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Some new Applications of Orbit Harmonics

by

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Abstract

We prove a new result in the Theory of Orbit Harmonics and derive from it a new proof of the Cohen–Macauliness of the ring $\mathcal{QI}_m(G)$ of m-Quasi-Invariants of a Coxeter Group G. Using the non-degeneracy of the fundamental bilinear form on $\mathcal{QI}_m(G)$, this approach yields also a direct and simple proof that the quotient of $\mathcal{QI}_m(G)$ by the ideal generated by the homogeneous G-invariants affords a graded version of the left regular representation of G. Originally all of these results were obtained as a combination of some deep work of Etingof–Ginzburg [3], Feigin–Veselov [6] and Felder–Veselov [5]. The arguments here are quite elementary and self contained, except those using the non-degeneracy of the fundamental bilinear form.

Introduction

Throughout this paper we let G be a finite reflection group of $n \times n$ matrices, $\Sigma(G)$ will denote its class of reflections and for each $s \in \Sigma(G)$ we choose once and for all a vector α_s perpendicular to the reflecting hyperplane of s. In this manner the linear form giving the equation of this reflecting hyperplane is given by the scalar product (x, α_s). This given, a polynomial $P(x) = P(x_1, x_2, \ldots, x_n)$ is said to be G-m-Quasi-Invariant if and only if for all $s \in \Sigma(G)$ the polynomial $(1-s)P(x)$ is divisible by $(x, \alpha_s)^{2m+1}$. It easily shown that G-m-Quasi-Invariants form a finitely generated G-invariant graded subalgebra of the polynomial ring $\mathbf{Q}[X_n]$, where X_n is short for $x_1, x_2, \ldots, x_n$. Denoting this algebra by $\mathcal{QI}_m[G]$, we have the proper inclusions

$$\mathcal{QI}_1[G] \supset \mathcal{QI}_2[G] \supset \cdots \supset \mathcal{QI}_m[G] \supset \cdots.$$

Clearly, $\mathcal{QI}_0[G] = \mathbf{Q}[X_n]$, and $\mathcal{QI}_\infty[G]$ may be viewed as the algebra Λ_G of G-invariant polynomials. Our goal here is to derive new proofs of some of the basic results on m-Quasi-Invariants by means of the Theory of Orbit Harmonics. More generally, using this theory, we will prove the following basic result.

Theorem I.1

Let $\mathbf{A}$ be a degree-graded G-invariant subalgebra of $\mathbf{Q}[X_n]$, and suppose that

(i) $\mathbf{A} \supseteq \Lambda_G$,

(ii) *for some non-trivial homogeneous G-invariant $B(x)$ we have $B(x)\mathbf{Q}[X_n] \subseteq \mathbf{A}$,*

(iii) *$\mathbf{A}$ has a G-invariant, non-degenerate, symmetric bilinear form $\langle\ ,\ \rangle_{\mathbf{A}}$, graded by degree†,*

(iv) *the orthogonal complement $\mathbf{H}_G(\mathbf{A})$, with respect to $\langle\ ,\ \rangle_{\mathbf{A}}$, of the ideal $\mathcal{J}_G(\mathbf{A})$ generated in $\mathbf{A}$ by the homogeneous G-invariants has dimension bounded by the order of G.*

Then both $\mathbf{H}_G(\mathbf{A})$ and $\mathbf{A}/\mathcal{J}_G(\mathbf{A})$ afford the regular representation of G and $\mathbf{A}$ is free over Λ_G.

To state a further application of this Theorem we need to introduce further notation and make some preliminary observations. To begin with, we should note that within G-m-Quasi-Invariants we find m-analogues of all the ingredients that occur in the relationship between the polynomial ring $\mathbf{Q}[X_n]$ and the ring of invariants Λ_G. For instance, let us recall that the space H_G of "*G-Harmonics*" is defined as the orthogonal complement of the ideal $\mathcal{J}_G$ generated by the homogeneous G-invariants. Now, it is well known that, for a Coxeter group G of $n \times n$ matrices, Λ_G is a free polynomial ring on n homogeneous generators $q_1(x), q_2(x), \dots q_n(x)$. It follows from this that we have

$$H_G \;=\; \big\{ P \in \mathbf{Q}[X_n] \;:\; q_k(\partial_x)P(x) = 0 \;\;\text{for all}\quad k = 1, 2, \dots, n \big\}, \tag{I.1}$$

where for a polynomial $P(x)$ we set $P(\partial_x) = P(\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$. It is also shown in [12] that H_G is the linear span of all the partial derivatives of the discriminant

$$\Pi_G(x) \;=\; \prod_{s \in \Sigma(G)} (x, \alpha_s)\,. \tag{I.2}$$

† Homogeneous elements of $\mathbf{A}$ of different degrees are orthogonal with respect to $\langle\ ,\ \rangle_{\mathbf{A}}$.

In symbols,

$$H_G \;=\; \big\{ Q(\partial_x)\Pi_G(x)\; Q \in \mathbf{Q}[X_n]\, \big\}. \tag{I.3}$$

We have m-analogues of both (I.1) and (I.3). To describe them we need to recall that in [2] Chalykh and Veselov show that to each homogeneous m-Quasi-Invariant $Q(x)$ of degree d there corresponds a unique homogeneous differential operator, acting on $\mathcal{QI}_m[G]$, of the form

$$\gamma_Q(x,\partial_x) \;=\; Q(\partial_x) + \sum_{|q|<d} c_q(x)\partial_x^q, \tag{I.4}$$

where $\partial_x^q = \partial_{x_1}^{q_1}\partial_{x_2}^{q_2}\cdots\partial_{x_n}^{q_n}$ and $|q| = q_1+q_2+\cdots+q_n$, with $c_q(x)$ a rational function in $x_1, x_2, \ldots, x_n$ with a denominator which factors into a product of the linear forms (x,α_s). In fact, there is even an explicit formula for $\gamma_q(x,\partial_x)$ which is due to Berest [1]. This is

$$\gamma_Q(x,\partial_x) \;=\; \frac{1}{2^d d!}\sum_{k=0}^{d}\binom{d}{k} L_m(G)^{d-k}\underline{Q}L_m(G)^k \tag{I.5}$$

where $\underline{Q}$ denotes the operator "*multiplication by* $Q(x)$",

$$L_m(G) \;=\; \Delta_2 \;-\; 2m\sum_{s\in\Sigma(G)}\frac{1}{(x,\alpha_s)}\partial_{\alpha_s} \tag{I.6}$$

with Δ_2 the ordinary Laplacian and ∂_{α_s} the directional derivative corresponding to α_s. In fact, it develops that the linear extension of the map $Q \to \gamma_Q(x,\partial_x)$ defined by (I.5) yields an algebra isomorphism of $\mathcal{QI}_m[G]$ onto the algebra of operators of the form (I.4) that commute with $L_m(G)$. In particular for all $P, Q \in \mathcal{QI}_m[G]$ we have

$$\gamma_{PQ}(x,\partial_x) \;=\; \gamma_P(x,\partial_x)\gamma_Q(x,\partial_x). \tag{I.7}$$

This given, a deep result of Opdam [11] implies that the bilinear form defined by setting, for $P, Q \in \mathcal{QI}_m[G]$

$$\big\langle P\;,\;Q\big\rangle_m \;=\; \gamma_P(s,\partial_x)Q(x)\big|_{x=0} \tag{I.8}$$

is non-degenerate on $\mathcal{QI}_m[G]\times\mathcal{QI}_m[G]$. Now, the space $\mathbf{H}_G(m)$ of m-Harmonics is simply defined as the orthogonal complement, with respect to $\big\langle\;,\;\big\rangle_m$, of the ideal

$\mathcal{J}_G(m)$ generated in $\mathcal{QI}_m[G]$ by the homogeneous G invariants. This given, the m-analogue of (I.1) is simply

$$\mathbf{H}_G(m) \;=\; \big\{ P \in \mathbf{Q}[X_n] \;:\; \gamma_{q_k}(x,\partial_x)P(x) = 0 \;\text{ for all }\quad k = 1,2,\ldots,n \big\} \tag{I.9}$$

It should be mentioned that it follows from this that $\mathbf{H}_G(m) \subseteq \mathcal{QI}_m[G]$. This is an immediate consequence of the remarkable property of the operator $L_m(G)$ to the effect that for any two polynomials P, Q we have $L_m(G)P = Q$ with $Q \in \mathcal{QI}_m[G]$ if and only if $P \in \mathcal{QI}_m[G]$. In particular, any polynomial in the kernel of $L_m(G)$ is necessarily in $\mathcal{QI}_m[G]$.

Now, it develops that a beautiful m-analogue of (I.3) was conjectured by Feigin and Veselov in [6] and proved by Etingov and Ginsburg in [3]. In the present notation, this result may be stated as follows.

Theorem I.2 (Theorem 6.20 of [3])

$$\mathbf{H}_G(m) \;=\; \big\{ \gamma_Q(x,\partial_x)\Pi_G^{2m+1}(x) \;:\; Q \in \mathcal{QI}_m[G] \,\big\}. \tag{I.10}$$

In fact, if $\mathcal{B} \subset \mathcal{QI}_m[G]$ is any basis for the quotient of $\mathcal{QI}_m[G]/\mathcal{J}_G(m)$, then the collection

$$\mathcal{F} \;=\; \big\{ \gamma_b(x,\partial_x)\,\Pi_G^{2m+1}(x) \;:\; b \in \mathcal{B} \big\} \tag{I.11}$$

is a basis for $\mathbf{H}_G(m)$.

We shall show here that, by combining the theory of orbit harmonics with a Hilbert series result of Felder–Veselov, we can also obtain a rather nice new proof of this result.

This paper consists of 6 sections. In the first four sections we establish Theorem I.1 by a sequence of five steps as follows. In the first step we introduce, for a given G-regular vector $a = (a_1, a_2, \ldots, a_n)$, the ring $\mathbf{A}_{[a]}$ of the G-orbit of a and show that it affords the regular representation of G. In the second step we introduce its graded version $\mathbf{gr}\,\mathbf{A}_{[a]}$ and show that it carries a graded version of the regular representation of G. In the third step, we introduce the space $H_{[a]}(\mathbf{A})$ of orbit $\mathbf{A}$-harmonics and use the non-degeneracy of the form $\langle\;,\;\rangle_{\mathbf{A}}$ to show that $H_{[a]}(\mathbf{A})$ and $\mathbf{gr}\,\mathbf{A}_{[a]}$ are equivalent as graded G-modules. In the fourth step we use the dimension bound in (iv) to show that $H_{\mathbf{A}}$ and $H_{[a]}(\mathbf{A})$ are one and the same. In the fifth step we use again the non-degeneracy of the form to show that $H_{\mathbf{A}}$ and the quotient ring $\mathbf{A}/\mathcal{I}_G(\mathbf{A})$ are equivalent as graded G-modules. In the sixth and final step we combine property (ii) with a simple Hilbert series argument and derive that $\mathbf{A}$ is free over Λ_G.

In the fifth section we use some basic facts about Coxeter groups and their corresponding m-Quasi-Invariants to show that the hypotheses of Theorem I.1 are satisfied when $\mathbf{A} = \mathcal{QI}_m[G]$ and thereby derive that $\mathcal{QI}_m[G]$ is a Cohen–Macaulay module over the ring of G-invariants. We believe that the resulting proof is simpler, more elementary and more revealing than the previous proofs.

In the sixth section we prove Theorem I.2. This last section has a substantially different character than the previous ones. It makes crucial use of deep results such as the symmetry of the Hilbert series of m-Harmonics and properties of the Baker–Akhiezer function of $\mathcal{QI}_m[G]$ whose proofs, to this date, are far from being elementary.

Finally, we should mention that some of the methods of the Theory of Orbit Harmonics we use here were developed and successfully used in [7] in the study of the Garsia–Haiman modules. This may not be an accident since, in a sense, m-Quasi-Invariants may be viewed as the Jack polynomial case of the so-called $n!$-conjecture. In particular, since m-Quasi-Invariants arose in the study of operators which commute with $L_m(G)$, by analogy, there must be spaces arising from a study of operators that commute with the Macdonald operator. We believe that we have only seen here the tip of a mathematical iceberg gravid with combinatorial implications. To dispell any doubts we may have on this score, we urge the reader to view the surprising facts that emerged in [9] in the study of the simplest possible cases. Namely when the underlying reflection group reduces to the symmetric group S_2 or the dihedral group D_2.

1. The orbit Ring

In this paper we adopt the convention that an $n \times n$ matrix $A = \|a_{ij}\|_{1\le i,j\le n}$ acts on a point $x = (x_1, x_2, \ldots, x_n)$ by right multiplication. In this manner the action of A on a polynomial $P(x) = P(x_1, x_2, \ldots, x_n)$ is simply expressed in the form

$$T_A P(x) = P(xA).$$

We will work with a fixed finite Coxeter group G of $n \times n$ matrices and a general algebra $\mathbf{A}$ which satisfies the hypotheses of Theorem I.1. The specialization $\mathbf{A} = \mathcal{QI}_m[G]$ will only take place in Section 5. In the first four sections we will make use of the invariant $B(x)$ which satisfies hypothesis in (ii). This given, for our developments it is necessary that we choose once and for all a point $a = (a_1, a_2, \ldots, a_n)$ which satisfies the following two conditions

$$\text{a)} \quad B(a) \neq 0\,, \qquad \text{and} \qquad \text{b)} \quad \Pi_G(a) = \prod_{s \in \Sigma(G)} (a, \alpha_s) \neq 0. \tag{1.1}$$

Throughout the paper we denote by $[a]_G$ the G-orbit of a. In symbols,

$$[a\,]_G \;=\; \{aA \,:\, A \in G\}\,. \tag{1.2}$$

Note that because of (1.1) b) the point a cannot lie in any of the reflecting hyperplanes of G. Since G is a Coxeter group, it follows that the stabilizer of a, namely the subgroup

$$G_a \;=\; \{A \in G \,:\, aA = a\}$$

reduces to the identity. This assures that $[a\,]_G$ consists of $|G|$ distinct points.

Next we need a polynomial $\phi_a(x)$ such that

$$\phi_a(x) \;=\; \begin{cases} 1 & \text{if } x = a, \\ \\ 0 & \text{if } x = b \text{ with } b \in [a\,]_G \text{ and } b \neq a. \end{cases} \tag{1.3}$$

The construction of this polynomial can be carried out in many ways. For the moment it is immaterial how we pick $\phi_a(x)$. However, to get the best results in Section 5, it will be necessary that $\phi_a(x)$ is constructed to have the smallest possible degree. It turns out that we can never do better than the degree of Π_G. Indeed, suppose (1.3) holds true and set

$$F(x) \;=\; \sum_{B \in G} \det(B)\phi_a(xB)\,.$$

Now note that from (1.3) it follows that $F(a) = 1 \neq 0$. Moreover we also have

$$F(xA^{-1}) \;=\; \det(A)\,F(x) \qquad\qquad \text{(for all } A \in G)$$

and this implies that $F(x)$ is a G-invariant multiple of $\Pi_G(x)$. Thus $degree\big(F(x)\big) \geq degree\big(\Pi_G(x)\big)$ and this can only happen when $degree\big(\phi_a(x)\big) \geq degree\big(\Pi_G(x)\big)$.

