots\sigma_{i-1}\gamma_j\gamma_{m+1}\ldots\gamma_{m+\ell}\sigma_{i+\ell+1}\ldots\sigma_k \tag{3}$$

is also a $1k\ldots2$ occurrence.

Since this is quite a bit to require of γ_j, we must prove its existence. Assume, for a contradiction, that no such γ_j exists. Then, for all γ_i, $1\leq i\leq k$, we have no such ℓ satisfying (1) and (2), or there exist $\gamma_{m+1},\ldots,\gamma_{m+\ell}$ such that (3) is also a $1k\ldots2$ occurrence. In either case, there exists a $1k\ldots2$ occurrence of the form $\sigma_1\ldots\sigma_x\gamma_y\ldots\gamma_z$ in $\mathcal{P}\tau$ with $\{\sigma_1,\ldots,\sigma_x\}\cap\{\gamma_1,\ldots,\gamma_k\}=\emptyset$ and hence, $\sigma_1\ldots\sigma_x\gamma_y\ldots\gamma_z$ is also in τ, a contradiction since such an occurrence is smaller than $\gamma_1\ldots\gamma_k$.

We rewrite (1) as

$$\sigma_1\ldots\sigma_{i-1}\gamma_j\sigma_{i+1}\ldots\sigma_{i+\ell}\gamma_{j+m+1}\ldots\sigma_k. \tag{4}$$

From (4) we must have, by assumption (3), $|\{\gamma_{j+1},\ldots,\gamma_{j+m}\}|<|\{\sigma_{i+1},\ldots,\sigma_{i+\ell}\}|$. But then

$$\gamma_1\ldots\gamma_{j-1}\sigma_{i+1}\ldots\sigma_{i+\ell}\gamma_{j+m+1}\ldots\gamma_k$$

contains a smaller $1k\ldots2$ occurrence in τ than $\gamma_1\ldots\gamma_k$, a contradiction. Thus, $\mathcal{P}\tau$ contains no $1k\ldots2$ occurrence $\sigma_1\ldots\sigma_k$ with $\sigma_1^{-1}<\gamma_2^{-1}$ (order in $\mathcal{P}\tau$).

To finish proving that Γ_k is well-defined, assume, for a contradiction, that $\gamma_2\beta_1\ldots\beta_{k-1}$ in $\mathcal{P}\tau$ is a smaller $1k\ldots2$ occurrence (with respect to the indices) than $\gamma_1\ldots\gamma_k$ is in τ. Then we must have $\beta_1^{-1}<\gamma_3^{-1}$ (order in $\mathcal{P}\tau$). However, this implies that $\gamma_1\beta_1\gamma_3\ldots\gamma_k$ is a smaller $1k\ldots2$ occurrence in τ than $\gamma_1\ldots\gamma_k$, a contradiction. This concludes the proof that Γ_k is well-defined.

To show that Γ_k is a bijection, note that $\Gamma_k{}^{-1}$ is found by using the operation $\mathcal{Q}$ where $\mathcal{Q}\pi_i=\pi_i$ if $i\notin\{p_1,p_2,\ldots,p_k\}$ and $\mathcal{Q}\pi_{p_i}=\pi_{p_{i-1\,(\mathrm{mod}\,k)}}$ for $i=1,2,\ldots,k$.

Using Γ_k, along with the standard bijections, we immediately have the following theorem.

Theorem. Let $k \geq 2$. The following patterns are Wilf equivalent.

$$\begin{array}{llll} A. & k(k-1)\cdots 1 & B. & 1k(k-1)\cdots 2 \\ C. & 12\cdots k & D. & 23\cdots k1 \\ E. & k12\cdots(k-1) & F. & (k-1)(k-2)\cdots 1k \end{array}$$

Remark. Γ_4 proves the Wilf equivalence of 1234 and 4123, a result of Stankova [28].

