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HOOKS AND POWERS OF PARTS IN PARTITIONS

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Abstract. This paper shows that the number of hooks of length k contained in all partitions of n equals k times the number of parts of length k in partitions of n. It contains also formulas for the moments (under uniform distribution) of k-th parts in partitions of n.

1. Introduction and Main Results

Many textbooks contain material on partitions. Two standard references are **[A]** and **[S]**.

A *partition* of a natural integer n with *parts* $\lambda_1, \dots, \lambda_k$ is a finite decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ of natural integers $\lambda_1, \dots, \lambda_k > 0$ such that $n = \sum_{i=1}^k \lambda_i$. We denote by $|\lambda|$ the *content* n of λ. Partitions are also written as sums: $n = \lambda_1 + \cdots + \lambda_k$ and one uses also the (abusive) multiplicative notation

$$\lambda = 1^{\nu_1} \cdot 2^{\nu_2} \cdots n^{\nu_n}$$

where ν_i denotes the number of parts equal to i in the partition λ.

A partition is graphically represented by its *Young diagram* obtained by drawing λ_1 adjacent boxes of identical size on a first row, followed by λ_2 adjacent boxes of identical size on a second row and so on with all first boxes (of different rows) aligned along a common first column. In the sequel we identify a partition with its Young diagram. A *hook* in a partition is a choice of a box H in the corresponding Young diagram together with all boxes at the right of the same row and all boxes below of the same column. The total number of boxes in a hook is its *hooklength*, the number of boxes in a hook to the right of H is its *armlength* and the number of boxes of a hook below H is the *leglength*. The Figure below displays the Young diagram of the partition $(5, 4, 3, 1)$ of 13 together with a hook of length 4 having armlength 2 and leglength 1. We call the couple (armlength,leglength) of a hook its *hooktype* and denote it by $\tau = \tau(\alpha, k-1-\alpha)$ if its armlength is α and its leglength $k - 1 - \alpha$. Such a hook has hence total length k and there are exactly k different hooktypes for hooks of length k.

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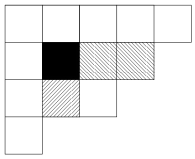

The partition $(5, 4, 3, 1)$ of 13 together with a hook of type $\tau(2, 1)$ and length 4.

Let k be a natural integer and let $\tau = \tau(\alpha, k-1-\alpha)$ be the hooktype of a hook of length k with armlength α and leglength $k - 1 - \alpha$. Given a partition λ of n, set

$$\tau(\lambda) = \sharp\{\text{hooks of type } \tau \text{ in (the Young diagram of) } \lambda\}$$

and

$$\tau(n) = \sum_{\lambda,\ |\lambda|=n} \tau(\lambda)$$

where the sum is over all partitions of n.

Theorem 1.1. *One has*

$$\begin{aligned}\sum_{n=1}^{\infty} \tau(n) z^n &= \frac{z^k}{1-z^k} \prod_{i=1}^{\infty} \frac{1}{1-z^i} \\ &= \sum_{\lambda=1^{\nu_1} 2^{\nu_2} \dots} \nu_k \ z^{|\lambda|}\end{aligned}$$

where the last sum is over all partitions of integers.

In other terms, the number of hooks of given type and length k appearing in all partitions of n equals the number of parts of length k in all partitions of n.

