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PATTERN AVOIDANCE IN COLOURED PERMUTATIONS

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Abstract. Let S_n be the symmetric group, C_r the cyclic group of order r, and let $S_n^{(r)}$ be the wreath product of S_n and C_r; which is the set of all coloured permutations on the symbols $1, 2, \ldots, n$ with colours $1, 2, \ldots, r$, which is the analogous of the symmetric group when $r = 1$, and the hyperoctahedral group when $r = 2$. We prove, for every 2-letter coloured pattern $\phi \in S_2^{(r)}$, that the number of ϕ-avoiding coloured permutations in $S_n^{(r)}$ is given by the formula $\sum_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$. Also we prove that the number of Wilf classes of restricted coloured permutations by two patterns with r colours in $S_2^{(r)}$ is one for $r = 1$, is four for $r = 2$, and is six for $r \geq 3$.

1. Introduction

The goal of this note is to give analogies of enumerative results on certain classes of permutations characterized by pattern-avoidance in the symmetric group, and in the hyperoctahedral group. In $S_n^{(r)}$, the natural analogue of the symmetric group and of the hyperoctahedral group, we identify classes of restricted coloured permutations with enumerative properties analogous to results in the symmetric group and hyperoctahedral group. In the remainder of this section we present a brief account of earlier work which motivated out investigation, summarize the main results, and present the basic definitions used throughout the note.

Pattern avoidance in the symmetric group proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [K, T, W] to the theory of Kazhdan-Lusztig polynomials [Br], singularities of Schubert varieties [LS, Bi], Chebyshev polynomials [CW, MV1, Kr, MV2, MV3], and rook polynomials [MV4]. Signed pattern avoidance in the hyperoctahedral group proved to be a useful language in combinatorial statistics defined in type-B noncrossing partitions, enumerative combinatorics [S, BS], algebraic combinatorics [FK, BK, Be, M, R].

Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations. An *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\pi_{i_1}, \ldots, \pi_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ*-avoiding*, if there is no occurrence of τ in π. The set of all τ-avoiding permutations in S_n is denoted by $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids any $\tau \in T$; the corresponding subset of S_n is denoted by $S_n(T)$. The first case examined was the case of permutations avoiding one pattern of length 3. Knuth [K] found that $|S_n(\tau)| = C_n$ for all $\tau \in S_3$, where C_n is the nth Catalan number. Later, Simion and Schmidt [SS] found the cardinalities of $|S_n(T)|$ for all $T \subset S_3$.

The hyperoctahedral group B_n is an analog of the symmetric group S_n. Let us view the elements of B_n as signed permutation $b = b_1 b_2 \dots b_n$ in which each of the symbols $1, 2, \dots, n$ appears once, possibly barred. Thus, the cardinality of B_n is $n!2^n$. Simion [S] was looking for the analogs of Knuth's results for B_n; she discovered that for every 2-letter signed pattern τ; the number of τ-avoiding signed permutations in B_n is $\sum_{j=0}^{n} \binom{n}{j}^2 j!$. Also Simion [S] found the number of all coloured permutations in B_n avoiding double 2-letter signed patterns in B_2. This invites us to define a further generalizations for avoiding a pattern in the symmetric group S_n and avoiding a signed pattern in the hyperoctahedral group B_n.

The group $S_n^{(r)} = S_n \wr C_r$ where C_r is the cyclic group of order r, is an analog of the symmetric group (S_n) and of the hyperoctahedral group (B_n). We will view the elements of the set $S_n^{(r)}$ as coloured permutations $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ in which each of the symbols $1, 2, \dots, n$ appears once, coloured by one of the colours $1, 2, \dots, r$ (more generally, we denote by $S_{\{a_1,\dots,a_n\}}^{\{s_1,\dots,s_r\}}$ the set of all permutations of the symbols $a_1, \dots, a_n$ where each symbol appears once and is coloured by one of the colours $s_1, \dots, s_r$). Thus, $S_n^{(1)} = S_n$, $S_n^{(2)} = B_n$, and the cardinality of $S_n^{(r)}$ is $n!r^n$. The absolute value notation means $|\phi|$ is the permutation $(|\phi_1|, \dots, |\phi_n|)$ where $|\phi_j|$ is the symbol which appear in ϕ at the position j. An example $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$ is a coloured permutation in $S_3^{(2)}$, and $|\phi| = (1, 3, 2)$.

Let $\phi = (\tau_1^{(s_1)}, \dots, \tau_k^{(s_k)}) \in S_k^{(r)}$, and $\psi = (\alpha_1^{(v_1)}, \dots, \alpha_n^{(v_n)}) \in S_n^{(r)}$; we say that ψ *contains* ϕ (or is ϕ-containing) if there is a sequence of k indices, $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that the following two conditions hold:

(i) $(\alpha_{i_1}, \dots, \alpha_{i_k})$ is order-isomorphic to $|\phi|$;
(ii) $v_{i_j} = s_j$ for all $j = 1, 2, \dots, k$.

Otherwise, we say that ψ *avoids* ϕ (or is ϕ-avoiding). The set of all ϕ-avoiding coloured permutations in $S_n^{(r)}$ is denoted by $S_n^{(r)}(\phi)$, and in this context ϕ is called a *coloured pattern*. For an arbitrary finite collection of coloured patterns T, we say that ψ avoids T if ψ avoids any $\phi \in T$; the corresponding subset of $S_n^{(r)}$ is denoted by $S_n^{(r)}(T)$. As an example, $\psi = (3^{(1)}, 2^{(2)}, 1^{(2)}) \in S_3^{(2)}$ avoids $(2^{(1)}, 1^{(1)})$; that is, $\psi \in S_3^{(2)}((2^{(1)}, 1^{(1)}))$.