To show that this minimum can be achieved, we use the well known fact that if $\mathcal{B} = \{h_1, h_2, \ldots, h_{|G|}\}$ is a basis for the G-Harmonics then every polynomial $P(x)$ has an expansion of the form

$$P(x) \;=\; \sum_{i=1}^{|G|} h_i(x) A_i(x) \qquad\qquad (\text{with } A_i(x) \in \Lambda_G). \tag{1.4}$$

This given, we start by constructing in any manner we please an initial polynomial $P_a(x)$ satisfying

$$P_a(x) \;=\; \begin{cases} 1 & \text{if } x = a, \\ \\ 0 & \text{if } x = b \text{ with } b \in [a\,]_G \text{ and } b \neq a, \end{cases}$$

then use (1.4) and obtain the expansion

$$P_a(x) \;=\; \sum_{i=1}^{|G|} h_i(x) A_{i,a}(x) \qquad\qquad (\text{with } A_{i,a}(x) \in \Lambda_G)\,.$$

This done, we claim that we can take

$$\phi_a(x) \;=\; \sum_{i=1}^{|G|} h_i(x) A_{i,a}(a)\;. \tag{1.5}$$

In fact, note that the G-invariance of the coefficients $A_{i,a}(x)$ gives

$$P_a(b) \;-\; \phi_a(b) \;=\; \sum_{i=1}^{|G|} h_i(b)\big(A_{i,a}(b) - A_{i,a}(a)\big) \;=\; 0 \qquad\qquad \text{for all} \quad b \in [a\,]_G.$$

Thus this choice of $\phi_a(x)$ will also satisfy (1.3). Note next that since (1.5) defines $\phi_a(x)$ to be a G-harmonic polynomial, it follows from (I.3) that

$$\mathit{degree}\big(\phi_a(x)\big) \leq \mathit{degree}\big(\Pi_G(x)\big)\,.$$

Since we already have the reverse inequality, equality must hold true for this choice of ϕ_a.

Now for each $b \in [a]_G$ with $b = aA$ we set

$$\phi_b(x) \;=\; \phi_a(xA^{-1}) \qquad \text{and} \qquad \epsilon(b) \;=\; \det(A)\,.$$

Note that from (1.3) it immediately follows that

$$\phi_b(x) = \begin{cases} 1 & \text{if } x = b, \\ \\ 0 & \text{if } x = b' \text{ with } b' \in [a]_G \text{ and } b' \neq b. \end{cases} \tag{1.6}$$

Next, we let $\mathcal{J}_{[a]_G}(\mathbf{A})$ denote the ideal generated in $\mathbf{A}$ by the elements of $\mathbf{A}$ that vanish on $[a]_G$. In symbols,

$$\mathcal{J}_{[a]_G}(\mathbf{A}) \;=\; \big\{ P(x) \in \mathbf{A} \,:\, P(b) = 0 \;\text{ for all }\; b \in [a]_G \big\}. \tag{1.7}$$

This given, it follows that

Proposition 1.1

The quotient ring

$$\mathbf{A}_{[a]_G} \;=\; \mathbf{A}/\mathcal{J}_{[a]_G}(\mathbf{A})$$

has dimension $|G|$ and affords the left regular representation of G.

Proof

In view of (1.1), we can set

$$\psi_b(x) \;=\; \phi_b(x)\frac{B(x)}{B(a)}\,. \tag{1.8}$$

Note that, since $B(x)$ is G-invariant, we derive from (1.6) that

$$\psi_b(x) = \begin{cases} 1 & \text{if } x = b, \\ \\ 0 & \text{if } x = b' \text{ with } b' \in [a]_G \text{ and } b' \neq b. \end{cases} \tag{1.9}$$

Moreover, from (ii) we also derive that

$$\psi_b(x) \in \mathbf{A} \qquad \text{for all} \quad b \in [a]_G\,.$$

Now note that, for any $P \in \mathbf{A}$, (1.9) gives

$$P(x) \,-\, \sum_{b \in [a]_G} P(b)\, \psi_b(x) \in \mathcal{J}_{[a]_G}(\mathbf{A})\,. \tag{1.10}$$

It will be convenient to express this relation by writing

$$P(x) \;\cong_{[a]_G}\; \sum_{b\in[\,a\,]_G} P(b)\,\psi_b(x)\,. \tag{1.11}$$

Thus the collection

$$\big\{\psi_b(x)\big\}_{b\in[\,a\,]_G} \tag{1.12}$$

is a basis for the quotient ring $\mathbf{A}_{[\,a\,]_G}$.

Since the ideal $\mathcal{J}_{[\,a\,]_G}(\mathbf{A})$ is G-invariant it immediately follows that G acts on $\mathbf{A}_{[\,a\,]_G}$. Thus to complete our proof we only need to compute the character of this action. To this end, note that, for all σ and β in G, we derive from (1.11) that

$$\begin{aligned} T_\sigma\psi_{a\beta}(x) \;=\; \psi_a(x\sigma\beta^{-1}) \;\cong_{[\,a\,]_G}\; & \sum_{\alpha\in G}\psi_a(a\alpha\sigma\beta^{-1})\psi_{a\alpha}(x) \\ \cong_{[\,a\,]_G}\; & \sum_{\alpha\in G}\chi\big(\alpha\sigma\beta^{-1}=id\big)\psi_{a\alpha}(x) \\ \cong_{[\,a\,]_G}\; & \sum_{\alpha\in G}\chi\big(\alpha\sigma=\beta\big)\psi_{a\alpha}(x), \end{aligned}$$

where $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise. Thus the action of G on the basis $\big\{\psi_b(x)\big\}_{b\in[\,a\,]_G}$ is given by the matrix $A(\sigma)=\|\chi\big(\alpha\sigma=\beta\big)\|_{\alpha,\beta\in G}$. It follows that the character of the G action on $\mathbf{A}_{[\,a\,]_G}$ is given by

$$trace\,A(\sigma) \;=\; \sum_{\alpha\in G}\chi\big(\alpha\sigma=\alpha\big) \;=\; \begin{cases} |G| & \text{if } \sigma=id, \\ 0 & \text{if } \sigma\neq id, \end{cases}$$

and this is the character of the left regular representation of G, precisely as asserted.

Remark 1.1

The ring $\mathbf{A}_{[\,a\,]_G}$ is not graded but it has a filtration given by the subspaces

$$\mathcal{H}_{\leq k}(\mathbf{A}_{[\,a\,]_G}) \;=\; \mathcal{L}_{[\,a\,]_G}\big[P\;:\;P\in\mathcal{H}_{\leq k}\big(\mathbf{A}\big)\big] \tag{1.13}$$

where "$\mathcal{L}_{[\,a\,]_G}$" denotes "*Linear Span*" modulo $\mathcal{J}_{[\,a\,]_G}(\mathbf{A})$ and $\mathcal{H}_{\leq k}\big(\mathbf{A}\big)$ is the subspace of $\mathbf{A}$ spanned by its elements of degree $\leq k$. Now note that, since by construction we have

$$degree\,\psi_b(x) \;=\; degree\big(B(x)\big)\times degree(\phi_a), \tag{1.14}$$

it immediately follows from the expansion in (1.11) that

$$\mathbf{A}_{[a]_G} = \mathcal{H}_{\leq d_{\mathbf{A}}}\big(\mathbf{A}_{[a]_G}\big) \tag{1.15}$$

where for convenience we have set

$$d_{\mathbf{A}} = degree\big(B(x)\big) \times degree(\phi_a). \tag{1.16}$$

It will be convenient here and after to adopt the convention that if V is a graded vector space then $\mathcal{H}_{=k}(V)$ denotes the subspace spanned by homogeneous elements of degree k. Likewise $\mathcal{H}_{\leq k}(V)$ denotes the subspace spanned by the homogeneous elements of degree $\leq k$.

2. The graded version of $\mathbf{A}_{[a]_G}$

For each polynomial P we shall here and after denote by $h(P)$ the homogeneous component of highest degree in P. This given, we let $\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})$ be the ideal in $\mathbf{A}$ generated by the highest degree components of elements of $\mathcal{J}_{[a]_G}(\mathbf{A})$. In symbols,

$$\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A}) = \big(h(P) \,:\, P \in \mathcal{J}_{[a]_G}(\mathbf{A})\big)_{\mathbf{A}}\,. \tag{2.1}$$

This brings us to define the "*graded version*" of $\mathbf{A}_{[a]_G}$ as the quotient

$$\mathbf{gr}\,\mathbf{A}_{[a]_G} = \mathbf{A}/\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})\,. \tag{2.2}$$

Since $\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})$ is generated by homogeneous polynomials, $\mathbf{gr}\,\mathbf{A}_{[a]_G}$ is necessarily a graded ring. Note further that if P is any homogeneous polynomial of degree $> d_{\mathbf{A}}$ then the fact that

$$P(x) - \sum_{b \in [a]_G} P(b)\,\psi_b(x) \quad \in \; \mathcal{J}_{[a]_G}(\mathbf{A})$$

immediately implies that $P \in \mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})$. Thus it follows that $\mathbf{gr}\,\mathbf{A}_{[a]_G}$ has the direct sum decomposition

$$\mathbf{gr}\,\mathbf{A}_{[a]_G} = \bigoplus_{k=0}^{d_{\mathbf{A}}} \mathcal{H}_{=k}(\mathbf{gr}\,\mathbf{A}_{[a]_G}). \tag{2.3}$$

It will be convenient to choose once and for all, for each $1 \le k \le d_{\mathbf{A}}$, a collection $\mathcal{B}^{=k} \subset \mathcal{H}_{=k}(\mathbf{A})$ yielding a basis for the subspace $\mathcal{H}_{=k}(\mathbf{gr}\,\mathbf{A}_{[a]_G})$. This given, we have the following useful fact.

Proposition 2.1

The collection

$$\mathcal{B}^{\le k} \;=\; \mathcal{B}^{=0} \uplus \mathcal{B}^{=1} \uplus \cdots \uplus \mathcal{B}^{=k}$$

is also a basis for the subspace $\mathcal{H}_{\le k}(\mathbf{A}_{[a]_G})$. In particular, $\mathcal{B}^{\le d_{\mathbf{A}}}$ is a basis of $\mathbf{A}_{[a]_G}$.

Proof

Clearly the result holds true for $k = 0$ since $\mathcal{B}_{=0}$ reduces to the single constant $\mathbf{1}$. So we may proceed by induction on k. Let us then assume that each $P \in \mathcal{H}_{\le k-1}(\mathbf{A}_{[a]_G})$ has an expansion in terms of $\mathcal{B}^{\le k-1}$. This given, note that any $P \in \mathcal{H}_{\le k}(\mathbf{A})$, viewed as a representative of an element of $\mathbf{gr}\,\mathbf{A}_{[a]_G}$, may be expanded in terms of the basis $\mathcal{B}^{\le k}$. In other words, we will have constants c_ϕ such that

$$P \;=\; \sum_{\phi\in\mathcal{B}^{\le k}} c_\phi\, \phi \;+\; R\;, \tag{2.4}$$

for a suitable $R \in \mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})$. Now the definition of $\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})$ yields that there must be elements $f_i \in \mathcal{J}_{[a]_G}(\mathbf{A})$ and $A_i \in \mathbf{A}$ giving

$$R \;=\; \sum_i A_i\, h(f_i)\;. \tag{2.5}$$

Since R is necessarily of degree $\le k$ we see that there is no loss in assuming that we have

$$degree\, A_i \;\le\; k - degree\, h(f_i)\,. \tag{2.6}$$

Using (2.5) we may rewrite (2.4) in the form

$$P \;=\; \sum_{\phi\in\mathcal{B}^{\le k}} c_\phi\, \phi \,+\, \sum_i A_i\, f_i \;-\; \sum_i A_i\,(f_i - h(f_i))\,, \tag{2.7}$$

and this implies that

$$P \;-\; \sum_{\phi\in\mathcal{B}^{\le k}} c_\phi\, \phi \;\cong_{[a]_G}\; -\sum_i A_i\,(f_i - h(f_i))\,. \tag{2.8}$$

But now (2.6) and the definition of $h(f_i)$ yield that

$$degree \sum_i A_i\,(f_i - h(f_i)) \;\leq\; k-1 \;. \tag{2.9}$$

Thus our inductive hypothesis yields that we have constants d_ϕ giving

$$\sum_i A_i\,(f_i - h(f_i)) \;\cong_{[a]_G} \sum_{\phi\in\mathcal{B}^{\leq k-1}} d_\phi\,\phi \;,$$

and this combined with (2.8) completes the induction. This proves the first assertion. The second assertion follows from the identity in (1.15).

Proposition 2.1 has the following remarkable corollary.