This result implies in particular that the total number of hooks of given type $\tau = \tau(\alpha, k - 1 - \alpha)$ occuring in all partitions of n depends only on the length k and not on the particular hooktype $\tau(\alpha, k-1-\alpha)$ itself. Since there are exactly k distinct hooktypes for hooks of length k, the total number of hooks of length k in partitions of n is given by the coefficient of z^n of the series

$$k \frac{z^k}{1-z^k} \prod_{i=1}^{\infty} \frac{1}{1-z^i} \ .$$

C. Bessenrodt pointed out to us that this Theorem follows directly from Theorem 1.1 in [**B**]. Indeed, Theorem 1.1 of [**B**] states that the number of k-hooks of given leglength which can be added to a Young diagram always exceeds by 1 the number of k-hooks of the same leglength which can be removed from the same Young diagram. This implies the identity

$$\sum_{n=1}^{\infty} \tau(n) z^n = z^k \left(\sum_{n=1}^{\infty} \tau(n) z^n + \prod_{i=1}^{\infty} \frac{1}{1-z^i} \right)$$

on the generating series appearing in Theorem 1.1. The obvious initial conditions $\tau(n) = 0$ for $n < k$ determine now the generating series which is easily checked to be

$$\frac{z^k}{1-z^k}\prod_{i=1}^{\infty}\frac{1}{1-z^i} \ .$$

Our initial proof of Theorem 1.1 was based on the $q-$binomial theorem.

As remarked previously, partitions of a natural integer n can be written in (at least) two different ways: either by considering the finite decreasing sequence

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$$

of its parts or by considering the vector

$$\nu = (\nu_1, \nu_2, \ldots, \nu_n)$$

where ν_i counts the multiplicity of parts with length i in λ. We still denote by λ the vector

$$(\lambda_1, \lambda_2, \ldots \lambda_k, 0, \ldots, 0) \in \mathbf{Z}^n$$

of length n obtained by appending $n-k$ zero-coordinates at the end of the vector $(\lambda_1, \ldots, \lambda_k)$ defining a partition λ.

We consider moreover the vector $\gamma = (\gamma_1, \ldots, \gamma_n)$ where γ_i equals the number of coordinates equal to i among $\nu_1, \ldots, \nu_n$. The vector γ of a partition encodes "multiplicities of multiplicities (of parts)" and does no longer encode the partition since for instance the γ-vectors of the two partitions $5 = 3+2$ and $5 = 4+1$ give both rise to $\gamma = (2, 0, 0, 0, 0)$.

We introduce also the vectors

$$\begin{aligned}
\lambda(n) &= (\lambda_1(n), \lambda_2(n), \ldots, \lambda_n(n)) = \textstyle\sum_{|\lambda|=n} \lambda \ , \\
\nu(n) &= (\nu_1(n), \nu_2(n), \ldots, \nu_n(n)) = \textstyle\sum_{|\lambda|=n} \nu \ , \\
\gamma(n) &= (\gamma_1(n), \gamma_2(n), \ldots, \gamma_n(n)) = \textstyle\sum_{|\lambda|=n} \gamma
\end{aligned}$$

of $\mathbf{Z}^n$ obtained by summing up the vectors λ, ν or γ over all partitions of n. The coordinates of the vector $\lambda(n)$ are of course related to the mean length (under uniform distribution) of the k-th part in partitions of n. Similarly, coordinates of $\nu(n)$ relate to the mean multiplicity of parts equal to k and coordinates of $\gamma(n)$ measure the mean number of distinct part-lengths appearing with common multiplicity k.

The following Table displays all five partitions of 4 together with the corresponding λ-, ν- and γ-vectors.

Partition	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
λ-vector	$(1,1,1,1)$	$(2,1,1,0)$	$(2,2,0,0)$	$(3,1,0,0)$	$(4,0,0,0)$
ν-vector	$(4,0,0,0)$	$(2,1,0,0)$	$(0,2,0,0)$	$(1,0,1,0)$	$(0,0,0,1)$
γ-vector	$(0,0,0,1)$	$(1,1,0,0)$	$(0,1,0,0)$	$(2,0,0,0)$	$(1,0,0,0)$

Summing up all λ-, ν- and γ-vectors associated to the five partitions of 4 we get hence

$$\begin{aligned}\lambda(4) &= (12,5,2,1),\\ \nu(4) &= (7,3,1,1),\\ \gamma(4) &= (4,2,0,1)\ .\end{aligned}$$

One has the following result.