Let T_1, T_2 be two subsets of coloured patterns; we say that $T_1, T_2 \subset S_k^{(r)}$ are in the same *Wilf class* if $|S_n^{(r)}(T_1)| = |S_n^{(r)}(T_2)|$ for $n \ge 0$ (see [W]).

In the symmetric group S_n, for every 2-letter pattern τ the number of τ-avoiding permutations is one, and for every pattern $\tau \in S_3$ the number of τ-avoiding permutations is given by the Catalan number [K]. Also Simion [S] proved there are similar results for the hyperoctahedral group B_n. Here we are looking for similar results for $S_n^{(r)}$. We show that for every 2-letter coloured pattern ϕ the number of ϕ-avoiding coloured permutations in $S_n^{(r)}$ is given by $\sum_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$, which generalizes the results of [S, Sec. 3].

The paper is organized as follows: the elementary definitions, and the symmetric operations, are treated in **Section** 2. In **Section** 3 we give two relations between avoidance of patterns in S_k and avoidance of coloured patterns in $S_k^{(r)}$. In **Section** 4 we present two sets of coloured patterns, and produce a bijection which gives a combinatorial geometric explanation for one of these results. Finally, in **Sections** 5 **and** 6 respectively, we prove the first and second part of the Main Theorem.

Main Theorem:
(i) For every 2-letter coloured pattern ϕ, the number of ϕ-avoiding coloured permutations in $S_n^{(r)}$ is given by the expression: $\sum_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$.
(ii) A double restrictions by 2-letter coloured patterns gives one Wilf class for $r = 1$, four Wilf classes for $r = 2$, and six Wilf classes for $r \geq 3$.

2. **Symmetries on coloured permutations**

As on the symmetric group S_n there are three natural symmetric operations: the reversal, the complement, and the inversion (see [SS]). On $S_n^{(r)}$ we define:

(i) the *reversal* $br : S_n^{(r)} \to S_n^{(r)}$ by

$$br : (\alpha_1^{(s_1)}, \ldots, \alpha_n^{(s_n)}) \mapsto (\alpha_n^{(s_n)}, \ldots, \alpha_1^{(s_1)});$$

(ii) the *complement* $bc : S_n^{(r)} \to S_n^{(r)}$ by

$$bc : (\alpha_1^{(s_1)}, \ldots, \alpha_n^{(s_n)}) \mapsto ((n+1-\alpha_1)^{(s_1)}, \ldots, (n+1-\alpha_n)^{(s_n)});$$

(iii) the *colour-complement* $cc : S_n^{(r)} \to S_n^{(r)}$ by

$$cc : (\alpha_1^{(s_1)}, \ldots, \alpha_n^{(s_n)}) \mapsto (\alpha_1^{(r+1-s_1)}, \ldots, \alpha_n^{(r+1-s_n)}).$$

Example 2.1. *Let* $\psi = (1^{(1)}, 3^{(2)}, 2^{(1)}) \in S_3^{(2)}$*, then* $br(\psi) = (2^{(1)}, 3^{(2)}, 1^{(1)})$*,* $bc(\psi) = (3^{(1)}, 1^{(2)}, 2^{(1)})$*, and* $cc(\psi) = (1^{(2)}, 3^{(1)}, 2^{(2)})$.

Proposition 2.2. *The group* $< br, bc, cc >$ *is isomorphic to* D_8.

Remark 2.3. *More generally, we extend these symmetric operations to* $T \subseteq S_n^{(r)}$ *by* $g(T) = \{g(\psi) | \psi \in T\}$*, where* $g = br$*,* bc*, or* cc*. Therefore for any* $T \subseteq S_n^{(r)}$*,* $n \geq 0$

$$|S_n^{(r)}(T)| = |S_n^{(r)}(br(T))| = |S_n^{(r)}(bc(T))| = |S_n^{(r)}(cc(T))|.$$

Also there are other symmetric operations. The first is the *inverse* $< \cdot >^{-1}$: $S_n^{(r)} \to S_n^{(r)}$ defined by

$$< \cdot >^{-1}: (\alpha_1^{(s_1)}, \ldots, \alpha_n^{(s_n)}) \mapsto (\beta_1^{(s_n)}, \ldots, \beta_n^{(s_n)});$$

where $\beta = \alpha^{-1}$ in S_n (see [W]). The second is *colour-permutation* $cp_\delta : S_n^{(r)} \to S_n^{(r)}$ where $\delta \in S_r$, defined by

$$cp_\delta(\alpha_1^{(s_1)}, \ldots, \alpha_n^{(s_n)}) = (\alpha_1^{(\delta_{s_1})}, \ldots, \alpha_n^{(\delta_{s_n})}).$$

More generally, for any $T \subset S_n^{(r)}$ we define $cp_\delta(T) = \{sp_\delta(\psi) | \psi \in T\}$.

Remark 2.4. *Let $T \subset S_k^{(r)}$, $\delta \in S_r$, then $|S_n^{(r)}(T)| = |S_n^{(r)}(cp_\delta(T))|$. An example, for $r \geq 3$, $|S_n^{(r)}((1^{(1)}, 2^{(2)}), (1^{(2)}, 2^{(3)}))| = |S_n^{(r)}((1^{(2)}, 2^{(1)}), (1^{(1)}, 2^{(3)}))|$, by the symmetric operation $cp_{(2,1,3,4,\ldots,r)}$.*

3. Avoidance patterns and coloured patterns

We say a coloured permutation $\phi \in S_k^{(r)}$ is *homogeneous* if $\phi_i = \alpha_i^{(u)}$ for all $i = 1, 2, \ldots, k$ where $1 \leq u \leq r$; in this case we denote ϕ by $[\alpha]_{(u)}$. More generally, we write $T_{(u)} = \{[\alpha]_{(u)} | \alpha \in T\}$, where $T \subset S_k$.