Theorem 2.1

The ring $\mathbf{gr}\,\mathbf{A}_{[a]_G}$ *yields a graded version of the regular representation. In fact, for every* $0\leq k\leq d_{\mathbf{A}}$*, we have the character relation*

$$char\,\mathcal{H}_{=k}(\mathbf{gr}\,\mathbf{A}_{[a]_G}) \;=\; char\,\mathcal{H}_{\leq k}(\mathbf{A}_{[a]_G}) \;-\; char\,\mathcal{H}_{\leq k-1}(\mathbf{A}_{[a]_G}). \tag{2.10}$$

In particular the characters of the subspaces $\mathcal{H}_{\leq k}(\mathbf{A}_{[a]_G})$ *are related to the graded character of* $\mathbf{gr}\,\mathbf{A}_{[a]_G}$ *by the following identity*

$$\sum_{k=0}^{d_{\mathbf{A}}} q^k char\,\mathcal{H}_{=k}(\mathbf{gr}\,\mathbf{A}_{[a]_G}) \;=\; q^{d_{\mathbf{A}}} char\,\mathbf{A}_{[a]_G} \;+\; (1-q)\sum_{k=0}^{d_{\mathbf{A}}-1} q^k\,char\,\mathcal{H}_{\leq k}(\mathbf{A}_{[a]_G}). \tag{2.11}$$

Proof

Let $\mathcal{B}^{\leq k}=\{\phi_i\}_{i=1}^{m_k}$ so that $\mathcal{B}^{=k}=\{\phi_i\}_{m_{k-1}<i\leq m_k}$, and let

$$A(\sigma) \;=\; \|a_{i,j}(\sigma)\|_{i,j=1}^{|G|}$$

be the matrix expressing the action of G on the basis $\mathcal{B}^{\leq d_{\mathbf{A}}}$ as elements of $\mathbf{A}_{[a]_G}$. Since $\mathcal{H}_{\leq k}(\mathbf{A}_{[a]_G})$ is G-invariant it follows that for any $j\leq m_k$ we have the expansion

$$T_\sigma\phi_j \;=\; \sum_{i=1}^{m_k}\phi_i\,a_{i,j}(\sigma) + R_{\sigma,j} \qquad\qquad \text{(with } R_{\sigma,j}\in\mathcal{J}_{[a]_G}(\mathbf{A}) \text{ of degree} \leq k). \tag{2.12}$$

This means that the action of G on the subspace $\mathcal{H}_{\leq k}(\mathbf{A}_{[a]_G})$ induces a representation given by the matrix

$$A^{\leq k}(\sigma) \;=\; \big\| a_{i,j}(\sigma) \big\|_{1\leq i,j\leq m_k}\,. \tag{2.13}$$

Note further that when $m_{k-1} < j \leq m_k$, then ϕ_j is homogeneous of degree k and equating homogeneous terms of degree k in (2.12) we get

$$T_\sigma \phi_j \;=\; \sum_{m_{k-1}<i\leq m_k} \phi_i\, a_{i,j}(\sigma) \;+\; \begin{cases} h(R_{\sigma,j}) & \text{if } \mathit{degree}\ R = k, \\ 0 & \text{if } \mathit{degree}\ R_{\sigma,j} < k, \end{cases}$$
$$(\text{with } R_{\sigma,j} \in \mathcal{J}_{[a]_G}(\mathbf{A})).$$

In either case we derive that

$$T_\sigma \phi_j \;\cong\; \sum_{m_{k-1}<i\leq m_k} \phi_i\, a_{i,j}(\sigma) \qquad\qquad (\text{modulo } \mathbf{gr}\, \mathcal{J}_{[a]_G}(\mathbf{A})).$$

This shows that the action of G on the subspace $\mathcal{H}_{=k}\big(\mathbf{gr}\,\mathbf{A}_{[a]_G}\big)$ induces a representation given by the matrix

$$A^{=k}(\sigma) \;=\; \big\| a_{i,j}(\sigma) \big\|_{m_{k-1}<i,j\leq m_k}\,.$$

Thus

$$\mathit{char}\ \mathcal{H}_{=k}\big(\mathbf{gr}\,\mathbf{A}_{[a]_G}\big) \;=\; \mathit{trace}\ A^{=k} \;=\; \mathit{trace}\ A^{\leq k} \;-\; \mathit{trace}\ A^{\leq k-1}\,.$$

This proves (2.10). This given, multiplying (2.10) by q^k and summing for $0 \leq k \leq d_{\mathbf{A}}$, the identity in (1.15) yields (2.11).

3. The orbit A-Harmonics

Recall that by the hypothesis in (iii) of Theorem I.1, the algebra $\mathbf{A}$ has a non-degenerate bilinear form $\langle\ ,\ \rangle_{\mathbf{A}}$. Using this form we define the $\mathbf{A}$-harmonics of the orbit $[a]_G$ to constitute the orthogonal complement of the ideal $\mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A})$. In symbols

$$\mathbf{H}_{[a]_G}(\mathbf{A}) \;=\; \big\{ P \in \mathbf{A} : \big\langle Q\ ,\ P \big\rangle_{\mathbf{A}} \;=\; 0 \quad \text{for all } Q \in \mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A}) \big\}\,. \tag{3.1}$$

Note that, since the bilinear form $\langle\ ,\ \rangle_{\mathbf{A}}$ is also graded, it immediately follows from (3.1) that all the homogeneous components of any $P \in \mathbf{H}_{[a]}(\mathbf{A})$ must be also in $\mathbf{H}_{[a]_G}(\mathbf{A})$. Thus $\mathbf{H}_{[a]_G}(\mathbf{A})$ is a graded vector space. However we can establish a considerably stronger result.

Theorem 3.1

$\mathbf{H}_{[a]_G}(\mathbf{A})$ *is a graded G-module equivalent to* $\mathbf{gr}\,\mathbf{A}_{[a]_G}$. *In particular it also carries the regular representation and thus it has dimension the order of G.*

Proof

Note that the G-invariance of the bilinear form $\langle\ ,\ \rangle_{\mathbf{A}}$ may be also expressed in the form

$$\big\langle T_\sigma P\ ,\ Q\big\rangle_{\mathbf{A}} = \big\langle P\ ,\ T_\sigma^{-1} Q\big\rangle_{\mathbf{A}} \qquad \text{(for all } \sigma \in G), \tag{3.2}$$

and, since the ideal $\mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A})$ is G-invariant, it follows from (3.2) and the definition in (3.1) that also $\mathbf{H}_{[a]_G}(\mathbf{A})$ is G-invariant. Thus to prove the assertion we need only compute its character. To this end, for a given $1 \le k \le d_{\mathbf{A}}$ let $\{\psi_1, \psi_2, \ldots, \psi_s\} \subseteq \mathbf{A}$ be a basis for $\mathcal{H}_{=k}\big(\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})\big)$, and let $\{\phi_1, \phi_2, \ldots, \phi_r\} \subseteq \mathbf{A}$ be constructed so that, together with $\{\psi_1, \psi_2, \ldots, \psi_s\}$ they yield a basis for $\mathcal{H}_{=k}\big(\mathbf{A}\big)$. In particular, $\{\phi_1, \phi_2, \ldots, \phi_r\} \subseteq \mathbf{A}$ are independent modulo $\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})$, and therefore the elements of $\mathbf{gr}\,\mathbf{A}_{[a]_G}$ which they represent are a basis for $\mathcal{H}_{=k}\big(\mathbf{gr}\,\mathbf{A}_{[a]_G}\big)$.

Now note that the non-degeneracy of the form $\langle\ ,\ \rangle_{\mathbf{A}}$ on the subspace $\mathcal{H}_{=k}\big(\mathbf{A}\big)$ assures the non-singularity of the block matrix

$$M = \begin{bmatrix} \|\langle \phi_i, \phi_j\rangle_{\mathbf{A}}\|_{1\le i,j\le r} & \|\langle \phi_i, \psi_j\rangle_{\mathbf{A}}\|_{\substack{1\le i\le r \\ 1\le j\le s}} \\ \|\langle \psi_j, \phi_i\rangle_{\mathbf{A}}\|_{\substack{1\le i\le r \\ 1\le j\le s}} & \|\langle \psi_i, \psi_j\rangle_{\mathbf{A}}\|_{1\le i,j\le s} \end{bmatrix}. \tag{3.3}$$

It then follows that the matrix product

$$\big\langle \phi_1, \phi_2, \ldots, \phi_r, \psi_1, \psi_2, \ldots, \psi_s\big\rangle M^{-1}$$

yields a basis for $\mathcal{H}_{=k}\big(\mathbf{A}\big)$ that is "*dual*" to $\big\langle \phi_1, \phi_2, \ldots, \phi_r, \psi_1, \psi_2, \ldots, \psi_s\big\rangle$ with respect to the bilinear form $<\ ,\ >_{\mathbf{A}}$. That means that if this new basis is

$$\big\langle \eta_1, \eta_2, \ldots, \eta_r, \gamma_1, \gamma_2, \ldots, \gamma_s\big\rangle$$

then we shall have

$$\text{a)}\quad \big\langle \phi_i,\eta_j\big\rangle_{\mathbf{A}} \;=\; \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \quad ,\qquad \text{b)}\quad \big\langle \psi_i,\gamma_j\big\rangle_{\mathbf{A}} \;=\; \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \tag{3.4}$$

and

$$\text{a)}\quad \big\langle \phi_i,\gamma_j\big\rangle_{\mathbf{A}} = 0 \ \text{ for all } \ i,j\ ,\qquad \text{b)}\quad \big\langle \psi_i,\eta_j\big\rangle_{\mathbf{A}} = 0 \ \text{ for all } \ i,j \tag{3.5}$$

Note that, since $\psi_1,\psi_2,\ldots,\psi_s$ span $\mathcal{H}_{=k}\big(\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})\big)$, the relations in (3.5) b) imply that $\eta_1,\eta_2,\ldots,\eta_r$ lie in $\mathcal{H}_{=k}\big(\mathbf{H}_{[a]_G}(\mathbf{A})\big)$. We claim that $\big\langle \eta_1,\eta_2,\ldots,\eta_r\big\rangle$ are in fact a basis of $\mathcal{H}_{=k}\big(\mathbf{H}_{[a]_G}(\mathbf{A})\big)$. To show this, note that the duality relations in (3.4) and (3.5) yield that for any polynomial $P\in\mathcal{H}_{=k}\big(\mathbf{A}\big)$ we have the expansions

$$P \;=\; \sum_{i=1}^{r}\big\langle P,\eta_i\big\rangle_{\mathbf{A}}\phi_i \;+\; \sum_{j=1}^{s}\big\langle P,\gamma_j\big\rangle_{\mathbf{A}}\psi_i\ , \tag{3.6}$$

$$P \;=\; \sum_{i=1}^{r}\big\langle P,\phi_i\big\rangle_{\mathbf{A}}\eta_i \;+\; \sum_{j=1}^{s}\big\langle P,\psi_j\big\rangle_{\mathbf{A}}\gamma_i\ . \tag{3.7}$$

In particular, if $P\in\mathcal{H}_{=k}\big(\mathbf{H}_{[a]_G}(\mathbf{A})\big)$, the latter expansion reduces to

$$P \;=\; \sum_{i=1}^{r}\big\langle P,\phi_i\big\rangle_{\mathbf{A}}\eta_i\,. \tag{3.8}$$

This shows that $\big\langle \eta_1,\eta_2,\ldots,\eta_r\big\rangle$ span $\mathcal{H}_{=k}\big(\mathbf{H}_{[a]_G}(\mathbf{A})\big)$. Since they are independent by construction, it follows that they are a basis as asserted. This given, let $A(\sigma)=\|a_{i,j}(\sigma)\|_{i,j=1}^{r}$ be the matrix that expresses the action of G on $\big\langle \eta_1,\eta_2,\ldots,\eta_r\big\rangle$. That is, for all $\sigma\in G$ we have

$$T_\sigma\,\eta_i \;=\; \sum_{u=1}^{r}\eta_u\ a_{u,i}(\sigma) \qquad\qquad \text{for } 1\leq i\leq r. \tag{3.9}$$

Using the expansion in (3.6) for $P=T_\sigma\phi_v$, we get

$$\begin{aligned} T_\sigma\phi_v \;&=\; \sum_{i=1}^{r}\big\langle T_\sigma\phi_v,\eta_i\big\rangle_{\mathbf{A}}\phi_i \;+\; \sum_{j=1}^{s}\big\langle T_\sigma\phi_v,\gamma_j\big\rangle_{\mathbf{A}}\psi_j \\ (\text{by (3.2)})\;&=\; \sum_{i=1}^{r}\big\langle \phi_v,T_\sigma^{-1}\eta_i\big\rangle_{\mathbf{A}}\phi_i \;+\; \sum_{j=1}^{s}\big\langle T_\sigma\phi_v,\gamma_j\big\rangle_{\mathbf{A}}\psi_j \\ &\cong\; \sum_{i=1}^{r}\big\langle \phi_v,T_\sigma^{-1}\eta_i\big\rangle_{\mathbf{A}}\phi_i \qquad\qquad (\text{modulo }\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})). \end{aligned} \tag{3.10}$$

But, using (3.9), we derive (from (3.4) a)):

$$\big\langle \phi_v, T_\sigma^{-1}\eta_i \big\rangle_{\mathbf{A}} \;=\; \sum_{u=1}^{r} \big\langle \phi_v, \eta_u \big\rangle_{\mathbf{A}} a_{u,i}(\sigma^{-1}) \;=\; a_{v,i}(\sigma^{-1}) \;,$$

and thus (3.10) reduces to

$$T_\sigma \phi_v \;\cong\; \sum_{i=1}^{r} a_{v,i}(\sigma^{-1})\phi_i \qquad\qquad (\text{modulo } \mathbf{gr}\, \mathcal{J}_{[a]_G}(\mathbf{A})).$$

In other words, the action of G on the basis $\big\langle \phi_1, \phi_2, \ldots, \phi_r \big\rangle$ of $\mathcal{H}_{=k}\big(\mathbf{gr}\, \mathbf{A}_{[a]_G}\big)$ is given by the matrix

$$A^\top\big(\sigma^{-1}\big)\,.$$

Since

$$\mathit{trace}\; A(\sigma) \;=\; \mathit{trace}\; A^\top\big(\sigma^{-1}\big)$$

we can thereby conclude that

$$\mathit{char}\; \mathcal{H}_{=k}\big(\mathbf{H}_{[a]_G}(\mathbf{A})\big) \;=\; \mathit{char}\; \mathcal{H}_{=k}\big(\mathbf{gr}\, \mathbf{A}_{[a]_G})\big)\,.$$

This proves that $\mathbf{H}_{[a]_G}(\mathbf{A})$ and $\mathbf{gr}\, \mathbf{A}_{[a]_G}$ are equivalent as graded G-modules and completes the proof of the theorem.

Remark 3.1

We should note that if V is a finite dimensional vector space with a non-degenerate, symmetric, bilinear form $\big\langle\; ,\; \big\rangle_V$, and $U \subset V$ is a proper subspace, and if we set

$$U^\perp \;=\; \big\{ P \in V \;:\; \big\langle P\;,\; Q \big\rangle_V = 0 \;\; \text{for all} \;\; Q \in U \,\big\} \;,$$

then

$$U^{\perp^\perp} \;=\; U\,. \tag{3.11}$$

Note that, since the form $\big\langle\; ,\; \big\rangle_{\mathbf{A}}$ is graded we can apply this result with $V = \mathcal{H}_{=k}(\mathbf{A})$ and $U = \mathcal{H}_{=k}\big(\mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A})\big)$ and deduce the relations

$$\mathcal{H}_{=k}\big(\mathbf{H}_{[a]_G}(\mathbf{A})\big) \;=\; \Big(\mathcal{H}_{=k}\big(\mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A})\big)\Big)^\perp$$

and

$$\mathcal{H}_{=k}\big(\mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A})\big) \;=\; \Big(\mathcal{H}_{=k}\big(\mathbf{H}_{[a]_G}(\mathbf{A})\big)\Big)^{\perp}\,.$$

Similarly we get

$$\mathbf{H}_{[a]_G}(\mathbf{A}) \;=\; \big(\mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A})\big)^{\perp} \quad \text{and} \quad \mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A}) \;=\; \big(\mathbf{H}_{[a]_G}(\mathbf{A})\big)^{\perp}\,. \tag{3.12}$$

4. The A-Harmonics of G

In the classical case the space $\mathbf{H}_G$ of "*Harmonics*" of G is defined as the orthogonal complement of the ideal generated by the homogeneous G-invariants. For $\mathbf{A}$-harmonics the definition is entirely analogous. We simply let $\mathcal{J}_G(\mathbf{A})$ be the ideal generated in $\mathbf{A}$ by the homogeneous G-invariants and set, (using the notation in Remark 3.1):

$$\mathbf{H}_G(\mathbf{A}) \;=\; \big(\mathcal{J}_G(\mathbf{A})\big)^{\perp} \tag{4.1}$$

Note that, since the bilinear form $\langle\ ,\ \rangle_{\mathbf{A}}$ is graded, $\mathbf{H}_G(\mathbf{A})$ will necessarily be a graded vector space, as was the case for $\mathbf{H}_{[a]_G}(\mathbf{A})$ itself. However, the hypothesis in (iv) immediately yields the following remarkable result.