Theorem 1.2. *For all $n \geq 1$ we have $\lambda_n(n) = \nu_n(n) = \gamma_n(n) = 1$ and*

$$\begin{aligned}\lambda_k(n) &= \nu_k(n) + \lambda_{k+1}(n)\ ,\\ \nu_k(n) &= \gamma_k(n) + \nu_{k+1}(n)\end{aligned}$$

for $k = 1, \ldots, n-1$.

These equalities can be restated as

$$\lambda_k(n) = \sum_{i=k}^{n} \nu_i(n) \text{ and } \nu_k(n) = \sum_{i=k}^{n} \gamma_i(n)\ .$$

This last equality states for instance that the sum over all partitions of n of the number of distinct parts arising with multiplicity at least k equals the number of parts equal to k in all partitions of n. Our proof of this fact uses generating series. C. Bessenrodt ([**B1**]) communicated to us a beautiful bijective proof which we reproduce with her permission.

The coordinates of the vectors $\nu(n)$ are of course given by the generating series mentioned in Theorem 1.1, i.e., the k-th coordinate $\nu_k(n)$ of $\nu(n)$ equals the coefficient of z^n in the generating series

$$\frac{z^k}{1-z^k}\prod_{i=1}\frac{1}{1-z^i}\ .$$

The coordinates of the vectors $\lambda(n)$ and $\nu(n)$ are then easily computed using Theorem 1.2. More precisely, one has the following result:

Corollary 1.3. *(i) The k-th coordinate of the vector $\lambda(n)$ is the coefficient of z^n in the generating series*

$$\prod_{i=1}^{\infty}\frac{1}{1-z^i}\ \sum_{j=k}^{\infty}\frac{z^j}{1-z^j}\ .$$

(ii) The k-th coordinate of the vector $\gamma(n)$ is the coefficient of z^n in the generating series

$$\frac{(1-z)\ z^k}{(1-z^k)(1-z^{k+1})}\prod_{i=1}^{\infty}\frac{1}{1-z^i}\ .$$

Given a partition

$$\lambda = (\lambda_1, \ldots, \lambda_n) = (1^{\nu_1}\cdots n^{\nu_n})$$

and an integer $d \geq 0$ we introduce the vectors $\binom{\lambda}{d}$ and $\binom{\nu}{d} \in \mathbf{Z}^n$ by setting

$$\binom{\lambda}{d} = \left(\binom{\lambda_1}{d}, \ldots, \binom{\lambda_n}{d}\right) \text{ and } \binom{\nu}{d} = \left(\binom{\nu_1}{d}, \ldots, \binom{\nu_n}{d}\right)$$

and define

$$\binom{\lambda(n)}{d} = \sum_{|\lambda|=n} \binom{\lambda}{d}, \quad \binom{\nu(n)}{d} = \sum_{|1^{\nu_1}\ 2^{\nu_2}\cdots|=n} \binom{\nu}{d}$$

with coordinates

$$\binom{\lambda_k(n)}{d} = \sum_{|\lambda|=n} \binom{\lambda_k}{d}, \quad \binom{\nu_k(n)}{d} = \sum_{|1^{\nu_1}\ 2^{\nu_2}\cdots|=n} \binom{\nu_k}{d}.$$

The following example shows the vectors $\binom{\lambda}{1} = \lambda, \binom{\lambda}{2}, \binom{\lambda}{3}$ and $\binom{\nu}{1} = \nu, \binom{\nu}{2}, \binom{\nu}{3}$ associated to all five partitions of 4.