4. Enumeration Results

In this section we give some enumeration results along with their relevant references. The first table gives the number of permutations that avoid a single pattern.

Avoid 1

Class $\mathcal{W}$	$s_n(T), T \in \mathcal{W}$	References
$\overline{(123)}$	$C_n = \frac{1}{n+1}\binom{2n}{n}$	[14], [12]
$\overline{(1234)}$	$2\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2\frac{3k^2-(2k+1)(n-1)}{(k+1)^2(k+2)(n-k+1)}$	[10]
$\overline{(1342)}$	$\frac{(-1)^{n-1}(7n^2-3n-2)}{2} + 3\sum_{k=2}^{n}(-1)^{n-k}2^k\binom{2k-4}{k-2}\frac{(n-k+2)(n-k+1)}{k(k+1)}$	[1]
$\overline{(1324)}$	?	open

$\overline{(123)} = S_3$; $\overline{(1234)} = \{1234, 1243, 2134, 2143, 3412, 3421, 4312, 4321\}$; $\overline{(1342)} = \{1342, 1423, 2341, 2413, 2431, 3124, 3142, 3241, 4132, 4213\}$; $\overline{(1324)} = \{1324, 4231\}$.

We continue with a table giving the number of permutations that contain a given pattern exactly once.

Contain 1

Class $\mathcal{W}$	$s_n(T), T \in \mathcal{W}$	Reference
$\overline{(\emptyset;123)}$	$\frac{3}{n}\binom{2n}{n-3}$ for $n \geq 3$	[20]
$\overline{(\emptyset;132)}$	$\binom{2n-3}{n-3}$ for $n \geq 3$	[2]

$\overline{(\emptyset;123)} = \{(\emptyset;123), (\emptyset;321)\}$; $\overline{(\emptyset;132)} = \{(\emptyset;132), (\emptyset;213), (\emptyset;231), (\emptyset;312)\}$.

The penultimate table gives the number of permutations that avoid a given pattern and contain a different given pattern exactly once. The last table gives the number of permutations that contain each of two given (not necessarily distinct) patterns exactly once.

Avoid 1, Contain 1

Class $\mathcal{W}$	$s_n(T), T \in \mathcal{W}$	Reference
$\overline{(123;321)}$	0 for $n \geq 6$	——
$\overline{(123;132)}$	$(n-2)2^{n-3}$ for $n \geq 3$	[20]
$\overline{(123;231)}$	$2n-5$ for $n \geq 3$	[17]
$\overline{(132;213)}$	$n2^{n-5}$ for $n \geq 4$	[16]
$\overline{(132;231)}$	2^{n-3} for $n \geq 3$	[16]

$\overline{(123;321)} = \{(123;321),(321;123)\}$; $\overline{(123;132)} = \{(123;132),(123;213),(132;123),(213;123),(231;321),(312;321),(321;231),(321;312)\}$; $\overline{(123;231)} = \{(123;231),(123;312),(132;321),(213;321),(231;123),(312;123),(321;132),(321;213)\}$; $\overline{(132;213)} = \{(132;213),(213;132),(231;312),(312;231)\}$; $\overline{(132;231)} = \{(132;231),(132;312),(213;231),(213;312),(231;132),(231;213),(312;132),(312;213)\}$.

Contain 2

Class $\mathcal{W}$	$s_n(T), T \in \mathcal{W}$	Reference
$\overline{(\emptyset;\{123,321\})}$	0 for $n \geq 6$	——
$\overline{(\emptyset;\{123,231\})}$	$2n-5$ for $n \geq 5$	[23]
$\overline{(\emptyset;\{123,132\})}$	$\binom{n-3}{2}2^{n-4}$ for $n \geq 5$	[22]
$\overline{(\emptyset;\{132,213\})}$	$(n^2+21n-28)2^{n-9}$ for $n \geq 7$	[23]
$\overline{(\emptyset;\{132,231\})}$	2^{n-3} for $n \geq 4$	[23]
$\overline{(\emptyset;\{123,123\})}$	$\frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$ for $n \geq 4$	[9]
$\overline{(\emptyset;\{132,132\})}$	$\frac{(n-2)^2(n+21)-4}{2n(n-1)}\binom{2n-6}{n-4}$ for $n \geq 4$	[15]