Theorem 3.1. *Let $1 \leq u \leq r$, $T \subseteq S_k$. For all $n \geq 0$*

$$|S_n^{(r)}(T_{(u)})| = \sum_{j=0}^{n} j!(r-1)^j |S_{n-j}(T)| \binom{n}{j}^2.$$

Proof. Since all the patterns in $T_{(u)}$ are homogeneous with the same colour, then we can choose a coloured permutation in $S_n^{(r)}(T_{(u)})$ by choosing $n-j$ symbols with colour u, and $n-j$ positions where $0 \leq j \leq n$, and in the other positions we put any coloured permutation with the other symbols and without colour u. Hence

$$|S_n^{(r)}(T_{(u)})| = \sum_{j=0}^{n} \binom{n}{j}^2 |S_{\{1,2,\ldots,j\}}^{\{u\}}(T_{(u)})| |S_{\{j+1,\ldots,n\}}^{\{1,\ldots,u-1,u+1,\ldots,r\}}|.$$

□

Example 3.2. (see [S, Eq. 46]) *Let $a = 1, 2$. For $r = 2$, by Theorem 3.1 we get*

$$|S_n^{(2)}((1^{(a)}, 2^{(a)}), (2^{(a)}, 1^{(a)}))| = (n+1)!,$$

$$|S_n^{(2)}((1^{(a)}, 2^{(a)}))| = |S_n^{(2)}((2^{(a)}, 1^{(a)}))| = \sum_{j=0}^{n} j! \binom{n}{j}^2.$$

Remark 3.3. *Let $r \geq 1$, $\tau \in S_k$. For all $n \geq 0$, $|S_n^{(r)}(F_\tau)| = r^n |S_n(\tau)|$, where $F_\tau = \{(\tau_1^{(v_1)}, \ldots, \tau_k^{(v_k)}) | 1 \leq v_1, v_2, \cdots, v_k \leq r\}$. As an example, $|S_n^{(r)}(T)| = r^n$ for all $n \geq 0$, where $T = \{(1^{(a)}, 2^{(b)}) | a, b = 1, 2, \ldots, r\}$.*

4. Restricted sets

In this section, we calculate cardinalities of $S_n^{(r)}(T)$ for two special subsets $T \subset S_2^{(r)}$. The first special subset is defined by $T_{b;a_1,a_2,\ldots,a_l} = \{(1^{(b)}, 2^{(a_j)}) | j = 1, 2, \ldots, l\}$.

Theorem 4.1. *Let $1 \leq l \leq r$, and $1 \leq b \leq a_1 < a_2 < \cdots < a_l \leq r$. Then*

$$\sum_{n \geq 0} \frac{|S_n^{(r)}(T_{b;a_1,a_2,\ldots,a_l})|}{n!} x^n = \left(\frac{1-(r-l)x}{(1-(r-1)x)^l} \right)^{\frac{1}{l-1}};$$

when $l = 1$ we take the limit of the right hand side which equals $\frac{e^{\frac{x}{1-(r-1)x}}}{1-(r-1)x}$.

Proof. Let $\phi \in S_n^{(r)}(T_{b;a_1,\ldots,a_l})$, $p_r(n) = |S_n^{(r)}(T_{b;a_1,\ldots,a_l})|$, and let us consider the possible values of ϕ_1:

(1) Let $\phi_1 = i^{(c)}$, $c \neq b$, and $1 \leq i \leq n$; so $\phi \in S_n^{(r)}(T_{b;a_1,\dots,a_l})$ if and only if $(\phi_2,\dots,\phi_n)$ is $T_{b;a_1,\dots,a_l}$ -avoiding, hence in this case there are $(r-1)np_r(n-1)$ coloured permutations.

(2) Let $\phi_1 = i^{(b)}$; since ϕ is $T_{b;a_1,\dots,a_l}$-avoiding, the symbols $i+1,\dots,n$ must appear without colours $a_1,\dots,a_l$. Also the symbols $1,\dots,i-1$ are $T_{b;a_1,\dots,a_l}$-avoiding, and can be placed anywhere at positions $2,\dots,n$, hence there are

$$\sum_{i=1}^{n} \binom{n-1}{i-1}(n-i)!(r-l)^{n-i}p_r(i-1)$$

coloured permutations in this case.

By the above two cases we obtain a recurrence relation satisfied by $p_r(n)$

$$p_r(n) = (r-1)np_r(n-1) + \sum_{i=1}^{n} \binom{n-1}{i-1}(n-i)!(r-l)^{n-i}p_r(i-1),$$

for $n \geq 1$, and $p_r(0) = 1$. Let $q_n = p_r(n)/n!$. By multiplying the recurrence by $x^{n-1}/(n-1)!$, and summing up over all $n \geq 1$, we obtain

$$\frac{d}{dx}q(x) = (r-1)\frac{d}{dx}(xq(x)) + \frac{q(x)}{1-(r-l)x},$$

where $q(x) = \sum_{n\geq 0} q_n x^n$. Since $q(0) = 1$, the theorem holds. □

Corollary 4.2. *For all $n \geq 0$, $|S_n^{(r)}(T_{1;1,2,\dots,r})| = \prod_{j=0}^{n}(j(r-1)+1)$.*