Example:

Partition		$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
$\binom{\lambda}{1}$	$=$	$(1,1,1,1)$	$(2,1,1,0)$	$(2,2,0,0)$	$(3,1,0,0)$	$(4,0,0,0)$
$\binom{\lambda}{2}$	$=$	$(0,0,0,0)$	$(1,0,0,0)$	$(1,1,0,0)$	$(3,0,0,0)$	$(6,0,0,0)$
$\binom{\lambda}{3}$	$=$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(1,0,0,0)$	$(4,0,0,0)$

Partition		$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
$\binom{\nu}{1}$	$=$	$(4,0,0,0)$	$(2,1,0,0)$	$(0,2,0,0)$	$(1,0,1,0)$	$(0,0,0,1)$
$\binom{\nu}{2}$	$=$	$(6,0,0,0)$	$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$
$\binom{\nu}{3}$	$=$	$(4,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$

We have thus

$$\binom{\lambda(4)}{1} = (12,5,2,1), \quad \binom{\lambda(4)}{2} = (11,1,0,0), \quad \binom{\lambda(4)}{3} = (5,0,0,0)$$
$$\binom{\nu(4)}{1} = (7,3,1,1), \quad \binom{\nu(4)}{2} = (7,1,0,0), \quad \binom{\nu(4)}{3} = (4,0,0,0)$$

The following probably well-known result allows easy computations of the vectors $\binom{\lambda(n)}{d}$ and $\binom{\nu(n)}{d}$.

Proposition 1.4. *For any natural integer $d \geq 0$, the k-th coefficients $\binom{\lambda_k(n)}{d}$, respectively $\binom{\nu_k(n)}{d}$ (extended by $\binom{0}{d}$ for $k > n$) have generating series*

$$\sum_n^\infty \binom{\lambda_k(n)}{d} z^n = \left(\prod_{j=1}^{k-1} \frac{1}{1-z^j}\right)\left(\sum_{i=0}^\infty \binom{i}{d} z^{ik} \left(\prod_{j=1}^{i} \frac{1}{1-z^j}\right)\right)$$

and

$$\sum_n^\infty \binom{\nu_k(n)}{d} z^n = \left(\frac{z^k}{1-z^k}\right)^d \left(\prod_{j=1}^\infty \frac{1}{1-z^j}\right).$$

Remark 1.5. One has

$$\sum_{|\lambda|=n} \lambda_k^d = \sum_i i!\ \mathrm{Stirling}_2(d,i)\ \binom{\lambda_k(n)}{i}$$

and

$$\sum_{|1^{\nu_1}\ 2^{\nu_2}\cdots|=n} \nu_k^d = \sum_i i!\ \mathrm{Stirling}_2(d,i)\ \binom{\nu_k(n)}{i}$$

where $\mathrm{Stirling}_2(d,i)$ denote Stirling numbers of the second kind, defined by $x^d = \sum_i \mathrm{Stirling}_2(d,i)\ x(x-1)\cdots(x-i+1)$.

Asymptotics are not so easy to work out from the formula for $\binom{\lambda(n)}{d}$. Our last result is an equivalent expression for the above series on which asymptotics are easier to see.

We introduce the generating series $\sigma_r(k)$ defined as

$$\sigma_r(k) = \sum_{i=k}^{\infty} \left(\frac{z^i}{1-z^i}\right)^r$$

for $r \geq 1$ and $k \geq 1$ natural integers. We consider the series $\sigma_r(k)$ as beeing graded of degree r and define the homogeneous series $S_d(k)$ of degree d by

$$\begin{aligned} S_d(k) &= \sum_{|(1^{\nu_1}\ 2^{\nu_2}\cdots)|=d} \frac{d!}{(\sum_i \nu_i)!} \binom{(\sum_i \nu_i)}{\nu_1\ \nu_2 \ldots} \prod_{i=1}^{d} \left(\frac{\sigma_i(k)}{i}\right)^{\nu_i} \\ &= d! \sum_{|1^{\nu_1}2^{\nu_2}\cdots t^{\nu_t}|=d} \prod_{j=1}^{d} \frac{(\sigma_j(k))^{\nu_j}}{j^{\nu_j}\ \nu_j!} \end{aligned}$$

(i.e., the coefficient of the homogeneous "monomial" series $\sigma_\lambda(k) = \sigma_{\lambda_1}(k)\ldots\sigma_{\lambda_s}(k)$ equals the number of elements in the symmetric group on $|\lambda|$ elements of the conjugacy class with s cycles of length $\lambda_1, \lambda_2, \ldots, \lambda_s$).