$\overline{(\emptyset;\{123,321\})} = \{(\emptyset;\{123,321\}),(\emptyset;\{321,123\})\}$; $\overline{(\emptyset;\{123,231\})} = \{(\emptyset;\{123,231\}),(\emptyset;\{123,312\}),(\emptyset;\{132,321\}),(\emptyset;\{213,321\})\}$; $\overline{(\emptyset;\{123,132\})} = \{(\emptyset;\{123,132\}),(\emptyset;\{123,213\}),(\emptyset;\{231,321\}),(\emptyset;\{312,321\})\}$; $\overline{(\emptyset;\{132,213\})} = \{(\emptyset;\{132,213\}),(\emptyset;\{231,312\})\}$; $\overline{(\emptyset;\{132,231\})} = \{(\emptyset;\{132,231\}),(\emptyset;\{132,312\}),(\emptyset;\{213,231\}),(\emptyset;\{213,312\})\}$; $\overline{(\emptyset;\{123,123\})} = \{(\emptyset;\{123,123\}),(\emptyset;\{321,321\})\}$; $\overline{(\emptyset;\{132,132\})} = \{(\emptyset;\{132,132\}),(\emptyset;\{213,213\}),(\emptyset;\{231,231\}),(\emptyset;\{312,312\})\}$.

5. An Interesting Generating Function

An interesting generating function can be obtained when we consider $\{s_n(132; \{(123)^r\})\}_n$. To this end, let

$$f(x, y; k) = \sum_{n,r} s_n(132, \{(123\ldots k)^r\})x^n y^r.$$

In [25], the authors show that

$$f(x, y; 3) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{xy}{1 - \cfrac{xy^3}{1 - \cfrac{xy^6}{\ddots}}}}}}$$

in which the n^{th} numerator is $xy^{\binom{n-1}{2}}$. This result was the apparent impetus for several papers, of which we note a few.

Let $F_r(x; k) = [y^r]f(x, y; k)$, i.e., the coefficient of y^r in $f(x, y; k)$. Let $U_n(x)$ denote the Chebyshev polynomials of the second kind, defined by

$$U_n(\cos(x)) = \frac{\sin((n+1)x)}{\sin(x)}.$$

Mansour and Vainshtein [18] show that for any $k \geq 1$ and $1 \leq r \leq k(k+3)/2$,

$$F_r(x; k) = \frac{x^{\frac{r-1}{2}} U_{k-1}^{r-1}\left(\frac{1}{2\sqrt{x}}\right)}{U_k^{r+1}\left(\frac{1}{2\sqrt{x}}\right)} \sum_{j=0}^{\lfloor (r-1)/k \rfloor} \binom{r - kj + j - 1}{j} \left(\frac{U_k\left(\frac{1}{2\sqrt{x}}\right)}{x^{\frac{k-2}{2k}} U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}\right)^{kj}.$$

(The case $F_0(x; k)$ had already been determined in terms of Chebyshev polynomials by Chow and West [4].)

This result was followed closely by a result of Jani and Rieper [11], who give a bijection between ordered trees and $S_n(132; \{(123\ldots k)^r\})$.

We then come to a quote found in [13]:

> *Whenever you encounter generating functions which can be expressed in terms of continued fractions or Chebyshev polynomials, then expect that Dyck or Motzkin paths are at the heart of your problem, and will help to solve it.*
>
> \- Krattenthaler

In this paper, Krattenthaler reproves and extends all the above results as well as proves new ones with continued fractions, Chebyshev polynomials of the second kind, and Dyck and Motzkin paths.

