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ON MULTIPLICITIES OF POINTS ON SCHUBERT VARIETIES IN GRASSMANNIANS

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Abstract. We answer some questions related to multiplicity formulas by Rosenthal and Zelevinsky and by Lakshmibai and Weyman for points on Schubert varieties in Grassmannians. In particular, we give combinatorial interpretations in terms of nonintersecting lattice paths of these formulas, which makes the equality of the two formulas immediately obvious. Furthermore we provide an alternative determinantal formula for these multiplicities, and we show that they count semistandard tableaux of unusual shapes.

1. Introduction. The *multiplicity* of a point on an algebraic variety is an important invariant that "measures" singularity of the point. Recently, Rosenthal and Zelevinsky [8] gave a determinantal formula for the multiplicity of a point on a Schubert variety in a Grassmannian (see Theorem 1). This formula immediately raised three questions (and, indeed, they are asked in [8, Remark 5, paragraph after Theorem 1, Remark 7]):

(1) Is there a direct way to see that the formula yields positive integers (that is, leaving aside the fact that the Rosenthal–Zelevinsky theorem says that it gives multiplicities of singular points)?
(2) The formula is in form of a binomial determinant. Such determinants are very common in combinatorics. Is there a combinatorial interpretation?
(3) Lakshmibai and Weyman [5, Theorem 5.4] give a different determinantal formula in a special case (see Theorem 2). It is not immediately clear why it agrees with the formula by Rosenthal and Zelevinsky. Is there a straightforward explanation?

The purpose of this note is to answer these questions. In reply to Question 1 we show that, by means of the Lindström–Gessel–Viennot theorem [6, Lemma 1], [4, Theorem 1] (see Theorem 3), the formula of Rosenthal and Zelevinsky counts certain

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families of nonintersecting lattice paths. Clearly, this immediately explains why the formula yields positive integers. At the same time, this also provides a first answer to Question 2. In the special case considered by Lakshmibai and Weyman, we apply an easy combinatorial transformation to the families of nonintersecting lattice paths corresponding to the Rosenthal–Zelevinsky determinant formula and thus convert them into other families of nonintersecting lattice paths. These latter families of nonintersecting lattice paths directly yield the Lakshmibai–Weyman formula, again by means of the Lindström–Gessel–Viennot theorem. This answers Question 3.

In addition, we use the "dual path" idea by Gessel and Viennot [3, Sec. 4] to derive an alternative determinantal formula for the multiplicities in the general case. As a bonus, this enables us to also find a combinatorial description of the multiplicities as the numbers of semistandard tableaux of unusual shapes, thus providing another answer to Question 2.

In the next section we review the basic definitions and the formulas by Rosenthal and Zelevinsky and by Lakshmibai and Weyman. Then, in Section 3, we first recall the Lindström–Gessel–Viennot theorem on nonintersecting paths, and then explain how to interpret the formulas by Rosenthal and Zelevinsky and by Lakshmibai and Weyman in terms of nonintersecting lattice paths, and why this immediately explains that they are equivalent in the relevant special case. We conclude this section by showing how the description of the singular locus of a Schubert variety in a Grassmannian follows directly from the combinatorics of nonintersecting lattice paths. Finally, in Section 4, we derive the alternative determinantal formula for the multiplicities (see Theorem 5) and its interpretation in terms of semistandard tableaux (see Corollary 6).

2. The multiplicity formulas by Rosenthal and Zelevinsky and by Lakshmibai and Weyman. Let d and n be positive integers with $0 \le d \le n$. The *Grassmannian* $\mathrm{Gr}_d(V)$ is the variety of all d-dimensional subspaces in an n-dimensional vector space V (over some algebraically closed field of arbitrary characteristic). Given an integer vector $\mathbf{i} = (i_1, i_2, \dots, i_d)$, $1 \le i_1 < i_2 < \cdots < i_d \le n$ and a complete flag $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$, the *Schubert variety* $X_\mathbf{i}$ is defined by

$$X_\mathbf{i} = \{W \in \mathrm{Gr}_d(V) : \dim(W \cap V_{i_k}) \ge k \text{ for } k = 1, 2, \dots, d\}.$$