Theorem 4.1

The two subspaces $\mathbf{H}_{[a]_G}(\mathbf{A})$ and $\mathbf{H}_G(\mathbf{A})$ are identical. In particular, $\mathbf{H}_G(\mathbf{A})$ carries a graded version of the left regular representation of G. Moreover we also have

$$\mathcal{J}_G(\mathbf{A}) \;=\; \mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A}) \tag{4.2}$$

Proof

Note that if Q is a G-invariant polynomial then the polynomial $Q(x)-Q(a)$ vanishes at all points of the orbit $[a]_G$. That is $Q(x)-Q(a)\in\mathcal{J}_{[a]_G}(\mathbf{A})$. Thus if Q is also homogeneous of positive degree it necessarily follows that

$$Q(x)\in\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})\,.$$

This proves the containment

$$\mathcal{J}_G(\mathbf{A}) \;\subseteq\; \mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})\,.$$

Therefore we must also have the reverse containment

$$\big(\mathcal{J}_G(\mathbf{A})\big)^{\perp} \supseteq \big(\mathbf{gr}\,\mathcal{J}_{[a]_G}(\mathbf{A})\big)^{\perp}\ ,$$

and this can be written as

$$\mathbf{H}_G(\mathbf{A}) \supseteq \mathbf{H}_{[a]_G}(\mathbf{A})\,. \tag{4.3}$$

But now Theorem 3.1 and the hypothesis in (iv) give

$$|G| \,=\, \dim \mathbf{H}_G(\mathbf{A}) \,\geq \dim \mathbf{H}_{[a]_G}(\mathbf{A}) \,=\, |G|\ ,$$

and (4.3) forces the desired equality

$$\mathbf{H}_G(\mathbf{A}) \,=\, \mathbf{H}_{[a]_G}(\mathbf{A})\,.$$

Then it follows that we must also have

$$\mathbf{H}_G(\mathbf{A})^{\perp} \,=\, \mathbf{H}_{[a]_G}(\mathbf{A})^{\perp}\,,$$

and the equality in (4.2) follows immediately from Remark 3.1.

We should note that from (4.2) we can derive the following result.

Theorem 4.2

If $P \in \mathbf{A}$ is homogeneous then

$$\mathit{degree}(P) > d_{\mathbf{A}} \quad \Longrightarrow \quad P \in \mathcal{J}_G(\mathbf{A}) \tag{4.4}$$

Proof

We have seen that, in terms of the basis elements defined in (1.8), every element $P \in \mathbf{A}$ satisfies the identity in (1.10), that is

$$P(x) \,-\, \sum_{b \in [a]_G} P(b)\, \psi_b(x) \in \mathcal{J}_{[a]_G}(\mathbf{A})\,. \tag{4.5}$$

Since all the terms in the sum have degree $\leq d_{\mathbf{A}}$, if P is homogeneous of degree $> d_{\mathbf{A}}$ then (4.5) forces

$$P(x) \in \mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A})\ ,$$

and the equality in (4.2) proves (4.4).

But the most important consequence of (4.2) is given by the following truly remarkable result.

Theorem 4.3

The quotient ring

$$\mathbf{A}/\mathcal{J}_G(\mathbf{A})$$

carries the regular representation of G and therefore has dimension the order of G.

Proof

The equality $\mathcal{J}_G(\mathbf{A}) = \mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A})$ forces the equality

$$\mathbf{A}/\mathcal{J}_G(\mathbf{A}) \;=\; \mathbf{A}/\mathbf{gr}\mathcal{J}_{[a]_G}(\mathbf{A}) \;,$$

and therefore the assertion follows from Theorem 2.1.

Before we can complete the proof of Theorem I.1 we need to recall some basic facts about Cohen–Macaulay algebras. To begin with, let us recall that the Hilbert series of a finitely generated, graded algebra $\mathbf{A}$ is given by the formal sum

$$F_{\mathbf{A}}(t) \;=\; \sum_{k\geq 0} t^k \dim \mathcal{H}_k(\mathbf{A}) \;, \tag{4.6}$$

where $\mathcal{H}_k(\mathbf{A})$ denotes the subspace spanned by the elements of $\mathbf{A}$ that are homogeneous of degree k. It is well known that $F_{\mathbf{A}}(t)$ is a rational function of the form

$$F_{\mathbf{A}}(t) \;=\; \frac{P(t)}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})} \;,$$

with $d_1, d_2, \ldots, d_n$ positve integers and $P(t)$ a polynomial that does not vanish at $t = 1$. The minimum n for which this is possible characterizes the growth of $\dim \mathcal{H}_k(\mathbf{A})$ as $k{\to}\infty$. This integer is customarily called the "*Krull dimension*" of $\mathbf{A}$ and is denoted "$\dim_K \mathbf{A}$". It is easily shown that we can always find in $\mathbf{A}$ homogeneous elements $\theta_1, \theta_2, \ldots, \theta_n$ such that the quotient of $\mathbf{A}$ by the ideal generated by $\theta_1, \theta_2, \ldots, \theta_n$ is a finite dimensional vector space. In symbols

$$\dim\, \mathbf{A}/(\theta_1, \theta_2, \ldots, \theta_n)_{\mathbf{A}} < \infty \tag{4.7}$$

It is also a fact that $\dim_K \mathbf{A}$ is also equal to the minimum n for which this is possible. When (4.7) holds true and $n = \dim_K \mathbf{A}$ then $\{\theta_1, \theta_2, \ldots, \theta_n\}$ is called a "homogeneous system of parameters", $\mathcal{HSOP}$ in brief.

It follows from (4.7) that if $\eta_1, \eta_2, \ldots, \eta_N \in \mathbf{A}$ give a basis for the quotient in (4.7) then every element of $\mathbf{A}$ has an expansion of the form

$$P \;=\; \sum_{i=1}^{N} \eta_i P_i(\theta_1, \theta_2, \ldots, \theta_n) \;, \tag{4.8}$$

with coefficients $P_i(\theta_1, \theta_2, \ldots, \theta_n)$ polynomials in their arguments. The algebra $\mathbf{A}$ is said to be Cohen–Macaulay, when the coefficients $P_i(\theta_1, \theta_2, \ldots, \theta_n)$ are uniquely determined by P. This amounts to the requirement that the collection

$$\left\{\eta_i\, \theta_1^{p_1} \theta_2^{p_2} \cdots \theta_n^{p_n}\right\}_{i,p} \tag{4.9}$$

is a basis for $\mathbf{A}$ as a vector space and therefore $\mathbf{A}$ is a free module over $\mathbf{Q}[\theta_1, \theta_2, \ldots, \theta_n]$. Note that when this happens and $\theta_1, \theta_2, \ldots, \theta_n; \eta_1, \eta_2, \ldots, \eta_N$ are homogeneous of degrees $d_1, d_2, \ldots, d_n; r_1, r_2, \ldots, r_N$ then we must necessarily have

$$F_{\mathbf{A}}(t) \;=\; \frac{\sum_{i=1}^{N} t^{r_i}}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})} \;, \tag{4.10}$$

from which it follows that $\dim_K \mathbf{A} = n$.

This brings us to a useful criterion for assuring the Cohen–Macauliness of a finitely generated graded algebra.

Proposition 4.1

Let $\theta_1, \theta_2, \ldots, \theta_n$ be an $\mathcal{HSOP}$, and let $d_i = degree(\theta_i)$, Then $\mathbf{A}$ is free over $\mathbf{Q}[\theta_1, \theta_2, \ldots, \theta_n]$ and therefore Cohen–Macaulay if and only if

$$\lim_{t\to 1^-} (1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n}) F_{\mathbf{A}}(t) \;=\; \dim\, \mathbf{A}/(\theta_1, \theta_2, \ldots, \theta_n)_{\mathbf{A}} \tag{4.11}$$

Proof†

Note first that the necessity of the condition follows immediately from (4.10). To prove the sufficiency, let $\eta_1, \eta_2, \ldots, \eta_N$ be a homogeneous basis for the quotient $\mathbf{A}/(\theta_1, \theta_2, \ldots, \theta_n)_{\mathbf{A}}$ and set $r_i = degree(\eta_i)$. Next let $\mathcal{M}_i$ denote the subspace spanned by the collection

$$\left\{\eta_j \theta_1^{p_1} \theta_2^{p_2} \cdots \theta_n^{p_n} \;:\; 1 \le j \le i\,;\, p_j \ge 0\right\}. \tag{4.12}$$

† This is a known result but we include a proof for the sake of completeness.

It is easily seen that if $\mathcal{H}_m(\mathcal{M}_i)$ and $\mathcal{H}_m(\mathcal{M}_i/\mathcal{M}_{i-1})$ denote the subspaces of $\mathcal{M}_i$ and $\mathcal{M}_i/\mathcal{M}_{i-1}$ spanned by their homogeneous elements of degree m then we must have

$$\dim \mathcal{H}_m(\mathcal{M}_i) = \dim \mathcal{H}_m(\mathcal{M}_i/\mathcal{M}_{i-1}) + \dim \mathcal{H}_m(\mathcal{M}_{i-1}) .$$

Multiplying by t^m and summing, we derive the Hilbert series identities

$$F_{\mathcal{M}_i}(t) = F_{\mathcal{M}_i/\mathcal{M}_{i-1}}(t) + F_{\mathcal{M}_{i-1}}(t) \qquad (\text{for } 1 \le i \le N \text{ with } \mathcal{M}_0 = \{0\}) .$$

This implies that

$$F_{\mathbf{A}}(t) = F_{\mathcal{M}_1}(t) + F_{\mathcal{M}_2/\mathcal{M}_1}(t) + \cdots + F_{\mathcal{M}_N/\mathcal{M}_{N-1}}(t) . \tag{4.13}$$

Now, for a given $1 \le i \le N$, let ϕ be the map from the polynomial ring $\mathbf{Q}[x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n}]$ onto $\mathcal{M}_i/\mathcal{M}_{i-1}$ defined by setting for every polynomial $P(x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n})$

$$\phi P = \eta_i \, P(\theta_1, \theta_2, \ldots, \theta_n) .$$

Note that, a priori the kernel $\mathcal{J}$ of ϕ will be an ideal of $\mathbf{Q}[x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n}]$, and since ϕ preserves degrees, we will have

$$F_{\mathcal{M}_i/\mathcal{M}_{i-1}}(t) = t^{r_i} F_{\mathbf{Q}[x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n}]/\mathcal{J}}(t) . \tag{4.14}$$

Thus, if $\mathcal{J}$ happens to be trivial, it follows that

$$F_{\mathcal{M}_i/\mathcal{M}_{i-1}}(t) = \frac{t^{r_i}}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})} . \tag{4.15}$$

On the other hand, if $\mathcal{J}$ contains a non trivial homogeneous element P of degree d, from (4.14) we derive the coefficient-wise inequality of Hilbert series

$$F_{\mathcal{M}_i/\mathcal{M}_{i-1}}(t) << t^{r_i} F_{\mathbf{Q}[x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n}]/(P)}(t) , \tag{4.16}$$

where the symbol "<<" is to indicate that the inequality is coefficient-wise. Since the ring $\mathbf{Q}[x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n}]$ has no zero divisors, we will have

$$F_{\mathbf{Q}[x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n}]/(P)}(t) = \frac{1-t^d}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})} ,$$

and from (4.16) we derive that

$$F_{\mathcal{M}_i/\mathcal{M}_{i-1}}(t) \;<<\; t^{r_i}\frac{1-t^d}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})}\,. \tag{4.17}$$

In conclusion, we see that we will have

$$\lim_{t\to 1^-}(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})F_{\mathcal{M}_i/\mathcal{M}_{i-1}}(t) \;=\; \epsilon_i\ , \tag{4.18}$$

with

$$\epsilon_i \;=\; \begin{cases} 1 & \text{if (4.15) holds true,} \\ 0 & \text{if (4.17) holds true.} \end{cases}$$

Thus, passing to the limit as $t{\to}1^-$ in (4.13) and using (4.18) together with the hypothesis in (4.11), we finally obtain

$$\epsilon_1+\epsilon_2+\cdots+\epsilon_N \;=\; \dim\,\mathbf{A}/(\theta_1,\theta_2,\ldots,\theta_n)_{\mathbf{A}} \;=\; N\,. \tag{4.19}$$

This forces all the ϵ_i to be equal to one. However, this can only hold true when the collection

$$\big\{\eta_j\theta_1^{p_1}\theta_2^{p_2}\cdots\theta_n^{p_k}\big\}_{j,p}$$

forms an independent set. Indeed any relation of the form

$$\sum_{j=1}^N \eta_j P_j(\theta_1,\theta_2,\ldots\theta_n) \;=\; 0\,.$$

would force one of the ϵ_i to vanish and contradict (4.19). This shows that

$$F_{\mathbf{A}}(t) \;=\; \frac{F_{\mathbf{A}/(\theta_1,\theta_2,\ldots,\theta_n)_{\mathbf{A}}}(t)}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})} \tag{4.20}$$

and proves the Cohen–Macauliness of $\mathbf{A}$.

We are now ready to complete the proof of Theorem I.1.