6. Refined Restricted Permutations

Combinatorics without algebra ... is like sex without love.

- Anthony Joseph

We now move on to some very recent results on restricted permutations. In [24], the study of restricted permutations refined by the number of fixed points was initiated (to the best of my knowledge) in an attempt to combine the "purely combinatorial" with the algebraic properties of permutations.

We first look at restricted permutations classified by cycle structure.

Cycle structure	$S_6(123)$	$S_6(132)$	$S_6(213)$	$S_6(321)$	$S_6(231)$	$S_6(312)$
1^6	0	1	1	1	1	1
$1^4 2^1$	0	5	5	5	9	9
$1^3 3^1$	0	8	8	8	12	12
$1^2 2^2$	10	9	9	9	18	18
$1^2 4^1$	8	12	12	12	17	17
$1^1 2^1 3^1$	24	20	20	20	24	24
$1^1 5^1$	24	20	20	20	20	20
2^3	10	5	5	5	4	4
$2^1 4^1$	26	20	20	18	12	12
3^2	6	8	8	10	3	3
6^1	24	24	24	24	12	12
Sum	132	132	132	132	132	132

In this table, we see that $S_6(132)$ and $S_6(213)$ have the same cycle structure breakdown, and that the same holds true for $S_6(231)$ and $S_6(312)$. This is fairly easy to explain.

Theorem (Robertson, Saracino, Zeilberger [24]) Let $\gamma \in S_n$ be given by $\gamma_i = n+1-i$ for $1 \leq i \leq n$. For $\pi \in S_n$, let $\pi^\star = \gamma\pi\gamma^{-1}$. The number of occurrences of the pattern 213 (resp. 312) in π equals the number of occurrences of the pattern 132 (resp. 231) in $\pi^\star$.

This is just the reversal bijection followed by the complement bijection. However, written in the above form as a conjugation, we see that the cycle structure is preserved.

Digging a little deeper into the cycle structure breakdown, we notice that $S_6(321)$ has a breakdown very close to that of $S_6(132)$. In fact, the only place where a difference occurs in the table is boxed in. And, if we look at the number of fixed points, we have the same breakdown according to the number of fixed points.

Let $S_n^k(S;T)$ be the elements of $S_n(S;T)$ with exactly k fixed points and let $s_n^k(T) = |S_n^k(T)|$.

Robertson, Saracino, and Zeilberger [24] prove that for all n and all k, $0 \leq k \leq n$,

- $s_n^k(132) = s_n^k(231) = s_n^k(321)$
- $s_n^k(231) = s_n^k(312)$
- $s_n^k(123)$

are the three classes, where we have

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	2	2	0	1					
4	6	4	3	0	1				
5	18	13	6	4	0	1			
6	57	40	21	8	5	0	1		
7	186	130	66	30	10	6	0	1	
8	622	432	220	96	40	12	7	0	1

$\mathbf{s_n^k(132) = s_n^k(213) = s_n^k(321)}$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	1	3	0	1					
4	4	4	5	0	1				
5	10	16	8	7	0	1			
6	31	44	35	12	9	0	1		
7	94	146	102	59	16	11	0	1	
8	303	464	362	180	87	20	13	0	1

$\mathbf{s_n^k(231) = s_n^k(312)}$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	2	3	0	0					
4	7	4	3	0	0				
5	20	20	2	0	0	0			
6	66	48	18	0	0	0	0		
7	218	183	28	0	0	0	0	0	
8	725	552	153	0	0	0	0	0	0

$\mathbf{s_n^k(123)}$

It is quite amazing that $s_n^k(132) = s_n^k(321)$. Robertson, Saracino, and Zeilberger [24] give a fairly technical proof. This was followed by a proof due to Elizalde [7], a student of Stanley, who refines this result further by showing that we have "number of excedances" in addition to the number of fixed points. Pak and Elizalde [21] follow this up by giving a nice bijective proof.