Proof. Immediately, Case 1 of the proof of Theorem 4.1 gives $(r-1)np_r(n-1)$ coloured permutations, and Case 2 of the proof of Theorem 4.1 (because $i = n$) gives $p_r(n-1)$ coloured permutations. Then for $n \geq 2$,

$$p_r(n) = |S_n^{(r)}(T_{1;1,2,\dots,r})| = ((r-1)n+1)p_r(n-1).$$

Since $p_r(0) = 1$, the corollary holds. □

Example 4.3. (see [S, Eq. 47]) *By Corollary 4.2,*

$$|S_n^{(2)}(T_{1;1,2})| = |S_n^{(2)}((1^{(1)},2^{(1)}),(1^{(1)},2^{(2)}))| = (n+1)!.$$

Now we present the second special subset. Consider a subset $T \subset S_k^{(r)}$; we say that T is *good* if it is the union of disjoint homogeneous subsets; that is, $T = \bigcup_{j=1}^{p}(T_j)_{(u_j)}$. As an example, $T = \{(1^{(1)},2^{(1)},3^{(1)}),(1^{(1)},3^{(1)},2^{(1)}),(2^{(2)},1^{(2)},3^{(2)})\}$ is a good set.

Theorem 4.4. *Let $T = \bigcup_{j=1}^{p}(T_j)_{(u_j)}$ be a good set. Then $|S_n^{(r)}(T)|$ for $n \geq 0$ is given by*

$$\sum_{j_1=0}^{n}\sum_{j_2=0}^{n-j_1}\cdots\sum_{j_p=0}^{n-j_1\cdots-j_{p-1}}(r-p)^{n-j_1\cdots-j_p}\binom{n}{j_1,\dots,j_p,n-j_1\cdots-j_p}^2 \cdot(n-j_1\cdots-j_p)!\prod_{i=1}^{p}|S_{j_i}(T_i)|.$$

Proof. The theorem holds for $p=1$ by Theorem 3.1. Now let $p>1$, so by definitions
$$|S_n^{(r)}(T)| = \sum_{j_1=0}^{n} |S_{n-j_1}^{\{1,\dots,u_1-1,u_1+1,\dots,r\}}(T\backslash(T_1)_{(u_1)})||S_j(T_1)|\binom{n}{j_1}^2,$$
therefore,
$$|S_n^{(r)}(T)| = \sum_{j_1=0}^{n} |S_{n-j_1}^{(r-1)}(T\backslash(T_1)_{(u_1)})||S_{j_1}(T_1)|\binom{n}{j_1}^2.$$
Hence, by the inductive assumption, and
$$|S_{n-j_1-\cdots-j_p}^{(r-p)}| = (n-j_1-\cdots-j_p)!(r-p)^{n-j_1-\cdots-j_p},$$
the theorem holds. $\square$

Let $T_{l;a_1,\dots,a_d;a_{d+1},\dots,a_l}$ be a subset of $S_2^{(r)}$ defined by
$$T_{l;a_1,\dots,a_d;a_{d+1},\dots,a_l} = \bigcup_{i=1}^{d}\{(1^{(a_i)},2^{(a_i)})\} \cup \bigcup_{i=d+1}^{l}\{(2^{(a_i)},1^{(a_i)})\},$$
hence by Theorem 4.4 we obtain the following corollary:

Corollary 4.5. *Let $a_1,\dots,a_l$ be l different numbers integers between 1 and r. Then $|S_n^{(r)}(T_{l;a_1,\dots,a_d,a_{d+1},\dots,a_l})|$ for $n\geq 0$ is given by*
$$\sum_{j_1=0}^{n}\sum_{j_2=0}^{n-j_1}\cdots\sum_{j_l=0}^{n-j_1\cdots-j_{l-1}} (r-l)^{n-j_1\cdots-j_l}\binom{n}{j_1,\dots,j_l,n-j_1\cdots-j_l}^2(n-j_1\cdots-j_l)!.$$

Now we build a bijection, which gives for the set $S_n^{(r)}(T_{l;a_1,\dots,a_d;a_{d+1}\dots,a_l})$ a combinatorial geometric explanation. Consider l lines $L_1,\dots,L_l$ such that L_i contains all the points of the form $j^{(i)}$ for all $j=1,2,\dots,n$. We say L_i is *good* if the points $1^{(i)}$ to $n^{(i)}$ are decreasing, and the line L_i is *bad* if the points $1^{(i)},\dots,n^{(i)}$ are increasing, otherwise we say the line L_i is *free*.

Now we consider the following collection which represents the set $T_{l;a_1,\dots,a_d;a_{d+1},\dots,a_l}$. Let $L_{a_1},\dots,L_{a_d}$ be good lines, $L_{a_{d+1}},\dots,L_{a_l}$ be bad lines, and L_i be a free line for all $1\leq i\leq r$ such that $i\notin\{a_1,\dots,a_l\}$. For example, the representation of $T_{2;3;2}$ where $r=4$, is given by the following diagram.

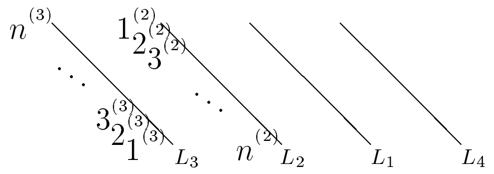

Figure 1: Representation of $T_{2;3;2}$

Here the lines L_1 and L_4 are free lines.

Now let us define a *path* between the points on the lines of the representation of $T_{l;a_1,\dots,a_d;a_{d+1},\dots,a_l}$. A path is a collection of steps, starting anywhere, where every step is one of the following steps (such that no two points in the collection have the same symbols):

(i) a decreasing step from a point to another point on a bad, or a good line,
(ii) a free step on the free line, or between the lines (from a point to another point).