Theorem 4.4

The algebra $\mathbf{A}$ *is a free* Λ_G*-module and therefore Cohen–Macaulay.*

Proof

Let $q_1(x), q_2(x), \ldots q_n(x)$ be a fundamental set of homogeneous generators of Λ_G and suppose that $d_1, d_2, \ldots, d_n$ are their respective degrees. It is well known (see [10]) that we must have the equality

$$d_1 d_2 \cdots d_n \;=\; |G|. \tag{4.21}$$

From the hypothesis (ii) it follows that we have the containments

$$B(x)\mathbf{Q}[X_n] \subset \mathbf{A} \subset \mathbf{Q}[X_n]\,, \tag{4.22}$$

and since $\mathbf{A}$ is degree graded, and B is homogeneous, it follows from (4.6) that the Hilbert series $F_{\mathbf{A}}(q)$ of $\mathbf{A}$ will satisfy the inequalities

$$\frac{t^{degree(B)}}{(1-t)^n} << F_{\mathbf{A}}(t) << \frac{1}{(1-t)^n}\;. \tag{4.23}$$

In particular, this shows that the Krull dimension of $\mathbf{A}$ is n.

This given, multiplying both sides of (4.23) by $(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})$ and passing to the limit as $t{\to}1^-$, the equality in (4.21) implies that

$$lim_{t\to 1^-}(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})F_{\mathbf{A}}(t) \;=\; |G|\,. \tag{4.24}$$

Moreover, since $q_1(x), q_2(x), \ldots q_n(x)$ are also generators of the ideal $\mathcal{J}_G(\mathbf{A})$ we see that Theorem 4.3 implies that

$$\dim\, \mathbf{A}/\big(q_1, q_2, \ldots q_n\big)_{\mathbf{A}} \;=\; |G|\;,$$

and Proposition 4.1 then yields that $\mathbf{A}$ is Cohen–Macaulay over Λ_G.

Remark 4.1

We should note that (4.20) yields the Hilbert series identity

$$F_{\mathbf{A}}(t) \;=\; \frac{F_{\mathbf{A}/(q_1,q_2,\ldots q_n)_{\mathbf{A}}}(t)}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_n})}\;.$$

In particular it follows that if $\mathcal{B}$ is any basis for the quotient

$$\mathbf{A}/(q_1, q_2, \ldots q_n)_{\mathbf{A}}$$

then the collection

$$\Big\{ b\, q_1^{p_1} q_2^{p_2} \cdots q_n^{p_n} \Big\}_{b\in\mathcal{B}\,,\, p_i\geq 0}$$

is a basis for $\mathbf{A}$.

5. $\mathbf{QI_m(G)}$ is a Cohen–Macaulay algebra.

In this section we show that $\mathcal{QI}_m(G)$ satisfies the hypotheses (i)-(iv) of Theorem I.1.

Now, (i) is no problem since we have seen that the definition gives $\mathcal{QI}_m(G) \supset \mathcal{QI}_\infty(G) = \Lambda_G$ for all $m \geq 0$. For property (ii), we can take

$$B(x) = \Pi_G(x)^{2m}. \tag{5.1}$$

To see this, we need only observe that for any polynomial $P(x) \in \mathbf{Q}[X_n]$ and any $s \in \Sigma(G)$, the G-invariance of $\Pi_G(x)^{2m}$ gives

$$(1-s)\Pi_G(x)^{2m}P(x) \;=\; \Pi_G(x)^{2m}(1-s)P(x).$$

Clearly, $\Pi_G(x)^{2m}$ yields the factor $(x,\alpha_s)^{2m}$ and another factor (x,α_s) comes from $(1-s)P(x)$. This proves that

$$\Pi_G(x)^{2m}\mathbf{Q}[X_n] \subset \mathcal{QI}_m(G)\ , \tag{5.2}$$

as desired.
For (iii) we need to make sure that the bilinear form

$$\big\langle P\ ,\ Q\big\rangle_G \;=\; \gamma_P(x,\partial_x)Q(x)\Big|_{x=0} \tag{5.3}$$

mentioned in the introduction has the required properties. To begin with, in the S_n case a reasonably elementary proof of non-degeneracy can be found in [8], but in the general case we will have to rely on Opdam's proof [11]. The G-invariance as expressed in (3.2) immediately follows from the identity

$$T_\sigma \gamma_Q(x,\partial_x)T_\sigma^{-1} \;=\; \gamma_{T_\sigma Q}(x,\partial_x) \qquad (\,\text{for all}\quad \sigma\in G\,)$$

satisfied by all the operators γ_Q. The latter in turn follows from the Berest formula (I.5) and the identity

$$T_\sigma L_m(G)T_\sigma^{-1} \;=\; L_m(G) \qquad (\,\text{for all}\quad \sigma\in G\,),$$

which can be easily verified from the definition in (I.6). The symmetry of $\langle\ ,\ \rangle_m$ is a consequence of the symmetry of the Baker–Akhiezer function (see Section 6 for more information on that function) of the algebra $\mathcal{QI}_m(G)$. A reasonably accessible proof of this result can be found in [8]. The fact that $\langle\ ,\ \rangle_m$ is graded is an immediate consequence of the fact that for Q homogeneous of degree d the differential operator $\gamma_Q(x,\partial_x)$ is also homogeneous of order d. Again this can be easily seen from the Berest formula.

We are thus left with the verification of property (iv). Before we do this, we need to fix some notation and establish a few auxiliary facts that are of interest by themselves. To begin with, we recall that the point $a=(a_1,a_2,\ldots,a_n)$ has to be chosen to have a trivial G-stabilizer. In this case we can satisfy all the conditions we need (including (1.1)) by requiring that

$$\Pi_G(a)\neq 0 \tag{5.4}$$

Here, as in Section 1, $\phi_b(x)$ denotes the polynomial defined by (1.6). But it will be helpful to assume that the polynomial $\phi_a(x)$ satifying (1.3) is chosen so as to have the minimal degree $|\Sigma(G)|$. Thus, since $degree\big(\Pi(x_G)\big)=|\Sigma(G)|$, the definition in (1.8) and (5.1) yield that the polynomial $\psi_b(x)$ satisfying (1.9) has degree

$$d_m(G)=(2m+1)|\Sigma(G)|\,. \tag{5.5}$$

Before we proceed with our arguments, it will be convenient to adopt a notation that is more adherent to the present choice $\mathbf{A}=\mathcal{QI}_m(G)$. To this end, for the rest of this paper we shall assume that

(1) *The ideal of G-m-Quasi-Invariants that vanish in* $[a]_G$ *is denoted* $\mathcal{J}_{[a]_G}(m)$.
(2) *The quotient of* $\mathcal{QI}_m(G)$ *by* $\mathcal{J}_{[a]_G}(m)$ *is denoted* $\mathbf{R}_{[a]_G}[m]$.
(3) *The graded version of* $\mathcal{J}_{[a]_G}(m)]$ *is denoted* $\mathbf{gr}\,\mathcal{J}_{[a]_G}(m)$.
(4) *The graded version of* $\mathbf{R}_{[a]_G}[m]$ *is denoted* $\mathbf{gr}\,\mathbf{R}_{[a]_G}[m]$
(5) *The orthogonal complement of* $\mathbf{gr}\,\mathcal{J}_{[a]_G}(m)$ *with respect to the bilinear form in* (5.3) *is denoted* $\mathbf{H}_{[a]_G}(m)$ *and its elements are called "orbit m-Harmonics".* (5.6)
(6) *The ideal generated in* $\mathcal{QI}_m(G)$ *by the G-invariants is denoted* $\mathcal{J}_G(m)$.
(7) *The orthogonal complement of* $\mathcal{J}_G(m)$ *with respect to the bilinear form in* (5.3) *is denoted* $\mathbf{H}_G(m)$, *and its elements are called "G-m-Harmonics".*

This given, since the only place we have used property (iv) is in the proof of Theorem 4.1, all the results obtained in Sections 1, 2 and 3 hold true with $\mathbf{A} = \mathcal{QI}_m(G)$. Therefore we can state

Theorem 5.1

(a) $\mathbf{R}_{[a]_G}[m]$ *is of dimension* $|G|$ *and affords the regular representation of* G.

(b) $\mathbf{R}_{[a]_G}[m]$ *and* $\mathbf{H}_{[a]_G}(m)$ *are equivalent G-modules affording a graded version of the regular representation of* G.

However, here we have the following two additional results.

Theorem 5.2

$$\mathbf{H}_{[a]_G}(m) = \left\{Q \in \mathcal{QI}_m(G) : \gamma_P Q = 0 \text{ for all } \quad P \in \mathbf{gr}\,\mathcal{J}_{[a]_G}(m)\right\}. \tag{5.7}$$

Proof

In view of the definition in (5.3), we clearly see that the condition $\gamma_P Q = 0$ is stronger than $\langle P\,,\, Q\rangle_m = 0$. Thus we need only establish the containment

$$\mathbf{H}_{[a]_G}(m) \subseteq \left\{P \in \mathcal{QI}_m(G) : \gamma_P Q = 0 \text{ for all } \quad P \in \mathbf{gr}\,\mathcal{J}_{[a]_G}(m)\right\}. \tag{5.8}$$

To this end, note that since $\mathbf{gr}\,\mathcal{J}_{[a]_G}(m)$ is an ideal of $\mathcal{QI}_m(G)$, the defining condition

$$\mathbf{H}_{[a]_G}(m) = \left\{Q \in \mathcal{QI}_m(G) : \langle P\,,\, Q\rangle_m = 0 \text{ for all } \quad P \in \mathbf{gr}\,\mathcal{J}_{[a]_G}(m)\right\}$$

can be also written in the form

$$\mathbf{H}_{[a]_G}(m) = \Big\{ Q \in \mathcal{QI}_m(G) \,:\, \big\langle RP \,,\, Q \big\rangle_m \,=\, 0 \quad \text{for all} \quad P \in \mathbf{gr}\, \mathcal{J}_{[a]_G}(m) \quad \text{and} \quad R \in \mathcal{QI}_m(G) \Big\}.$$

On the other hand from (I.7) and the definition in (5.3) of the bilinear form $\langle \;,\; \rangle_m$ we derive that

$$\big\langle RQ \,,\, P \big\rangle_m \,=\, \big\langle R \,,\, \gamma_Q P \big\rangle_m$$

Thus

$$P \in \mathbf{H}_{[a]_G}(m) \quad \text{implies} \quad \big\langle R \,,\, \gamma_Q P \big\rangle_m = 0 \quad \text{for all} \quad Q \in \mathbf{gr}\, \mathcal{J}_{[a]_G}(m) \quad \text{and} \quad R \in \mathcal{QI}_m(G)\ .$$

But then the non-degeneracy of $\langle \;,\; \rangle_m$ yields that

$$P \in \mathbf{H}_{[a]_G}(m) \quad \Longrightarrow \quad \gamma_Q P = 0 \quad \text{for all} \quad Q \in \mathbf{gr}\, \mathcal{J}_{[a]_G}(m)\,.$$

This proves (5.8) and completes the proof of the theorem.

The next result proves the identity in (I.9).

Theorem 5.3

$$\mathbf{H}_G(m) \,=\, \big\{ P \in \mathbf{Q}[X_n] \,:\, \gamma_{q_k}(x, \partial_x) P(x) = 0 \;\; \text{for all} \quad k = 1, 2, \ldots, n \big\}\ . \tag{5.9}$$

Proof

By definition,

$$\mathbf{H}_G(m) \,=\, \big\{ P \in \mathbf{Q}[X_n] \,:\, \big\langle Q \,,\, P \big\rangle_m = 0 \;\; \text{for all} \quad Q \in \mathcal{J}_G(m) \big\}\ ,$$

and, since the fundamental invariants $q_1, q_2, \ldots, q_n$ generate the ideal $\mathcal{J}_G(m)$, this is equivalent to

$$\mathbf{H}_G(m) \,=\, \big\{ P \in \mathbf{Q}[X_n] \,:\, \big\langle Rq_k \,,\, P \big\rangle_m = 0 \;\; \text{for all} \;\; R \in \mathcal{QI}_m(G) \;\; \text{and} \;\; k = 1, 2, \ldots, n \big\} \tag{5.10}$$

Now, from (5.3) and (I.7) we derive that

$$\big\langle Rq_k\ ,\ P\big\rangle_m\ =\ \big\langle R\ ,\ \gamma_{q_k}P\big\rangle_m\ . \tag{5.11}$$

But the non-degeneracy of $\langle\ ,\ \rangle_m$ yields that

$$\big\langle R\ ,\ \gamma_{q_k}P\big\rangle_m=0\quad\text{for all}\quad R\in\mathcal{QI}_m(G)\quad\text{implies}\quad\gamma_{q_k}P\ =\ 0. \tag{5.12}$$

Combining (5.10), (5.11) and (5.12) proves (5.9).

We are now finally in a position to establish property (iv) for $\mathbf{A}=\mathcal{QI}_m(G)$.

Theorem 5.4 We have

$$\dim\mathbf{H}_G(m)\ \leq\ |G|. \tag{5.13}$$

Proof

Notice to the reader. The following argument was provided to us by E-mail by Feigin and Veselov. We feel compelled to reproduce it here since it is not available in the present literature. Although this result is stated in many places (see [3], [4], [6]), proofs (when not omitted) give no indication that it could be established in such a simple and elementary manner.