The *Schubert cell* $X_\mathbf{i}^\circ$ is an open subset in $X_\mathbf{i}$ given by

$$X_\mathbf{i}^\circ = \{W \in X_\mathbf{i} : \dim(W \cap V_{i_k - 1}) = k - 1 \text{ for } k = 1, 2, \dots, d\}.$$

It is well-known (see e.g. [2, Sec. 9.4]) that the Schubert variety $X_\mathbf{i}$ is the disjoint union of Schubert cells $X_\mathbf{j}^\circ$ over all $\mathbf{j} \le \mathbf{i}$ (the latter inequality meaning $j_k \le i_k$ for $k = 1, 2, \dots, d$). The multiplicity of a point x in $X_\mathbf{i}$ is constant on each Schubert cell $X_\mathbf{j}^\circ \subset X_\mathbf{i}$. Following [8] we denote this multiplicity by $M_\mathbf{j}(\mathbf{i})$.

The determinantal formula by Rosenthal and Zelevinsky for the multiplicity $M_\mathbf{j}(\mathbf{i})$ is the following.

Theorem 1. ([8, Theorem 1]) *The multiplicity $M_\mathbf{j}(\mathbf{i})$ of a point $x \in X_\mathbf{j}^\circ \subset X_\mathbf{i}$ is given by*

$$M_\mathbf{j}(\mathbf{i}) = (-1)^{s_1 + \cdots + s_d} \det_{1 \le p,q \le d} \left(\binom{i_q}{p - 1 - s_q} \right), \tag{1}$$

where $s_q = |\{\ell : i_q < j_\ell\}|$.

In the special case that $\mathbf{j} = (1, 2, \dots, d)$, Lakshmibai and Weyman have given a different determinant formula. In the statement of their theorem we use standard partition terminology (see e.g. [7, Ch. I, Sec. 1]).

Theorem 2. ([5, Theorem 5.4]) *The multiplicity* $M_{(1,2,\dots,d)}(\mathbf{i})$ *of a point* $x \in X^\circ_{(1,2,\dots,d)} \subset X_{\mathbf{i}}$ *is given by*

$$M_{(1,2,\dots,d)}(\mathbf{i}) = \det_{1\le p,q\le r}\left(\binom{\alpha_p+\beta_q}{\alpha_p}\right), \tag{2}$$

where $(\alpha_1, \dots, \alpha_r \mid \beta_1, \dots, \beta_r)$ *is the Frobenius notation of the partition* $\lambda = (i_d - d, \dots, i_2 - 2, i_1 - 1)$.

3. Multiplicities count nonintersecting lattice paths. We start by recalling the main theorem on nonintersecting lattice paths, due to Lindström, and Gessel and Viennot.

Theorem 3. ([6, Lemma 1], [4, Theorem 1]) *Let* G *be any acyclic directed graph. Let* $A_1, A_2, \dots, A_d, E_1, E_2, \dots, E_d$ *be vertices of* G. *Then, with* S_d *denoting the group of permutations of* $\{1, 2, \dots, d\}$, *the following identity holds:*

$$\det_{1\le p,q\le d}(\mathcal{P}(A_p \to E_q)) = \sum_{\sigma\in S_d}(\operatorname{sgn}\sigma)\cdot\mathcal{P}(\mathbf{A}\to\mathbf{E}_\sigma, \text{nonint.}), \tag{3}$$

where $\mathcal{P}(A \to E)$ *denotes the number of paths from* A *to* E *in* G, *and where* $\mathcal{P}(\mathbf{A} \to \mathbf{E}_\sigma, \text{nonint.})$ *denotes the number of all families* $(P_1, P_2, \dots, P_d)$ *of paths in* G, P_ℓ *running from* A_ℓ *to* $E_{\sigma(\ell)}$, $\ell = 1, 2, \dots, d$, *which are* nonintersecting. *A family of paths is called nonintersecting if no two paths in the family have a point in common.*

The most commonly used instance of this rather general theorem arises when the starting and end points are in a position such that the only nonvanishing term on the right-hand side of (3) is the one for σ equal to the identity permutation.