We have then the following result.

Theorem 1.6. *For any natural integers $d \geq 1$ and $k \geq 1$, we have*

$$\sum_n^{\infty} \binom{\lambda_k(n)}{d} z^n = \frac{S_d(k)}{d!} \left(\prod_{j=1}^{\infty} \frac{1}{1-z^j}\right).$$

The first series $S_i = S_i(k)$ are given in terms of $\sigma_j = \sigma_j(k)$ as follows

$S_0 = 1$,
$S_1 = \sigma_1$,
$S_2 = \sigma_1^2 + \sigma_2$,
$S_3 = \sigma_1^3 + 3\sigma_1\sigma_2 + 2\sigma_3$
$S_4 = \sigma_1^4 + 6\sigma_1^2\sigma_2 + 3\sigma_2^2 + 8\sigma_1\sigma_3 + 6\sigma_4$
$S_5 = \sigma_1^5 + 10\sigma_1^3\sigma_2 + 20\sigma_1^2\sigma_3 + 15\sigma_1\sigma_2^2 + 30\sigma_1\sigma_4 + 20\sigma_2\sigma_3 + 24\sigma_5$

Let us remark that the analogous statement of Theorem 1.6 for the generating series $\sum_n \binom{\nu_k}{d} z^n$ boils down to a trivial identity.

The formulas of Theorem 1.6 ease the computations of asymptotics (in n) for $\lambda_k(n)$ and its moments and allow a rederivation of the results contained in **[EL]** and **[VK]**: Indeed, the asymptotics of the coefficients in $\sum_n^\infty \binom{\lambda_k(n)}{d} z^n$ are essentially given by the asymptotics of

$$\frac{\sigma_1^d(k)}{d!}\left(\prod_{j=1}^\infty \frac{1}{1-z^j}\right)$$

which can be worked out.

2. Proofs

Proof of Theorem 1.2. The partition 1^n yields the unique non-zero contribution to $\lambda_n(n)$ and $\gamma_n(n)$ and this contribution equals 1 in both cases. The partition n consisting of a unique part of length n yields the unique non-zero contribution to $\nu_n(n)$ and this contribution equals again 1.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, the *conjugate partition* $\lambda^t = (\lambda_1^t, \ldots, \lambda_{k'}^t)$ of λ is defined by

$$\lambda_j^t = \sharp\{i \mid \lambda_i \geq j\}$$

(this corresponds to a reflection of the Young diagramm of λ through the main diagonal $y = -x$). The difference $\lambda_k - \lambda_{k+1}$ (where non-existing parts are considered as parts of length 0) equals hence the number ν_k^t of parts having length k in the transposed partition $\lambda^t = (1^{\nu_1^t}\ 2^{\nu_2^t} \cdots)$ of λ. Summing over all partitions of n yields then the recursion relation $\lambda_k(n) = \nu_k(n) + \lambda_{k+1}(n)$.

The proof of the equality $\nu_k(n) = \gamma_k(n) + \nu_{k+1}(n)$ uses generating series. Introducing the numbers

$$m_k(\lambda) = \sharp\{i \mid \nu_i \geq k\}\ ,$$
$$m_k(n) = \sum_{|\lambda|=n} m_k(\lambda)$$

one has obviously $\gamma_k(n) = m_k(n) - m_{k+1}(n)$. We have hence to show the equality $m_k(n) = \nu_k(n)$ for $1 \leq k < n$ (the equalities $m_n(n) = \nu_n(n) = 1$ are easy).