The idea is to show that each m-Harmonic $Q\in\mathbf{H}_G(m)$ is completely determined by $|G|$ of its derivatives at the point a. To do this, we fix once and for all a fundamental set $q_1,q_2,\ldots,q_n$ of G-invariants and a monomial basis $\{x^{\epsilon_1},x^{\epsilon_2},\ldots,x^{\epsilon_N}\}$ for the quotient $\mathbf{Q}[X_n]/(q_1,q_2,\ldots,q_n)_{\mathbf{Q}[X_n]}$. Since it is well known that $\dim\mathbf{Q}[X_n]/(q_1,q_2,\ldots,q_n)_{\mathbf{Q}[X_n]}\ =\ |G|$, we must necessarily have

$$N=|G|.$$

This given, we need only show that every $Q(x)\in\mathbf{H}_G(m)$ is completely determined by the $|G|$ derivatives

$$\partial_x^{\epsilon_1}Q(x)\,\big|_{x=a}\,,\quad\partial_x^{\epsilon_2}Q(x)\big|_{x=a}\ ,\quad\ldots\ ,\ \partial_x^{\epsilon_N}Q(x)\,\big|_{x=a}\,.$$

That is we must show that

$$\partial_x^{\epsilon_1}Q(x)\,\big|_{x=a}=0\,,\quad\partial_x^{\epsilon_2}Q(x)\big|_{x=a}=0\ ,\quad\ldots\ ,\ \partial_x^{\epsilon_N}Q(x)\,\big|_{x=a}=0 \tag{5.14}$$

forces

$$Q \equiv 0. \tag{5.15}$$

To do this, the point of departure is the well known fact that every polynomial $P \in \mathbf{Q}[X_n]$ may be given an expansion of the form

$$P \;=\; \sum_{r=1}^{|G|} x^{\epsilon_r} A_{P,r}(q_1, q_2, \ldots, q_n)\ , \tag{5.16}$$

with the coefficients $A_{P,r}$ polynomials in their arguments. Specializing P to the monomial x^p (5.16) gives

$$x^p \;=\; \sum_{r=1}^{|G|} x^{\epsilon_r}\ a_{p,r}(x) \tag{5.17}$$

with

$$a_{p,r}(x) \;=\; A_{x^p,r}(q_1, q_2, \ldots, q_n) \tag{5.18}$$

Two important facts should be kept in mind here:

(1) We can assume that

$$degree(a_{p,r}(x)) \;=\; |p| - |\epsilon_r|\,. \tag{5.19}$$

(2) From (5.18), (5.9) and (I.7) it follows that

$$\gamma_{a_{p,r}}(x,\partial_x)Q(x) \;=\; A_{x^p,r}(0,0,\ldots,0)\,Q(x) \qquad \text{for all} \quad Q \in \mathbf{H}_G(m)\,. \tag{5.20}$$

Note first that, since it is well known that we must have

$$\sum_{r=1}^{N} q^{\epsilon_r} \;=\; \prod_{i=1}^{n}(1+q+\cdots+q^{d_i-1})$$

with $d_i = degree(q_i)$, it follows that one of the exponents ϵ_r vanishes. Thus one of the conditions in (5.14) reduces to

$$Q(a) \;=\; 0\,. \tag{5.21}$$

Our next task is to show that the remaining conditions in (5.14) force

$$\partial_x^p Q(x)\,\Big|_{x=a} \;=\; 0 \tag{5.22}$$

for all

$$p = (p_1, p_2, \ldots, p_n)\,.$$

In view of (5.21), we can proceed by induction on $|p| = p_1 + p_2 + \cdots + p_n$. We assume (5.22) to be true for $|p| < d$ and show that it holds true for $|p| = d$. To this end we use (5.17) and write for $|p| = d$,

$$\partial_x^p Q(x) \;=\; \sum_{|\epsilon_r|=d} a_{p,r}\;\partial_x^{\epsilon_r} Q(x) \;+\; \sum_{|\epsilon_r|<d} \partial_x^{\epsilon_r} a_{p,r}(\partial_x) Q(x)$$

Note that (5.19) says that $a_{p,r}$ in the first sum reduces to a scalar and in the second sum it must be a homogeneous polynomial of degree $d - |\epsilon_r| > 0$. In particular the conditions in (5.14) immediately give us that

$$\partial_x^p Q(x)\,\big|_{x=a} \;=\; \sum_{|\epsilon_r|<d} \partial_x^{\epsilon_r} a_{p,r}(\partial_x) Q(x)\big|_{x=a}\,. \tag{5.23}$$

Now, from (I.4) and the fact that $degree(a_{p,r}) = d - |\epsilon_r|$, we obtain an expansion of the form

$$\gamma_{a_{p,r}}(x, \partial_x) \;=\; a_{p,r}(\partial_x) \;+ \sum_{|q|<d-|\epsilon_r|} c_{q,p,r}(x)\partial_x^q\,.$$

Using this in (5.23), we derive that

$$\begin{aligned}\partial_x^p Q(x)\,\big|_{x=a} \;=\; &\sum_{|\epsilon_r|<d} \partial_x^{\epsilon_r} \gamma_{a_{p,r}}(x, \partial_x) Q(x)\big|_{x=a} \\ &- \sum_{|\epsilon_r|<d}\;\sum_{|q|<d-|\epsilon_r|} \partial_x^{\epsilon_r} c_{q,p,r}(x)\partial_x^q Q(x)\big|_{x=a}\,.\end{aligned}$$

But (5.20) and the inductive hypothesis reduces this to

$$\partial_x^p Q(x)\,\big|_{x=a} \;=\; -\sum_{|\epsilon_r|<d}\;\sum_{|q|<d-|\epsilon_r|} \partial_x^{\epsilon_r} c_{q,p,r}(x)\partial_x^q Q(x)\big|_{x=a}\,. \tag{5.24}$$

Note that this makes perfectly good sense since our assumption in 5.4 assures that the denominators that will be produced by the term

$$\partial_x^{\epsilon_r} c_{q,p,r}(x)\partial_x^q Q(x)$$

will not vanish at $x = a$. However, (5.24) completes the induction since the derivatives of Q that will be produced by these terms will necessarily be of order $< d$ and by the inductive hypothesis they will all vanish at $x = a$..

This completes the proof of the dimension bound in (5.13).

Theorem 5.3 has a number of immediate corollaries that are worth stating explicitly.

Theorem 5.5

The following remarkable equalities hold true for every integer $m \geq 0$:

$$\mathbf{H}_{[a]_G}(m) \,=\, \mathbf{H}_G(m) \quad \textit{and} \quad \mathcal{J}_G(m) \,=\, \mathbf{gr}\,\mathcal{J}_{[a]_G}(m)\,. \tag{5.25}$$

Thus the m-Harmonics $\mathbf{H}_G(m)$ and the quotient ring

$$\mathcal{QI}_m(G)/(q_1, q_2, \ldots, q_n)_{\mathcal{QI}_m(G)} \tag{5.26}$$

have both dimension $|G|$ and afford the same graded regular representation of $|G|$ as the the space of orbit harmonics $\mathbf{H}_{[a]_G}(m)$.

Theorem 5.6

For every integer $m \geq 0$ the algebra of G-m-Quasi-Invariants is a free module over the ring of invariants Λ_G.

Theorem 5.7

Every homogeneous G-m-Quasi-Invariant $Q \in \mathcal{QI}_m(G)$ of degree greater than $(2m+1)|\Sigma(G)|$ lies in the ideal $\mathcal{J}_G(m)$.

Since we have verified that the algebra $\mathbf{A} = \mathcal{QI}_m(G)$ satisfies conditions (i), (ii), (iii), (iv) of Theorem I.1, all of these results are simply specializations to $\mathbf{A} = \mathcal{QI}_m(G)$ of the corresponding results established in Section 4.

6. More on the G-m-Harmonics

The goal of this section is to establish Theorem I.2. The basic tool in this task is a space $\Gamma_{[a]_G}(m)$ of formal power series in $x_1, x_2, \ldots, x_n$ which may be viewed as the orthogonal complement of the ideal $\mathcal{J}_{[a]_G}(m)$. More precisely we set

$$\Gamma_{[a]_G}(m) \,=\, \Big\{\Phi(x) \,:\, \gamma_Q(x, \partial_x)\Phi(x) = 0 \quad \text{for all } \; Q \in \mathcal{J}_{[a]_G}(m)\Big\} \tag{6.1}$$

Perhaps a few words are necessary here to assure that this is a well defined space. To begin with, we shall view each formal power series $\Phi(x)$ as the formal sum

$$\Phi \,=\, \Phi^{(0)} \,+\, \Phi^{(1)} \,+\, \Phi^{(2)} \,+\, \cdots \,+\, \Phi^{(k)} \,+\, \cdots$$

with $\Phi^{(k)}$ a polynomial in $x_1, x_2, \ldots, x_n$ homogeneous of degree k. Moreover if $Q \in \mathcal{J}_{[a]_G}(m)$ has the decomposition

$$Q \,=\, Q_0 \,+\, Q_1 \,+\, \cdots \,+\, Q_h$$

with Q_r homogeneous of degree r, then the equation

$$\gamma_Q(x, \partial_x)\Phi(x) = 0 \tag{6.2}$$

simply means that we must have

$$\sum_{r=0}^{h} \gamma_{Q_r}(x, \partial_x)\Phi^{(r+k)}(x) \,=\, 0 \qquad\qquad (\text{for all } k \geq 0). \tag{6.3}$$

Thus no infinite sums are involved in checking containment in $\Gamma_{[a]_G}(m)$.

To deal with this space, we need an important ingredient which has been in the background up to this moment but which nevertheless is the most significant tool in the Theory of m-Quasi-Invariants. This is a formal power series $\Psi_m(x, y)$ referred to as the "*Baker–Akhiezer function*" of $\mathcal{QI}_m(G)$. To use $\Psi_m(x, y)$, we shall have to state a number of facts whose original proofs are in a series of papers scattered over several years. However, a reasonably self contained account of this material with detailed proofs of everything we use here (except for the non-degeneracy of the bilinear form) can be found in [8]. This is a monograph we have put together for the benefit of future researchers in this area.

To begin with, it will be good to see how $\Psi_m(x, y)$ is defined. Indeed, although this definition will play no role here, the novice in this area has great difficulty locating it in the literature. Remarkably, $\Psi_m(x, y)$ may be given an explicit (though quite forbidding) construction based on a truly remarkable family $\mathcal{SD}_n$ of "*Shift-Differential*" operators that act on polynomials by a combination of the ordinary G-action followed by differentiation. These operators are of the form

$$A \,=\, \sum_{\sigma \in G} a_\sigma(x, \partial_x)\, T_\sigma \ , \tag{6.4}$$

where

$$a_\sigma(x,\partial_x) \;=\; \sum_p a_p(x)\partial_x^p \qquad\qquad (\partial_x^p = \partial_{x_1}^{p_1}\partial_{x_2}^{p_2}\cdots\partial_{x_n}^{p_n}), \tag{6.5}$$

with each $a_p(x)$ a special rational function in the ring $\mathcal{SR}_n(x)$ generated by the variables x_i together with the fractions $1/(x,\alpha_s)$ for $s \in \Sigma(G)$. It it is easily shown that $\mathcal{SD}_n$ is in fact an algebra since the algebra of differental operators of the form given in (6.5) is invariant under conjugation by elements of G. The building blocks of $\mathcal{SD}_n$ are the m-Dunkl operators, which can be written in the form

$$\nabla_i(m) \;=\; \partial_{x_i} - m \sum_{s\in\Sigma(G)} (\alpha_s, e_i)\frac{1}{(x,\alpha_s)}\,(1-s) \qquad\qquad (\text{for } i = 1,2,\ldots,n), \tag{6.6}$$

with e_i the i^{th} coordinate unit vector. For fixed m, the operators $\{\nabla_i(m)\}_{i=1}^n$ are a commuting set, and thus it makes perfectly good sense to evaluate a polynomial at $\nabla_1(m), \nabla_2(m), \ldots, \nabla_n(m)$. We adopt the notation

$$Q[\nabla(m)] \;=\; Q\big(\nabla_1(m), \nabla_2(m), \ldots, \nabla_n(m)\big) \qquad\qquad (\text{for all } Q \in \mathbf{Q}[X_n]).$$

Clearly, all these operators belong to the family $\mathcal{SD}_n$. But once they are written in the form given in (6.4), we can simply "*forget*" the G action and set

$$\Gamma A \;=\; \sum_{\sigma\in G} a_\sigma(x,\partial_x)\,.$$

It develops that this seemingly innocent operation can achieve miracles. For instance one obtains that

$$\Gamma \sum_{i=1}^n \nabla_i(m)^2 \;=\; L_m(G). \tag{6.7}$$

More generally, it can be shown that for each G-invariant Q we have the beautiful identity

$$\gamma_Q(x,\partial_x) \;=\; \Gamma Q[\nabla(m)]. \tag{6.8}$$

This given, let us set

$$O_m \;=\; \Gamma\,\Pi_G(\nabla(m))\Pi_G(\underline{x})\ ,$$

where "$\Pi_G(\underline{x})$" denotes the operator multiplication by $\Pi_G(x)$. Remarkably, it can be shown that we have the Opdam commutation relation

$$L_m(G)\,O_m \;=\; O_m L_{m-1}(G)\,. \tag{6.9}$$

This given, the Baker–Akhiezer function $\Psi_m(x,y)$ is simply defined by setting

$$\Psi_m(x,y) \;=\; O_m O_{m-1}\cdots O_1 e^{(x,y)}\;. \tag{6.10}$$

Since the definition in (I.6) gives that $L_0(G)=\Delta_2$, we see that it follows from (6.9) that

$$L_m(G)\Psi_m(x,y) \;=\; (y,y)\Psi_m(x,y)\,. \tag{6.11}$$

The properties of $\Psi_m(x,y)$ we will use here may be stated as follows:

(a) *Symmetry:* $\Psi_m(x,y)=\Psi_m(y,x)$.

(b) $\gamma_Q(x,\partial_x)\Psi_m(x,y) \;=\; Q(y)\Psi_m(x,y)$ *for all* $Q\in\mathcal{QI}_m(G)$. (6.12)

(c) *For all* $\sigma\in G$ *we have* $\Psi_m(x\sigma,y)=\Psi_m(x,y\sigma^{-1})$.

(d) *We have the decomposition* $\Psi_m(x,y)=c_m(G)+\sum_{k\geq 1}\Psi_m^{(k)}(x,y)$ *with*

- (i) $\Psi_m(0,0)=\Psi_m(0,y)=\Psi_m(x,0)=c_m(G)\neq 0$,
- (ii) $\Psi_m^{(k)}(x,y)$ *a polynomial homogeneous of degree* k *in the* $x's$ *and* $y's$ *separately,*
- (iii) $\Psi_m^{(k)}(x,y)$ *in* $\mathcal{QI}_m(G)$ *in* $x_1,x_2,\ldots,x_n$ *and* $y_1,y_2,\ldots,y_n$.

We should note that (6.12) (d) (ii) immediately follows from the definition in (6.10) and the fact that the operator $\mathbf{\Omega}_m=O_mO_{m-1}\cdots O_1$ does not change degrees. In fact, from the expansion of the exponential $e^{(x,y)}$ we derive that

$$\Psi_m^{(k)}(x,y) \;=\; \sum_{|p|=k}\frac{y^p}{p!}\;\mathbf{\Omega}_m x^p\;.$$

Moreover, property (6.12) (d) (iii) is an immediate consequence of the following remarkable fact which will also be needed in the sequel.

We have the relation $L_m(G)P=Q$ *with* $Q\in\mathcal{QI}_m(G)$ *if and only if* $P\in\mathcal{QI}_m(G)$. (6.13)

We now have all the tools we need to proceed with our developments. We begin with some basic properties of $\Gamma_{[a]_G}(m)$.

Proposition 6.1

(1) *All the homogeneous components of every* $\Phi\in\Gamma_{[a]_G}(m)$ *are in* $\mathcal{QI}_m(G)$.

(2) *The collection* $\{\Psi_m(x,b)\}_{b\in[a]_G}$ *is a basis for* $\Gamma_{[a]_G}(m)$.

(3) $\Gamma_{[a]_G}(m)$ *is a* G*-module affording the regular representation of* G.