Corollary 4. ([4, Cor. 2]) *In addition to the assumptions in Theorem 3, assume that for any* $i < j$ *and* $k < l$ *any path from* A_i *to* E_l *intersects any path from* A_j *to* A_k. *Then the number of all families* $(P_1, P_2, \dots, P_d)$ *of nonintersecting paths in* G, P_ℓ *running from* A_ℓ *to* $E_{\sigma(\ell)}$, $\ell = 1, 2, \dots, d$, *is equal to*

$$\det_{1\le p,q\le d}(\mathcal{P}(A_p \to E_q)).$$

In view of the above corollary, the determinant (2) by Lakshmibai and Weyman has an obvious interpretation in terms of nonintersecting lattice paths consisting of horizontal and vertical steps in the positive direction: It counts the number of all families $(P_1, P_2, \dots, P_r)$ of nonintersecting lattice paths, the path P_ℓ running from $(-\beta_\ell, 0)$ to

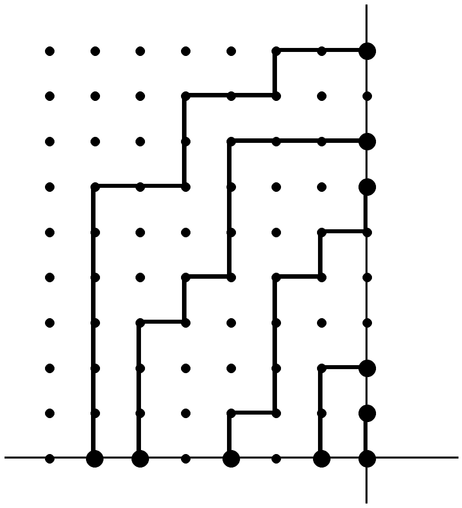


Figure 1

$(0, \alpha_\ell)$, $\ell = 1, 2, \dots, r$. See Figure 1 for an example, where $d = 7$, $(i_1, i_2, \dots, i_7) = (3, 5, 9, 10, 14, 15, 17)$, and, hence, $\lambda = (10, 9, 9, 6, 6, 3, 2) = (9, 7, 6, 2, 1 \mid 6, 5, 3, 1, 0)$.

The formula by Rosenthal and Zelevinsky can also be interpreted in terms of nonintersecting lattice paths. By Theorem 3, the determinant (1) counts the weighted sum of all families $(Q_1, Q_2, \dots, Q_d)$ of nonintersecting lattice paths, where the path Q_ℓ runs from $(-\ell + 1, \ell - 1)$, to $(-s_{\sigma(\ell)}, s_{\sigma(\ell)} + i_{\sigma(\ell)})$, $\ell = 1, 2, \dots, d$, for some permutation $\sigma \in S_d$, and where the weight of a path family is defined as the sign of σ. See Figure 2 for an example with the same parameters, in which $s_1 = 4$, $s_2 = 2$, and $s_3 = \cdots = s_7 = 0$. However, it is not difficult to see that the permutation σ is in fact always the same, and that the sign of σ is $(-1)^{s_1 + \cdots + s_d}$.

This gives a combinatorial interpretation of the Rosenthal–Zelevinsky formula for *any* $\mathbf{i}$ and $\mathbf{j}$, namely as the number of all families $(Q_1, Q_2, \dots, Q_d)$ of nonintersecting lattice paths, where the path Q_ℓ runs from $(-\ell + 1, \ell - 1)$, to $(-s_{\sigma(\ell)}, s_{\sigma(\ell)} + i_{\sigma(\ell)})$, $\ell = 1, 2, \dots, d$, for some permutation $\sigma \in S_d$ (which is uniquely determined).

In the special case, however, that $\mathbf{j} = (1, 2, \dots, d)$, we have $s_\ell = d - i_\ell$ as long as $i_\ell \le d$, so that the end points of the paths are either $(i_\ell - d, d)$ or $(0, i_\ell)$. It is now easy to bijectively map these path families to the former. The path Q_1 starts at $(0, 0)$. If the path family should be nonintersecting, then the only possibility for Q_1 is to run from $(0, 0)$ to $(0, i_m)$, where m is minimal such that $s_m = 0$. (This minimum exists since we must have $s_d = 0$.) In particular, since we must have $i_m > d$, the path Q_1 starts with $d + 1$ vertical steps. This forces all the other paths (if they want to be nonintersecting) to also be vertical until they reach height $d + 1$. If they do not reach height $d + 1$, then they terminate in a point $(i_\ell - d, d)$. All this is clearly visible in Figure 2.