We introduce the generating function

$$\psi_k(y, z) = \sum_\lambda y^{m_k(\lambda)} z^{|\lambda|}\ .$$

One has

$$\psi_k(y,z) = \sum_{I\subset\{1,2,3,\dots\},\ \sharp(I)<\infty} \left(\prod_{i\in I}\frac{yz^{ki}}{1-z^i}\right)\left(\prod_{1\leq j\notin I}\frac{1-z^{kj}}{1-z^j}\right)$$
$$= \prod_{j=1}^{\infty}\left(\frac{yz^{kj}}{1-z^j}+\frac{1-z^{kj}}{1-z^j}\right) = \prod_{j=1}^{\infty}\left(\frac{1}{1-z^j}-(1-y)\frac{z^{kj}}{1-z^j}\right) .$$

A small computation yields

$$\frac{\partial\psi_k(y,z)}{\partial y} = \psi_k(y,z)\sum_{j=1}^{\infty}\frac{z^{kj}}{1-(1-y)z^{kj}}$$

and we have hence

$$\sum_{n=0}^{\infty} m_k(n)\ z^n = \frac{\partial\psi_k}{\partial y}(1,z) = \left(\prod_{j=1}^{\infty}\frac{1}{1-z^j}\right)\ \frac{z^k}{1-z^k} = \sum_{n=0}^{\infty}\nu_k(n)\ z^n$$

(cf. Theorem 1.1 for the last equality) which finishes the proof. QED

C. Bessenrodt's bijective Proof of Theorem 1.2. We establish first the equality

$$\nu_k(n) = \sum_{i=k}^{n}\gamma_i(n)$$

which corresponds to the second statement of Theorem 1.2.

Define the sets

$$P_k(n) = \{(i,\lambda)\mid |\lambda|=n,\ 1\leq i\leq \nu_k(\lambda)\}$$

and

$$Q_k(n) = \{(j,\lambda)\mid |\lambda|=n,\ \nu_j(\lambda)\geq k\}$$

(recall that $\nu_i(\lambda)$ counts the number of parts with length i in the partition λ). The cardinality $\sharp(P_k(n))$ equals obviously $\nu_k(n)$ and one has similarly $\sharp(Q_k(n)) = \sum_{i=k}^{n}\gamma_i(n)$ (recall that $\gamma_i(\lambda)$ counts the number of distinct partlengths whose parts appear with multiplicity i in λ). An element $(i,\lambda)\in P_k(n)$ stems from a partition λ having at least i parts equal to k. Erasing i such parts and adding k parts of length i yields a partition μ of n such that μ contains at least k parts of length i. This shows that $(i,\mu)\in Q_k(n)$ and the application $(i,\lambda)\longmapsto(i,\mu)$ is clearly a bijection.

The first statement of Theorem 1.2 can be proven similarly by considering the sets

$$R_k = \{(i,\lambda)\mid |\lambda|=n,\ 1\leq i\leq \lambda_k(\lambda)\}$$

and

$$S_k(n) = \{(j,\lambda)\mid |\lambda|=n,\ j\leq \sum_{i=k}^{n}\nu_i(\lambda)\}$$

and the bijection $R_k(n)\longrightarrow S_k(n)$ defined by $(i,\lambda)\longmapsto(i,\lambda^t)$ (this is more or less the proof given above). QED

Corollary 1.3 results immediately from Theorem 1.2 and from the last equality in Theorem 1.1.