Proposition 4.6. *The set of paths of n steps is exactly the set of $T_{l;a_1,\dots,a_d;a_{d+1},\dots,a_l}$-avoiding coloured permutation in $S_n^{(r)}$.*

Proof. By definitions we see that every path of n steps is a $T_{l;a_1,\dots,a_d;a_{d+1},\dots,a_l}$-avoiding coloured permutation in $S_n^{(r)}$. On the other hand, if ϕ is $T_{l;a_1,\dots,a_d;a_{d+1},\dots,a_l}$-avoiding coloured permutation in $S_n^{(r)}$, then all the symbols in ϕ coloured by a_i are decreasing for $1 \leq i \leq d$ and increasing for $d+1 \leq i \leq l$, and the other symbols appear in any order coloured by any colour $u \neq a_i$ for all $1 \leq i \leq l$. Therefore, by reading ϕ from the left to the right, we obtain a path of n steps. Hence the proposition holds. $\square$

By using the above proposition we obtain a combinatorial proof of Corollary 4.5. This corollary produces a generalization of certain results in [S], in particular we get the following (because $i_1 + \cdots + i_r = n$).

Corollary 4.7. *Let $0 \leq d \leq r$; for $n \geq 0$,*

$$|S_n^{(r)}(T_{r;1,\dots,d;d+1,\dots,r})| = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \cdots \sum_{i_{r-1}=0}^{n-i_1-\cdots-i_{r-2}} \binom{n}{i_1,\dots,i_{r-1},n-i_1-\cdots-i_{r-1}}^2.$$

Example 4.8. (see [S, Eq. 49]) *Let $r=2$; Corollary 4.7 yields*

$$|S_n^{(2)}(T_{2;\emptyset;1,2})| = |S_n^{(2)}(T_{2;1;2})| = |S_n^{(2)}(T_{2;1,2;})| = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

5. Single restriction by a 2-letter coloured pattern

The length 2 coloured permutations give rise to some enumeratively interesting classes of coloured permutations, which we examine in this section. In the symmetric group S_n, patterns of length 2 are uninterestingly restrictive, and length 3 is the first interesting case. Also in $S_n^{(r)}$, restriction by patterns of length 1 is trivial, and given by the following formula $|S_n^{(r)}(1^a)| = n! \cdot (r-1)^n$, where $1 \leq a \leq r$. Let us write

$$d_r(n) = \sum_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2,$$

and let $d_r(x)$ be the generating function of the sequence $d_r(n)/n!$. From Theorems 4.1 and 4.4 it is easy to see that $d_r(x) = \frac{e^{\frac{x}{1-(r-1)x}}}{1-(r-1)x}$.

Now we prove the first case of the Main Theorem, that is, that there exists exactly one Wilf class of a single restriction by a 2-letter coloured pattern, for all $r \geq 1$.

Theorem 5.1. *Let $r \geq 1$, and $1 \leq a,b,c,d \leq r$. For $n \geq 0$*

$$|S_n^{(r)}((1^{(a)},2^{(b)}))| = |S_n^{(r)}((2^{(c)},1^{(d)}))| = d_r(n).$$

Proof. By Section 2 (symmetric operations) we have to verify two cases:

(1) Let $1 \leq a \leq r$; for $n \geq 0$, $|S_n^{(r)}((1^{(a)},2^{(a)}))| = d_r(n)$. But this follows from Theorem 3.1 and because $|S_m(12)| = 1$ for $m \geq 0$;

Case	ϕ	ϕ'	$\lvert S_n^{(5)}(\phi,\phi')\rvert$ for $n=0,1,2,3,4,5$	Reference
1	$(1^{(1)},2^{(1)})$	$(2^{(1)},1^{(1)})$	$1,5,48,672,12288,276480$	Theorem 6.1
2	$(1^{(1)},2^{(1)})$	$(1^{(1)},2^{(2)})$	$1,5,48,672,12288,276480$	Theorem 6.1
3	$(1^{(1)},2^{(2)})$	$(2^{(1)},1^{(2)})$	$1,5,48,672,12288,276480$	Theorem 6.1
4	$(1^{(1)},2^{(2)})$	$(2^{(2)},1^{(1)})$	$1,5,48,672,12288,276480$	Theorem 6.1
5	$(1^{(1)},2^{(2)})$	$(1^{(1)},2^{(3)})$	$1,5,48,672,12288,276480$	Theorem 6.1
6	$(1^{(1)},2^{(1)})$	$(1^{(2)},2^{(2)})$	$1,5,48,668,12046,265062$	Theorem 6.3
7	$(1^{(1)},2^{(1)})$	$(2^{(2)},1^{(2)})$	$1,5,48,668,12046,265062$	Theorem 6.3
8	$(1^{(1)},2^{(1)})$	$(1^{(2)},2^{(3)})$	$1,5,48,668,12046,265062$	Theorem 6.3
9	$(1^{(1)},2^{(1)})$	$(2^{(2)},1^{(3)})$	$1,5,48,668,12046,265062$	Theorem 6.3
10	$(1^{(1)},2^{(2)})$	$(1^{(3)},2^{(4)})$	$1,5,48,668,12046,265062$	Theorem 6.3
11	$(1^{(1)},2^{(2)})$	$(2^{(3)},1^{(4)})$	$1,5,48,668,12046,265062$	Theorem 6.3
12	$(1^{(1)},2^{(2)})$	$(2^{(1)},1^{(3)})$	$1,5,48,670,12168,270856$	Theorem 6.5
13	$(1^{(1)},2^{(2)})$	$(2^{(2)},1^{(3)})$	$1,5,48,670,12168,270856$	Theorem 6.5
14	$(1^{(1)},2^{(1)})$	$(2^{(1)},1^{(2)})$	$1,5,48,671,12288,273665$	Theorem 6.6
15	$(1^{(1)},2^{(2)})$	$(1^{(2)},2^{(3)})$	$1,5,48,669,12106,267867$	
16	$(1^{(1)},2^{(2)})$	$(1^{(2)},2^{(1)})$	$1,5,48,670,12166,270672$	