(4) *Every element* $\Phi \in \Gamma_{[a]_G}$ *may be written in the form*

$$\text{a)} \quad \Phi(x) = \gamma_Q \Theta_{[a]_G}(x) \qquad (\textit{for some } Q \in \mathcal{QI}_m(G)),$$

where

$$\text{b)} \quad \Theta_{[a]_G}(x) = \sum_{b \in [a]_G} \epsilon_b \Psi_m(x,b) = c\Pi(x)^{2m+1} + \cdots .$$

Proof

We have seen in (6.7) that $L_m(G) = \gamma_{p_2}(x, \partial_x)$ with $p_2(x) = (x,x)$. Since (x,x) is G-invariant, it follows that the difference $(x,x)-(a,a)$ belongs to $\Gamma_{[a]_G}(m)$. Thus, from the definition in (6.1) it follows that

$$L_m(G)\Phi = (a,a)\Phi \qquad (\text{ for all } \Phi \in \Gamma_{[a]_G}(m)). \tag{6.14}$$

But if $\Phi^{(i)}$ is the homogeneous component of degree i in Φ, then (6.14) yields that

$$L_m(G)\Phi^{(0)} = 0\,,\ L_m(G)\Phi^{(1)} = 0 \quad \text{and} \quad L_m(G)\Phi^{(i)} = (a,a)\Phi^{(i-2)} \quad \text{for all} \quad i \geq 2\,.$$

Thus property (1) follows from (6.13). To show (2) we note first that property (b) of the Baker–Akhiezer function implies that for all $Q \in \mathcal{J}_{[a]_G}(m)$ we have $\gamma_Q(x,\partial_x)\Psi_m(x,b) = 0$ for all $b \in [a]_G$. Thus

$$\big\{\Psi_m(x,b)\big\}_{b\in[a]_G} \subset \Gamma_{[a]_G}(m)\,.$$

To show that this collection spans $\Gamma_{[a]_G}(m)$, we will use the polynomials $\{\psi_b(x)\}_{[a]_G}$ defined in (1.8) with the specialization $\mathbf{A} = \mathcal{QI}_m(G)$ and $B(x) = \Pi_G(x)^{2m}$. Since (1.9) immediately gives the identity

$$1 \equiv \sum_{b\in[a]_G} \psi_b(x) \qquad (\text{mod } \mathcal{J}_{[a]_G}(m))$$

we derive from the definition in (6.1) that, for all $\Phi \in \Gamma_{[a]_G}(m)$ we have the decomposition

$$\Phi(x) = \sum_{b\in[a]_G} \Phi_b(x)\ , \tag{6.15}$$

with

$$\Phi_b(x) \;=\; \gamma_{\psi_b}(x, \partial_x)\Phi(x)\,. \tag{6.16}$$

We claim that the latter is none other than a scalar multiple of $\Psi_m(x, b)$. To prove this, note first that, for all $Q \in \mathcal{QI}_m(G)$ we have the relation

$$Q(x)\psi_b(x) \;-\; Q(b)\psi_b(x) \equiv 0 \qquad\quad (\mathrm{mod}\;\; \mathcal{J}_{[a]_G}(m)\,) \tag{6.17}$$

Thus

$$\begin{aligned} \gamma_Q(x, \partial_x)\Phi_b(x) \;&=\; \gamma_Q(x, \partial_x)\gamma_{\psi_b}(x, \partial_x)\Phi(x) \\ (\text{by (I.7)}) \;&=\; \gamma_{Q\psi_b}(x, \partial_x)\Phi(x) \\ (\text{by (6.17)}) \;&=\; Q(b)\gamma_{\psi_b}(x, \partial_x)\Phi(x) \;=\; Q(b)\Phi_b(x). \end{aligned} \tag{6.18}$$

Using this relation with $Q(x) = \Psi_m^{(k)}(x, y)$, we get

$$\gamma_{\Psi_m^{(k)}}\Phi_b\Big|_{x=0} \;=\; \Psi_m^{(k)}(b, y)\Phi_b(0). \tag{6.19}$$

On the other hand, since $\gamma_{\Psi_m^{(k)}}$ decreases degrees by k, denoting by $\Phi_b^{(k)}$ the k^{th} homogeneous component of Φ_b, it follows that

$$\begin{aligned} \gamma_{\Psi_m^{(k)}}\Phi_b(x)\Big|_{x=0} \;&=\; \big\langle \Psi_m^{(k)}\;,\;\Phi_b^{(k)} \big\rangle_m \\ &=\; \big\langle \Phi_b^{(k)}\;,\;\Psi_m^{(k)} \big\rangle_m \\ &=\; \gamma_{\Phi_b^{(k)}}\Psi_m^{(k)}(x, y)\Big|_{x=0} \\ &=\; \gamma_{\Phi_b^{(k)}}\Psi_m(x, y)\Big|_{x=0} \\ (\text{by (b) of (6.12) }) \;&=\; \Phi_b^{(k)}(y)\Psi_m(x, y)\Big|_{x=0} \\ (\text{by (d) (i) of (6.12)}) \;&=\; \Phi_b^{(k)}(y)c_m(G) \end{aligned} \tag{6.20}$$

Combining (6.19) and (6.20) we get that

$$\Psi_m^{(k)}(b, y)\Phi_b(0) \;=\; \Phi_b^{(k)}(y)c_m(G)\,.$$

This holding true for all k yields that

$$\Psi_m(b, y)\Phi_b(0) \;=\; \Phi_b(y)c_m(G)\,.$$

Solving for $\Phi_b(y)$ and using the symmetry of $\Psi_m(x,y)$, we now obtain

$$\Phi_b(y) \;=\; \frac{\Phi_b(0)}{c_m(G)}\;\Psi_m(y,b)\,, \tag{6.21}$$

as desired. Combining (6.15) with (6.21) proves that $\big\{\Psi_m(x,b)\big\}_{b\in[a]_G}$ spans $\Gamma_{[a]_G}(m)$.

To complete the proof of (2), we need to show independence. To this end, suppose that for some constants c_b we have

$$\Phi \;=\; \sum_{b\in[a]_G} c_b\Psi_m(x,b)\,. \tag{6.22}$$

Then the relations in (1.9) and (b) of (6.12) give, for $b'\in[a]_G$,

$$\big\langle \psi_{b'}\,,\,\Phi\big\rangle_m \;=\; \gamma_{\psi_{b'}}\Phi\Big|_{x=0} \;=\; \sum_{b\in[a]_G} c_b\,\psi_{b'}(b)\Psi_m(x,b)\Big|_{x=0} \;=\; c_{b'}\,c_m(G)\,. \tag{6.23}$$

Thus

$$\Phi=0 \;\Longrightarrow\; c_b=0 \quad \text{for all } b\in[a]_G\,,$$

proving independence. Incidentally, (6.23) yields that the expansion in (6.22) may be written in the form

$$\Phi \;=\; \sum_{b\in[a]_G} \frac{\big\langle \psi_b\,,\,\Phi\big\rangle_m}{c_m(G)}\;\Psi_m(x,b)\,.$$

Finally, note that property (c) of $\Psi_m(x,y)$ gives that

$$T_\sigma\Psi_m(x,b) \;=\; \Psi_m(x,b\sigma^{-1}) \qquad\qquad (\text{for all } \sigma\in G\,).$$

Thus the character χ of the action of G on the basis $\big\{\Psi_m(x,b)\big\}_{b\in[a]_G}$ has the expansion

$$\chi(\sigma) \;=\; \sum_{b\in[a]_G} \Psi_m(x,b\sigma^{-1})\Big|_{\Psi_m(x,b)} \;=\; \sum_{b\in[a]_G} \chi(b\sigma^{-1}=b) \;=\; \begin{cases} |G| & \text{if } \sigma=id,\\ 0 & \text{if } \sigma\neq id.\end{cases}$$

This gives (3).

Finally, note that if

$$\Phi(x) \;=\; \sum_{be\in[a]_G} c_b\Psi_m(x,b) \tag{6.24}$$

and we set

$$Q \;=\; \sum_{b'\in[a]_G} \tfrac{c_{b'}}{\epsilon_{b'}}\,\psi_{b'}(x)$$

then clearly $Q \in \mathcal{QI}_m(G)$, and we also have from (6.24)

$$\begin{aligned} \gamma_Q\Theta_{[a]_G} \;&=\; \sum_{b\in[a]_G}\epsilon_b \sum_{b'\in[a]_G} \tfrac{c_b}{\epsilon_{b'}}\,\gamma_{\psi_{b'}}\Psi_m(x,b) \\ \text{(by (6.12) (b))} \;&=\; \sum_{b\in[a]_G}\epsilon_b \sum_{b'\in[a]_G} \tfrac{c_b}{\epsilon_{b'}}\,\psi_{b'}(b)\Psi_m(x,b) \\ \text{(by (1.9))} \;&=\; \sum_{b\in[a]_G}\epsilon_b \tfrac{c_b}{\epsilon_b}\Psi_m(x,b) \;=\; \Phi(x). \end{aligned}$$

This proves (4) (a).

To prove (4) (b), note that from the definition of $\Theta_{[a]_G}(x)$ it follows that for any $\sigma \in G$ we have

$$T_\sigma\Theta_{[a]_G} \;=\; \det(\sigma)\,\Theta_{[a]_G}.$$

This forces all the homogeneous components of $\Theta_{[a]_G}$ to be G-invariant multiples of $\Pi_G(x)^{2m+1}$. Thus we can write

$$\Theta_{[a]_G}(x) \;=\; A(x)\Pi_G(x)^{2m+1} \;+\; \cdots \qquad\qquad \text{(for some } A(x)\in\Lambda_G).$$

Now, from the definition of $\Theta_{[a]_G}$ and (6.12) (b) we derive that

$$\gamma_{\Pi^{2m+1}}\Theta_{[a]_G}(x) \;=\; \sum_{b\in[a]_{S_n}} \epsilon(b)\Pi_G(b)^{2m+1}\Psi_m(x,b) \;=\; \Pi_G(a)^{2m+1}\sum_{b\in[a]_{S_n}}\Psi_m(x,b)\,. \tag{6.25}$$

Since $\gamma_{\Pi^{2m+1}}$ decreases degrees by $(2m+1)|\Sigma(G)|$, we have

$$degree\big(\gamma_{\Pi^{2m+1}}A(x)\Pi_G(x)^{2m+1}\big) \;=\; degree(A(x))\,.$$

Thus,

$$degree\left(\mu\big(\gamma_{\Pi^{2m+1}}\Theta_{[a]_G}(x)\big)\right) \geq degree(A(x))\,, \tag{6.26}$$

where, for each formal power series Φ, we let $\mu(\Phi)$ denote the homogeneous component of least degree in Φ. However, the right hand side of (6.25) (using (6.12) (d) (i)) yields that

$$\mu\big(\gamma_{\Pi^{2m+1}}\Theta_{[a]_G}(x)\big) \;=\; \Pi_G(a)^{2m+1}|G|c_m(G) \neq 0\,. \tag{6.27}$$

Combining (6.26) with (6.27) forces $A(x)$ to be a scalar.

Note that this argument also shows that for some constant C we must have

$$\gamma_{\Pi^{2m+1}}\Pi(x)^{2m+1} = C \neq 0 \tag{6.28}$$

To introduce our next construct, we need further notation. To begin with, we shall choose once and for all a degree lexicographic order of monomials. For instance, the one which corresponds to the total order $x_1 > x_2 > \ldots > x_n$. It will be convenient to denote this order by "$<_{dl}$". Recalling that $\mu(\Phi)$ denotes the homogeneous component of least degree in Φ, we will let $l(\Phi)$ denote the d-lex least monomial in $\mu(\Phi)$. We also set $l(f) = l(\Phi)$ when $f = \mu(\Phi)$.

Remark 6.1

It will be convenient in our further developments to use the symbol "$\Phi\big|_{=k}$" to denote the the homogeneous components of Φ of degree k. In the same vein, we let "$\Phi\big|_{<k}$" and "$\Phi\big|_{\leq k}$" the sum of all the homogeneous components of Φ of degree $< k$ and $\leq k$ respectively. In our previous notation

$$\Phi\big|_{=k} = \Phi^{(k)}\,, \qquad \Phi\big|_{<k} = \sum_{i=0}^{k-1}\Phi^{(i)} \qquad \text{and} \qquad \Phi\big|_{\leq k} = \sum_{i=0}^{k}\Phi^{(i)}\,. \tag{6.29}$$

Thus we have

$$\mu(\Phi) = \Phi\big|_{=k} \qquad \text{if and only if} \qquad \Phi\big|_{<k} = 0 \;\text{ and }\; \Phi\big|_{=k} \neq 0\,.$$

Thus we clearly see that the map $\Phi \mapsto \mu(\Phi)$ is not linear. Nevertheless, if $\mu(\Phi_1)$ and $\mu(\Phi_2)$ have the same degree then

$$\mu(\Phi_1) + \mu(\Phi_2) \neq 0 \quad \text{implies} \quad \mu(\Phi_1 + \Phi_2) = \mu(\Phi_1) + \mu(\Phi_2). \tag{6.30}$$

Keeping this in mind, we let $\mu(\Gamma_{[a]_G}(m))$ denote the linear span of the homogeneous components of least degree of elements of $\Gamma_{[a]_G}(m)$. In symbols

$$\mu(\Gamma_{[a]_G}(m)) \;=\; \mathcal{L}\big[\,\mu(\Phi)\;:\;\Phi\in\Gamma_{[a]_G}(m)\,\big]\,. \tag{6.31}$$

The following result shows the intimate relation between $\Gamma_{[a]_G}(m)$ and $\mu(\Gamma_{[a]_G}(m))$.