Proof of Proposition 1.4. A partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k = i \geq \lambda_{k+1} \geq \dots)$$

with $\lambda_k = i$ of $n = \sum_{j=1} \lambda_j$ can be written as

$$\lambda = (i+(\lambda_1 - i) \geq i+(\lambda_2 - i) \geq \cdots \geq i+(\lambda_{k-1} - i) \geq i \geq \lambda_{k+1} \geq \dots) .$$

Such partitions are hence in bijection with pairs of partitions

$$\begin{aligned}&\alpha = (\alpha_1 = (\lambda_1 - i) \geq \alpha_2 = (\lambda_2 - i) \geq \cdots \geq \alpha_{k-1} = (\lambda_{k-1} - i) \geq 0) ,\\ &\omega = (\omega_1 = \lambda_{k+1} \geq \omega_2 = \lambda_{k+2} \geq \dots)\end{aligned}$$

with α having at most $k-1$ non-zero parts and ω having all parts $\leq i$. The conjugate partition α^t of α has hence only parts $\leq k-1$. Such a pair α^t, ω of partitions yields hence a unique partition with $\lambda_k = i$ of the integer $n = ki + \sum_{j=1}^{k-1} \alpha^t_j + \sum_{j=k+1} \omega_j$ and contributes hence with $\binom{i}{d}$ to the k-th coordinate $\binom{\lambda_k(n)}{d}$ of $\binom{\lambda(n)}{d}$. Summing up over $i \in \mathbf{N}$ yields easily the generating series for $\binom{\lambda_k(n)}{d}$.

Considering the generating series for $\binom{\nu_k(n)}{d}$ one has

$$\sum_n \binom{\nu_k(n)}{d} z^n = \left(\sum_j \binom{j}{d} z^{jk}\right) \prod_{i \neq k} \frac{1}{1-z^i}$$

and the (easy) equality

$$\sum_j \binom{j}{d} Z^j = \frac{1}{Z} \left(\frac{Z}{1-Z}\right)^{d+1}$$

implies the result. QED

Proof of Theorem 1.6. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ the definition

$$\lambda^t_k = \sharp\{i \mid \lambda_i \geq k\}$$

for the k-th part of its transposed partition $\lambda^t = (\lambda^t_1, \lambda^t_2, \dots)$ shows the equalities

$$\sum_\lambda y^{\lambda_k}\ z^{|\lambda|} = \sum_\lambda y^{(\lambda^t)_k}\ z^{|\lambda|} = \left(\prod_{i=1}^{k-1} \frac{1}{1-z^i}\right) \left(\prod_{j=k}^{\infty} \frac{1}{1-yz^j}\right) .$$

Denote this series by $\varphi_k(y,z)$. An easy computation yields

$$\varphi_k(y,z) = \left(\prod_{i=1}^{\infty} \frac{1}{1-z^i}\right) \prod_{j \geq k} \left(1 - (y-1)\frac{z^j}{1-z^j}\right)^{-1} .$$

Applying the identity

$$\prod_i (1-x_i)^{-1} = \exp\left(\sum_{l=1}^{\infty} \frac{\sum_i x_i^l}{l}\right)$$

of formal power series to the last factor we get

$$\prod_{j\geq k}\left(1-(y-1)\frac{z^j}{1-z^j}\right)^{-1}=\exp\left(\sum_{l=1}^{\infty}\frac{(y-1)^l}{l}\sigma_l(k)\right)$$
$$=\sum_{n=0}^{\infty}\frac{(y-1)^n}{n!}\sum_{|(1^{\nu_1}\cdot 2^{\nu_2}\cdots)|=n}\frac{n!}{(\sum_i \nu_i)!}\binom{\sum_i \nu_i}{\nu_1\ \nu_2\ \ldots}\prod_i\left(\frac{\sigma_i(k)}{i}\right)^{\nu_i}$$
$$=\sum_{n=0}^{\infty}\frac{(y-1)^n}{n!}\ S_n(k)\ .$$

We have thus

$$d!\sum_{\lambda}\binom{\lambda_k}{d}z^{|\lambda|}=\frac{\partial^d\varphi_k}{\partial y^d}(1,z)=\left(\prod_{i=1}^{\infty}\frac{1}{1-z^i}\right)S_d(k)$$

which finishes the proof by comparison with Proposition 1.4. QED

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