Table 1. Pairs of 2-letter coloured patterns

(2) Let $b\le a$; for $n\ge 0$, $|S_n^{(r)}((1^{(a)},2^{(b)}))| = |S_n^{(r)}((1^{(a)},2^{(a)}))|$. But this follows from Theorem 4.1 by $|S_n^{(r)}(T_{a;b})| = |S_n^{(r)}(T_{a;a})|$.

□

6. Double restrictions by 2-letter coloured patterns

In this section, we find the number of Wilf classes for $r\ge 1$, of double restrictions by 2-letter coloured patterns. In $S_2^{(r)}$ there are $r^2(r^2-1)$ possibilities to choose two elements of the following forms: $(1^{(a)},2^{(b)}),(1^{(c)},2^{(d)})$, and there are r^4 possibilities to choose two elements of the following forms: $(1^{(a)},2^{(b)}),(2^{(c)},1^{(d)})$, where $1\le a,b,c,d\le r$. On the other hand, by symmetric operations (Section 2), the question of determining the $S_n^{(r)}(\phi,\phi')$ for $r^2(2r^2-1)$ choices for 2-letter coloured patterns ϕ, ϕ' reduces to the determination of the $S_n^{(r)}(\phi,\phi')$ where ϕ, ϕ' are from Table 1.

Theorem 6.1. *For $n\ge 0$, $|S_n^{(r)}(T)| = n!(n+r-1)(r-1)^{n-1}$ where*
(i) $T=\{(1^{(1)},2^{(1)}),(2^{(1)},1^{(1)})\}$ *for* $r\ge 1$*;*
(ii) $T=\{(1^{(1)},2^{(1)}),(1^{(1)},2^{(2)})\}$ *for* $r\ge 2$*;*
(iii) $T=\{(1^{(1)},2^{(2)}),(2^{(1)},1^{(2)})\}$ *for* $r\ge 2$*;*
(iv) $T=\{(1^{(1)},2^{(2)}),(2^{(2)},1^{(1)})\}$ *for* $r\ge 2$*;*
(v) $T=\{(1^{(1)},2^{(2)}),(1^{(1)},2^{(3)})\}$ *for* $r\ge 3$.

Proof. By Theorem 3.1 and because $|S_m(12,21)| = 1,1,0$ where $m=0$, $m=1$, $m\ge 2$ respectively, (i) holds, and Theorem 4.1 immediately yields (ii), and (v) respectively for $T_{1;1,2}$ and $T_{1;2,3}$. Now let us prove (iii) and (iv).

Case (iii)**:** Let $p_n = |S_n^{(r)}(T)|$, $\phi \in S_n^{(r)}(T)$, and let us consider the possible values of ϕ_1:

(1) Let $\phi_1 = i^{(c)}$, $c \neq 1$; $\phi \in S_n^{(r)}(T)$ if and only if $(\phi_2, \dots, \phi_n) \in S^{(r)}_{\{1,\dots,i-1,i+1,\dots,n\}}(T)$. Hence in this case there are $(r-1)np_{n-1}$ coloured permutations.

(2) Let $\phi_1 = i^{(1)}$; since ϕ is T-avoiding, the symbols $1, \dots, i-1, i+1, \dots, n$ are not coloured by 2, and can be replaced anywhere at positions $2, \dots, n$ for all $1 \le i \le m$. Hence, in this case there are $(n-1)!(r-1)^{n-1}$ coloured permutations.

Therefore by the above two cases p_n satisfies the following relation:

$$p_n = n(r-1)p_{n-1} + n!(r-1)^{n-1}.$$

Since $p_0 = 1$, (iii) holds.

Case (iv)**:** Let $p_n = |S_n^{(r)}(T)|$, $\phi \in S_n^{(r)}(T)$ such that $\phi_j = n^{(c)}$, and let us consider the possible values of j, c:

(1) Let $c \neq 2$; $\phi \in S_n^{(r)}(T)$ if and only if $(\phi_1, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_n) \in S^{(r)}_{n-1}(T)$. Hence in this case there are $(r-1)np_{n-1}$ coloured permutations.

(2) Let $c = 2$; $\phi \in S_n^{(r)}(T)$ if and only if $(\phi_1, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_n)$ is a coloured permutation with symbols $1, 2, \dots, n-1$ and colours $2, \dots, r$ for all $1 \le j \le n$. Hence, in this case there are $(n-1)!(r-1)^{n-1}$ coloured permutations.

So we obtain the same relations as for (iii), hence (iv) holds. □

Example 6.2. (see [S, Eq. 46, 47]) *As an example we get*

$$\begin{aligned} &|S_n^{(2)}((1^{(1)}, 2^{(1)}), (2^{(1)}, 1^{(1)}))| = |S_n^{(2)}((1^{(1)}, 2^{(1)}), (1^{(1)}, 2^{(2)}))| = \\ &|S_n^{(2)}((1^{(1)}, 2^{(2)}), (2^{(1)}, 1^{(2)}))| = |S_n^{(2)}((1^{(1)}, 2^{(2)}), (2^{(2)}, 1^{(1)}))| = \quad (n+1)! \end{aligned}$$

for $n \ge 0$, which was proved in [S].