An interesting sequence appears when we consider 132-avoiding derangements. We see that

$$\{s_n^0(132)\}_{n\geq 0} = 1, 0, 1, 2, 6, 18, 57, 186, 622, \ldots.$$

This is Fine's sequence, initially given in [8], which stems from similarity relations. (For a good survey of Fine's sequence, see [6].) Fine's sequence $\{F_n\}_{n\geq 0}$ is characterized by

$$2F_n + F_{n-1} = C_n, \tag{5}$$

which may be thought of as the defining equation for the sequence.

We now take a very brief sidestep to look at similarity relations.

Definition. A similarity relation, R, is a binary relation on an ordered set which is reflexive, symmetric, but not necessarily transitive, with the condition that if iRk and $i < j < k$ then

iRj and jRk. R is said to have k isolated points if k is the number of $i \in \{1, 2, \ldots, n\}$ such that there does not exist $j \neq i$ with iRj. If $k = 0$, we say that the similarity relation is nonsingular. We denote by $SR_n(k)$ the set of similarity relations on $\{1, 2, \ldots, n\}$ with k isolated points.

In [24], the authors show that, $s_n^k(321) = |SR_n(k)|$, a somewhat unexpected connection between two at-first-sight unrelated structures.

Stepping back to refined restricted permutations, in [24] we find that, for $\alpha \in \{132, 213, 321\}$, $0 \leq k \leq n$:

$$s_n^k(\alpha) = \sum_{j=0}^{n-k} (-1)^j \left(\frac{j+k+1}{n+1}\right) \binom{2n-k-j}{n} \binom{j+k}{k}.$$

In particular we obtain a formula for the Fine numbers:

$$F_n = \sum_{j=0}^{n} (-1)^j \left(\frac{j+1}{n+1}\right) \binom{2n-j}{n}.$$

The worst thing you can do to a problem is solve it completely.

- Dan Kleitman

As such, there are no nontrivial enumerations of $s_n^k(\alpha)$ for $\alpha \in \{123, 231, 312\}$ in [24].

Another unanswered question, begging for a solution, follows. For $\beta \in \{132, 213, 321\}$ we have from (5)

$$2s_n^0(\beta) + s_{n-1}^0(\beta) = s_n(\delta)$$

for any $\delta \in S_3$. Can this be explained bijectively by letting $\delta = \beta$?

The next set of results on refined restricted permutations comes from [19]: following the steps of Simion and Schmidt, Mansour and Robertson [19] provide enumerative results for $S_n^k(S)$ for all $S \subseteq S_3$, $|S| \geq 2$.

6.1 Refined Restricted Involutions

From the cycle structure breakdown table we notice that if we restrict our attention to involutions it appears that we may have some more unexpected structure.

Define $i_n^k(S;T)$ to be the number of elements of $S_n^k(S;T)$ that are involutions.

Deutsch, Robertson, and Saracino [5] prove that

- $i_n^k(132) = i_n^k(231) = i_n^k(321)$

- $i_n^k(231) = i_n^k(312)$
- $i_n^k(123)$

are the three classes. The proofs use the RSK correspondence as well as provide new bijections with Dyck paths.

One particularly nice formula follows. For $\alpha \in \{132, 213, 321\}$, $0 \le k \le n$,

$$i_n^k(\alpha) = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} & \textit{for } n+k \textit{ even} \\ 0 & \textit{for } n+k \textit{ odd}. \end{cases}$$

Catalan strikes again!

$$i_{2n}^0(132) = C_n = i_{2n-1}^1(132).$$

Hence, the number of 132-avoiding permutations on n letters is the Catalan number; the number of 132-avoiding derangements on n letters is the Fine number; and the number of 132-avoiding derangement involutions on $2n$ letters is the Catalan number.

Having gone from Catalan to Fine and back, we stop.

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