Proposition 6.2

We can find in $\Gamma_{[a]_G}(m)$ a collection of formal power series

$$\Phi_1\;,\;\Phi_2\;,\;\ldots\;,\;\Phi_{|G|} \tag{6.32}$$

with the property that

$$l(\Phi_1) <_{dl} l(\Phi_2) <_{dl} \cdots <_{dl} l(\Phi_{|G|}) \tag{6.33}$$

and such that each $f\in\mu(\Gamma_{[a]_G}(m))$ has a unique expansion of the form

$$f \;=\; \sum_{l(\Phi_i)\,\geq_{dl}\, l(f)} c_i\,\mu(\Phi_i)\,. \tag{6.34}$$

In particular, the collection

$$\mu(\Phi_1)\;,\;\mu(\Phi_2)\;,\;\ldots\;,\;\mu(\Phi_{|G|}) \tag{6.35}$$

is a basis of $\mu(\Gamma_{[a]_G}(m))$ and we have

$$\dim\,\mu(\Gamma_{[a]_G}(m)) \;=\; |G|\,. \tag{6.36}$$

Proof

The collection in (6.32) satisfying (6.33) can be constructed by starting with the basis

$$\big\{\Psi_\mu(x,b)\big\}_{b\in[a]_G}\;,$$

then progressively reducing it to echelon form with respect to the degree-lexicographic order of least monomials. This done, (6.33) yields that their minimum components in (6.35) are linearly independent. In fact, more than that is true. Note first that no element $\Phi\in\Gamma_{[a]_G}(m)$ can have a least monomial that is different from each of the monomials in (6.33). Indeed, such an element would necessarily be independent of the Φ_i's, and then $\Gamma_{[a]_G}(m)$ would have dimension greater than $|G|$. The existence of constants c_i giving (6.34) may be established by descent induction on the degree-lexicographic order of least monomials. To see this let

$$f = \mu(\Phi).$$

Now the result is immediate if $l(f) = l(\Phi_{|G|})$. In fact, let c be chosen so that $l(\Phi_{|G|})$ does not occur in $f - c\,\mu(\Phi_{|G|})$. Then the difference $\Phi - c\Phi_{|G|}$ must be identically zero, for otherwise its degree-lexicographically least monomial would necessarily be larger than $l(\Phi_{|G|})$ and this, as we have seen, is not possible. This gives

$$f \;=\; c\,\mu(\Phi_{|G|}),$$

and we are done in this case. So assume by induction that the expansion in (6.34) exists when $l(f) >_{dl} l(\Phi_{i_o})$. Let $l(f) = l(\Phi_{i_o})$. Again chose c so that $l(\Phi_{i_o})$ does not occur in $f - c\Phi_{i_o}$. Here there are two cases. If $f = c\,\mu(\Phi_{i_o})$ we are done. If, on the other hand, this difference does not vanish identically, then, since $l(f) = l(\Phi_{i_o})$ implies that $degree(\mu(\Phi)) = degree(\mu(\Phi_{i_o}))$, we can use (6.30) with $\Phi_1 = \Phi$ and $\Phi_2 = -c\,\Phi_{i_o}$ and conclude that

$$f \;=\; \mu(\Phi) \;=\; c\,\mu(\Phi_{i_o}) \;+\; \mu(\Phi - c\,\Phi_{i_o}) \;\;.$$

Since now $l(\Phi - c\,\Phi_{i_o}) \;>_{dl}\; l(\Phi_{i_o})$, we can use the induction hypothesis and deduce that, for some suitable constants c_i, we must have

$$f \;=\; c\,\mu(\Phi_{i_o}) \;+\; \sum_{i=i_o+1}^{|G|} \; c_i\;\mu(\Phi_i) \;\;.$$

This completes the induction and establishes (4.34). The remaining assertions are immediate consequences of (6.34).

The following remarkable fact provides us with a new tool for studying m-Harmonics.

Theorem 6.1

For all finite reflection groups G and all $m \geq 0$ we have

$$\mu\big(\Gamma_{[a]_G}(m)\big) \;=\; \mathbf{H}_G(m). \tag{6.37}$$

Proof

In view of the fact that from Theorem 5.5 and Proposition 6.2 it follows that these two spaces have the same dimension, to show the equality in (6.37) it is sufficient to derive the containment

$$\mu\big(\Gamma_{[a]_G}(m)\big) \;\subseteq\; \mathbf{H}_G(m)\,. \tag{6.38}$$

This in turn immediately follows from Theorem 5.3 once we verify that

$$P \in \mu\big(\Gamma_{[a]_G}(m)\big) \qquad \text{implies} \qquad \gamma_{q_k} P(x) = 0 \qquad (\text{for } k = 1, 2, \ldots, n)\,. \tag{6.39}$$

This given, let $Q(x)$ be a homogeneous G-invariant and note that since the difference $Q(x) - Q(a)$ vanishes throughout $[a]_G$, it follows from the definition in (6.1) that we have

$$\gamma_Q(x, \partial_x)\Phi(x) \;=\; Q(a)\Phi(x) \qquad (\text{for all} \quad \Phi \in \Gamma_{[a]_G}(m)\,). \tag{6.40}$$

Thus, if Q is of degree d, equating homogeneous components of both sides of (6.40), we derive that

$$\gamma_Q(x, \partial_x)\Phi^{(k)}(x) \;=\; \begin{cases} 0 & \text{if } k < d, \\ Q(a)\Phi^{(k-d)}(x) & \text{if } k \geq d. \end{cases} \tag{6.41}$$

Thus, if $\mu(\Phi(x))$ has degree d_o, then it follows from this that

$$\gamma_Q(x, \partial_x)\Phi^{(k)}(x) \;=\; 0 \qquad \text{for all} \quad k - d < d_o\,. \tag{6.42}$$

In particular, we must have

$$\gamma_Q(x, \partial_x)\mu\big(\Phi(x)\big) \;=\; 0\,,$$

and (6.39) then immediately follows from the definition in (6.31). This establishes (6.38) and completes our proof.

We now have all the tools we need for the proof of Theorem I.2. The argument is based on the following two basic observations

Proposition 6.3

Let $d_{mn} = maxdegree\big(\mathbf{H}_G(m)\big)$. *Then*

$\mathbf{Ob_1}$: *To show that an element $\Phi \in \Gamma_{[a\,]_G}(m)$ vanishes, it is sufficient to check that all its homogeneous components of degree $\leq d_{mn}$ vanish.*

$\mathbf{Ob_2}$: *To show that a polynomial $Q(x) \in \mathcal{QI}_m(G)$ belongs to $\mathcal{J}_{[a\,]_G}(m)$, it is sufficent to show that $\gamma_Q(x, \partial_x)$ kills the element $\Theta_{[a\,]_G}(m)$ defined in* (4) *of Proposition* 6.1.

Proof

Suppose that $\Phi \in \Gamma_{[a\,]_G}$ satisfies

$$\Phi^{(k)} = 0 \qquad (\text{for } k \leq d_{mn}), \tag{6.43}$$

and suppose if possible that $\Phi \neq 0$. Now we have shown that that for any $\Phi \in \Gamma_{[a\,]_G}$ we have $\mu(\Phi) \in \mathbf{H}_G(m)$. Thus (6.43) gives $\mu(\Phi) = 0$. But this is absurd since by definition $\Phi \neq 0$ implies $\mu(\Phi) \neq 0$. This proves $\mathbf{Ob_1}$. Note next that the equation $\gamma_Q \Theta_{[a\,]_G} = 0$ together with (6.12) (b) yields the identity

$$\sum_{b \in [a\,]_G} \epsilon(b) Q(b) \Psi_m(x, b) \;=\; 0\,.$$

However, the independence of $\big\{\Psi_m(x, b)\big\})_{b \in [a\,]_G}$ then forces $Q_(b) = 0$ for all $b \in [a\,]_G$, But that is $Q \in \mathcal{J}_{[a\,]_G}(m)$. This proves $\mathbf{Ob_2}$.

We shall now establish Theorem I.2 by proving a bit more.

Theorem 6.2

Let $\mathcal{B} \in \mathcal{QI}_m(G)$ yield a homogeneous basis for the quotient $\mathcal{QI}_m(G)/\mathcal{J}_G(m)$, and let $\mathcal{B}_k$ be the subset of elements of degree k in $\mathcal{B}$. Then the collection

$$\big\{\gamma_b(x, \partial_x) \Pi_G(x)^{2m+1}\big\}_{b \in \mathcal{B}_k}$$

is a basis of $\mathcal{H}_{=k}\big(\mathbf{H}_G(m)\big)$.

Proof

We shall have to use two facts, namely that

$$d_{mn} = degree\big(\Pi_G(x)^{2m+1}\big) \tag{6.44}$$

and that the Hilbert series $F_{\mathbf{H}_G(m)}(t)$ is palindromic. That is, we have

$$dim\big(\mathcal{H}_k(\mathbf{H}_{[a\,]_G}\big) \;=\; dim\big(\mathcal{H}_{d_{mn}-k}(\mathbf{H}_{[a\,]_G}\big)\,. \tag{6.45}$$

We should note that (6.44) immediately follows from the definition of the polynomial $\psi_b(x)$ given in (1.8), since we proved there that the maximum degree of the ordinary G-harmonics is equal to the degree of the discriminant $\Pi_G(x)$.

As for (6.45) we have to refer to the paper of Felder–Veselov [5] for a proof.

This given, we shall prove our result by induction on k. More precisely we shall assume that

1_k : $P(\partial_x)\Phi_{[a]})\,|_{<d_{mn}-k} \;=\; 0$ *implies that* P *is congruent mod* $\mathcal{J}_{[a\,]_G}(m)$ *to a polynomial* Q *of degree* $\leq k$*;*

and

2_k : $\big\{\gamma_b(x,\partial_x)\Pi_G(x)^{2m+1}\big\}_{b\in\mathcal{B}_k}$ *is a basis for* $\mathcal{H}_{=d_{mn}-k}\big(\mathbf{H}_G(m)\big)$*;*

hold true for all $k < k_o$ and complete the induction by showing that 1_{k_o} and 2_{k_o} must hold as well.
To this end, note first that, since $dim\big(\mathcal{H}_0(\mathbf{H}_{[a\,]_G}\big) = 1$, it follows from (6.39) that we also have

$$dim\big(\mathcal{H}_{d_{mn}}\big(\mathbf{H}_{[a\,]_G}\big)\big) = 1\,. \tag{6.46}$$

Since the polynomial $\Pi_G(x)^{2m+1}$ is in $\mathcal{QI}_m(G)$ and is clearly killed by all G-invariant differential operators, it follows that it lies in $\mathcal{H}_{d_{mn}}\big(\mathbf{H}_{[a\,]_G}\big)$. But then (6.44) yields that every element of $\mathcal{H}_{d_{mn}}\big(\mathbf{H}_{[a\,]_G}\big)$ is necessarily a multiple of $\Pi_G(x)^{2m+1}$. This proves 2_0. To start, we need also check the validity of 1_0. Note that 1_0 says that any P such that

$$\gamma_P(x,\partial_x)\Theta_{[a\,]_G}\,|_{<d_{mn}} \;=\; 0 \tag{6.47}$$

must be congruent to a constant mod $\mathcal{J}_{[a\,]_G}(m)$. Note that if $\gamma_P(x,\partial_x)\Theta_{[a\,]_G}\,|_{=d_{mn}}$ also vanishes then $\mathbf{Ob_1}$ gives that $\gamma_P(x,\partial_x)$ kills $\Theta_{[a\,]_G}$, and then $\mathbf{Ob_2}$ yields that P is congruent to zero mod $\mathcal{J}_{[a\,]_G}(m)$. On the other hand, if $\gamma_P(x,\partial_x)\Theta_{[a\,]_G}\,|_{=d_{mn}}$ does not vanish, then, since the homogeneous elements of degree d_{mn} in $\mathbf{H}_{[a]}$ are all multiples of $\Pi_G(x)^{2m+1}$, from (4) b) of Proposition 6.1 we derive that

$$\gamma_P(x,\partial_x)\Theta_{[a\,]_G}\,|_{=d_{mn}} \;=\; c\,\mu(\Theta_{[a\,]_G})$$

for a suitable constant c. But then all the homogeneous components of degree d_{mn} or less in

$$(\gamma_P(x,\partial_x) - c)\;\Theta_{[a\,]_G}$$

must vanish, and $\mathbf{Ob_1}$ and $\mathbf{Ob_2}$ again yield that $P - c \in \mathcal{J}_{[a]_G}(m)$. This gives 1_0.

We are thus in a position to proceed with our induction, and we shall assume that 1_k and 2_k hold for all $k < k_o$. We start by proving 2_{k_o}. To this end, note that by (6.45) for $k = k_o$ we need only show that $\{\gamma_b(x, \partial_x)\Pi_G(x)^{2m+1}\}_{b\in\mathcal{B}_{k_o}}$ is an independent set. So, let there be constants c_b such that

$$\sum_{b\in\mathcal{B}_{k_o}} c_b\, \gamma_b(x, \partial_x)\Pi_G(x)^{2m+1} \;=\; 0\,, \tag{6.48}$$

and set

$$P(x) \;=\; \sum_{b\in\mathcal{B}_{k_o}} c_b\, b(x)\,. \tag{6.49}$$

Now, (4.48) implies that

$$\gamma_P(x, \partial_x)\Theta_{[a]_G}\,|_{\le d_{mn}-k_o} \;=\; 0\ \ .$$

However, this brings us into 1_{k_o-1} and by induction we can find a polynomial Q of degree $< k_o$ congruent to P modulo $\mathcal{J}_{[a]_G}(m)$. But now, since *degree* $P = k_o >$ *degree* Q and P is homogeneous, we deduce that $P \in \mathbf{gr}\; \mathcal{J}_{[a]_G}(m)$. Now we have seen in (5.25) that $\mathbf{gr}\; \mathcal{J}_{[a]_G}(m) = \mathcal{J}_G(m)$, so it follows that $P \in \mathcal{J}_G(m)$. But this together with (6.49) contradicts the independence of $\mathcal{B}$ mod $\mathcal{J}_G(m)$. This forces the vanishing of all the coefficients c_b in (6.49). Thus 2_{k_o} must hold true as desired.

Next we show 1_{k_o}. So let

$$\gamma_P(x, \partial_x)\Theta_{[a]_G}\,|_{< d_{mn}-k_o} \;=\; 0\ \ . \tag{6.50}$$

If $\gamma_P(x, \partial_x)\Theta_{[a]_G} = 0$, then by $\mathbf{Ob_2}$ we must have $P \in \mathcal{J}_{[a]_G}(m)$ and we are done. If $\gamma_P(x, \partial_x)\Theta_{[a]_G}$ does not vanish, then (6.50) gives that

$$\textit{degree}\ \mu\big(\gamma_P(x, \partial_x)\Theta_{[a]_G}\big) \;=\; d_{mn} - k_1 \qquad\quad (\text{ with } k_1 \le k_o\ )\ \ .$$

By 2_{k_1}, which is now available up to and including k_o, we can find a homogeneous polynomial Q_1 of degree k_1 such that

$$\mu\big(\gamma_P(x, \partial_x)\Theta_{[a]_G}\big) \;=\; \gamma_{Q_1}(x, \partial_x)\, \Pi_G(x)^{2m+1}\ \ . \tag{6.51}$$

This in turn implies that for a suitable constant c

$$\gamma_{P-cQ_1}(x, \partial_x)\Theta_{[a]_G}\,|_{\le d_{mn}-k_1} \;=\; 0\ \ .$$

However, since $k_1 \leq k_o$, this brings us down into the domain of 1_{k_o-1}, so we can use the induction hypothesis and conclude that $P - cQ_1$ is congruent mod $\mathcal{J}_{[a]_G}(m)$ to a polynomial Q_2 of degree at most $k_o - 1$. In other words, we have shown that P is congruent mod $\mathcal{J}_{[a]_G}(m)$ to the polynomial $Q = cQ_1 + Q_2$ which is of degree at most k_o, which is precisely what we needed to show. This completes the induction and our proof.

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