Theorem 6.3. *Let $2 \le a \le b$, and $r \ge b$; for all $n \ge 1$*

$$|S_n^{(r)}(T)| = \sum_{i+j \le n} \binom{n}{i, j, n-i-j}^2 (n-i-j)!(r-2)^{n-i-j},$$

where

$$\begin{array}{llll} (i) & T = \{(1^{(1)}, 2^{(1)}), (1^{(a)}, 2^{(b)})\}; & (ii) & T = \{(1^{(1)}, 2^{(1)}), (2^{(a)}, 1^{(b)})\}; \\ (iii) & T = \{(1^{(1)}, 2^{(2)}), (1^{(3)}, 2^{(4)})\}; & (iv) & T = \{(1^{(1)}, 2^{(2)}), (2^{(3)}, 1^{(4)})\}. \end{array}$$

Proof. **Cases** (i), (ii): Similar to the proof of Theorem 3.1 we find that

$$|S_n^{(r)}(T)| = \sum_{j=0}^{n} \binom{n}{j}^2 |S_j(12)| S_{n-j}^{(r-1)}(\phi)|,$$

where either $\phi = (1^{(a)}, 2^{(b)})\}$ or $\phi = (2^{(a)}, 1^{(b)})\}$. Hence, since $S_j(12) = 1$ for all $j \ge 0$, the two claims follow from Theorem 5.1.

Cases (iii), (iv): Let $T_1 = \{(1^{(1)}, 2^{(2)}), (1^{(3)}, 2^{(4)})\}$, $T_2 = \{(1^{(1)}, 2^{(2)}), (2^{(3)}, 1^{(4)})\}$, and let $\phi \in S_n^{(r)}(T_1)$. Also let us define I_ϕ to be the set of all j such that ϕ_j is coloured by either 3 or 4. Now we define a function $f : S_n^{(r)}(T_1) \to S_n^{(r)}(T_2)$ by reversing all the ϕ_j where $j \in I_\phi$. Hence by definitions, f is a bijection, which means that

$|S_n^{(r)}((1^{(1)},2^{(2)}),(2^{(3)},1^{(4)}))| = |S_n^{(r)}((1^{(1)},2^{(2)}),(1^{(3)},2^{(4)}))|$.

On the other hand, by Theorem 5.1 there exist bijections, $f_{a,b;c} : S_n^{(r)}((1^{(a)},2^{(b)})) \to S_n^{(r)}((1^{(c)},2^{(c)})$, and $g_{a,b;c} : S_n^{(r)}((2^{(a)},1^{(b)})) \to S_n^{(r)}((2^{(c)},1^{(c)})$. Now let us define a bijection $g : S_n^{(r)}((1^{(1)},2^{(2)}),(2^{(3)},1^{(4)})) \to S_n^{\{1,3,5,6,\dots,r\}}((1^{(1)},2^{(1)}),(2^{(3)},1^{(3)}))$, as follows. Let $\phi \in S_n^{(r)}((1^{(1)},2^{(2)}),(2^{(3)},1^{(4)}))$, let I_ϕ the set of all j such that ϕ_j is coloured by either 1 or 2, and let J_ϕ the set of all j such that ϕ_j is coloured by either 3 or 4; we define $g(\phi)$ by operating the bijection $f_{1,2;1}$ on all ϕ_j where $j \in I_\phi$, by operating the bijection $f_{3,4;3}$ on all ϕ_j where $j \in J_\phi$, and leaving the other ϕ_j with $j \notin I_\phi \cup J_\phi$ in the same order. Hence, g is a bijection, and by Theorem 4.4 the Cases (iii) and (iv) follow. □

Example 6.4. (see [S, Eq. 47]) *As an example, by Theorem 6.3 for $n \geq 0$*

$$|S_n^{(2)}((1^{(1)},2^{(1)}),(2^{(2)},1^{(2)}))| = \binom{2n}{n}.$$

Theorem 6.5. *For $r \geq 3$,*

$$\sum_{n\geq 0} \frac{S_n^{(r)}(T)}{n!}x^n = \frac{\int d_{r-1}^2(x)dx}{1-(r-1)x},$$

where

$$(i) \quad T = \{(1^{(1)},2^{(2)}),(2^{(1)},1^{(3)})\}; \quad (ii) \quad T = \{(1^{(1)},2^{(2)}),(2^{(2)},1^{(3)})\}.$$

Proof. **Case** (i)**:** Let $T = \{(1^{(1)},2^{(2)}),(2^{(1)},1^{(3)}))\}$, $p_n = S_n^{(r)}(T)$, $\phi \in S_n^{(r)}(T)$, and let us consider the possible values of ϕ_1:

(1) Let $\phi_1 = i^{(c)}$, $c \neq 1$; so $\phi \in S_n^{(r)}(T)$ if and only if $(\phi_2,\dots,\phi_n) \in S^{(r)}_{\{1,\dots,i-1,i+1,\dots,n\}}(T)$. Hence in this case there are $(r-1)np_{n-1}$ coloured permutations.

(2) Let $\phi_1 = i^{(1)}$, $1 \leq i \leq n$; since ϕ is T-avoiding, the symbols $i+1,\dots,n$ are not coloured by 2, and the symbols $1,\dots,i-1$ are not coloured by 3. Hence there are $\binom{n-1}{i-1}|S_{n-i}^{(r-1)}((2^{(1)},1^{(3)}))||S_{i-1}^{(r-1)}((1^{(1)},2^{(2)}))|$ coloured permutations, so by Theorem 5.1 there are $\binom{n-1}{i-1}d_{r-1}(n-i)d_{r-1}(i-1)$ coloured permutations.

Therefore p_n satisfies the following relation:

$$p_n = n(r-1)p_{n-1} + \sum_{i=1}^n \binom{n-1}{i-1} d_{r-1}(n-i)d_{r-1}(i-1)$$

with $p_0 = 1$ and $p_1 = r$. Let $q_n = p_n/n!$, and $q(x) = \sum_{n\geq 0} q_n x^n$; by multiplying the last relation by $\frac{x^n}{(n-1)!}$, and summing over $n \geq 1$ we get

$$\sum_{n\geq 1}(nq_n - n(r-1)q_{n-1})x^n = xd_{r-1}^2(x),$$

hence $[(1-(r-1)x)q(x)]' = d_{r-1}^2(x)$, which means that Case (i) holds.

Case (ii)**:** Let $T = \{(1^{(1)},2^{(2)}),(2^{(2)},1^{(3)}))\}$, $p_n = S_n^{(r)}(T)$, and $\phi \in S_n^{(r)}(T)$ such that $\phi_j = n^{(c)}$. Let us consider the possible values of j, c:

(1) Let $c \neq 2$; $\phi \in S_n^{(r)}(T)$ if and only if $(\phi_1, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_n) \in S_{n-1}^{(r)}(T)$. Hence in this case there are $(r-1)np_{n-1}$ coloured permutations.

(2) Let $c = 2$; since ϕ is T-avoiding, all the symbols in $(\phi_1, \dots, \phi_{j-1})$ are not coloured by 1, and the symbols in $(\phi_{j+1}, \dots, \phi_n)$ are not coloured by 3. Hence there are $\binom{n-1}{j-1}|S_{j-1}^{(r-1)}((2^{(2)}, 1^{(3)}))||S_{n-j}^{(r-1)}((1^{(1)}, 2^{(2)}))|$ coloured permutations, so by Theorem 5.1 there are $\binom{n-1}{j-1}d_{r-1}(j-1)d_{r-1}(n-j)$ coloured permutations.

Therefore p_n satisfies the following relation:

$$p_n = n(r-1)p_{n-1} + \sum_{j=1}^{n} \binom{n-1}{j-1} d_{r-1}(n-j)d_{r-1}(j-1)$$

with $p_0 = 1$. Hence, by Case (i) we see that Case (ii) holds. □

Theorem 6.6. *For $r \geq 2$,*

$$\sum_{n\geq 0} \frac{S_n^{(r)}((1^{(1)}, 2^{(1)}), (2^{(1)}, 1^{(2)}))}{n!} x^n = \frac{\int \frac{d_{r-1}(x)}{1-(r-1)x} dx}{1-(r-1)x}.$$

Proof. Let $T = \{(1^{(1)}, 2^{(1)}), (2^{(1)}, 1^{(2)}))\}$, $p_n = S_n^{(r)}(T)$, $\phi \in S_n^{(r)}(T)$, and let us consider the possible values of ϕ_1:

(1) If $\phi_1 = i^{(c)}$ where $c \neq 1$, then $\phi \in S_n^{(r)}(T)$ if and only if $(\phi_2, \dots, \phi_n) \in S_{\{1,\dots,i-1,i+1,\dots,n\}}^{(r)}(T)$. Hence in this case there are $(r-1)np_{n-1}$ coloured permutations.

(2) If $\phi_1 = i^{(1)}$ then, since ϕ avoids T, the symbols $i+1, \dots, n$ are not coloured by 1, and the symbols $1, \dots, i-1$ are not coloured by 2. Hence there are $\binom{n-1}{i-1}|S_{n-i}^{(r-1)}||S_{i-1}^{(r-1)}((1^{(1)}, 2^{(1)}))|$ coloured permutations, so by Theorem 5.1 there are $\binom{n-1}{i-1}(n-i)!(r-1)^{n-i}d_{r-1}(i-1)$ coloured permutations.

Therefore p_n satisfies the following relation: $p_0 = 1$, and for $n \geq 1$

$$p_n = n(r-1)p_{n-1} + \sum_{i=1}^{n} \binom{n-1}{i-1} (n-i)!(r-1)^{n-i} d_{r-1}(i-1).$$

Let $q_n = p_n/n!$, and $q(x) = \sum_{n\geq 0} q_n x^n$, by multiplying the last relation by $\frac{x^n}{(n-1)!}$, and summing over $n \geq 1$ we get

$$\sum_{n\geq 1} (nq_n - n(r-1)q_{n-1})x^n = \frac{xd_{r-1}(x)}{1-(r-1)x},$$

hence $[(1-(r-1)x)q(x)]' = \frac{d_{r-1}(x)}{1-(r-1)x}$, which proves the theorem. □

Example 6.7. (see [S, Eq. 48]) *Let us write $a_n = |S_n^{(2)}((1^{(1)}, 2^{(1)}), (2^{(1)}, 1^{(2)}))|$; by symmetric operations and by Theorem 6.6, $p_n = np_{n-1} + (n-1)! \sum_{j=0}^{n-1} \frac{1}{j!}$ for $n \geq 1$, hence $n! < p_n < (n+1)!$ for $n \geq 3$.*

Let $wc(r)$ be the number of Wilf classes of a double restriction by 2-letter coloured patterns with r colours; then by Theorems 6.1–6.6, and by Table 1 we may formulate part (ii) of the Main Theorem as follows.

Corollary 6.8. *$wc(r) = 1, 4, 6$ for $r = 1, 2$, $r \geq 3$ respectively.*

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