Hence, we may without loss of any information cut off the path portions until height $d + 1$, respectively remove the paths that even do not reach that height. (The dotted line in Figure 2 indicates the line of height $d + 1 = 8$ along which the cut is performed.) What remains is a family of paths that is familiar from the formula of

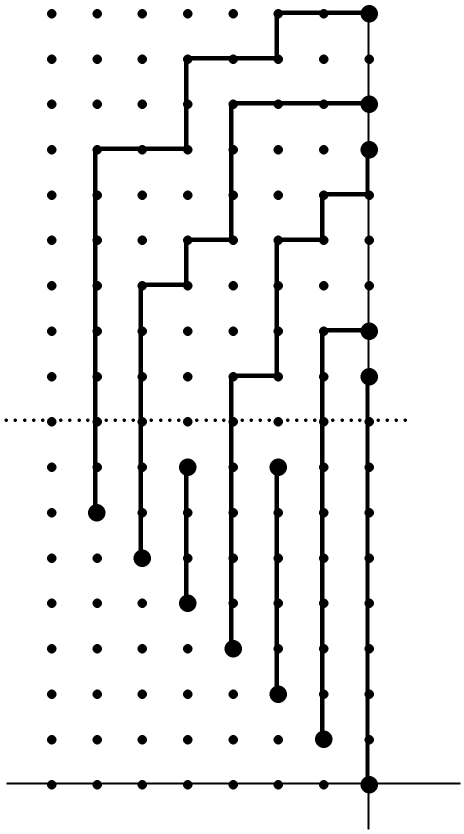


Figure 2

Lakshmibai and Weyman, see Figure 1.

This makes it obvious why formula (1) with $\mathbf{j} = (1, 2, \dots, d)$ and formula (2) agree.

Remark. The above nonintersecting lattice path interpretation of the multiplicity $M_{\mathbf{j}}(\mathbf{i})$ can be used to determine all Schubert cells $X^\circ_{\mathbf{j}}$ in the Schubert variety $X_{\mathbf{i}}$ which consist of singular points. For, it is well-known that this amounts to determine, for fixed $\mathbf{i}$, those $\mathbf{j}$ which satisfy $M_{\mathbf{j}}(\mathbf{i}) > 1$. In terms of the lattice path picture, this means to find those $\mathbf{j}$ which guarantee that the end points $(-s_\ell, s_\ell + i_\ell)$, $\ell = 1, 2, \dots, d$, of the paths are located in a manner such that there is more than one family of nonintersecting lattice paths connecting the starting points $(-\ell + 1, \ell - 1)$, $\ell = 1, 2, \dots, d$, to these end points. This is easily done, as it is easy to characterize the location of end points which yield just one such family of nonintersecting lattice paths. (In contrast, it is not possible to do this directly from the Rosenthal–Zelevinsky determinant.) Thus one would recover the well-known description of the singular locus of the Schubert variety $X_{\mathbf{i}}$ (cf. e.g. [1, Theorem 9.3.1]). We leave the details to the reader for the sake of brevity.

4. Dual paths and multiplicities as numbers of semistandard tableaux. In this section we derive an alternative determinantal formula for the multiplicities, and we show that they count certain semistandard tableaux.

If we would try to use the ideas of the previous section in the general case (i.e., for arbitrary $\mathbf{j} \le \mathbf{i}$), then we first discover that, in general, it is not true that $s_\ell + i_\ell$ is equal to d or i_ℓ. We can just say that the numbers $s_\ell + i_\ell$, $\ell = 1, 2, \dots, d$, are weakly increasing. Yet, we try the same construction.

Recall from the previous section that the multiplicity $M_{\mathbf{j}}(\mathbf{i})$ is equal to the number of all families $(Q_1, Q_2, \dots, Q_d)$ of nonintersecting lattice paths, where the path Q_ℓ runs from $(-\ell+1, \ell-1)$, to $(-s_{\sigma(\ell)}, s_{\sigma(\ell)} + i_{\sigma(\ell)})$, $\ell = 1, 2, \dots, d$, for some permutation $\sigma \in S_d$. (As we remarked, there is a unique permutation σ for which such families of nonintersecting lattice paths exist).

Let us consider an example: $d = 9$, $\mathbf{i} = (4, 6, 7, 13, 14, 17, 19, 20, 21)$ and $\mathbf{j} = (1, 2, 4, 7, 10, 12, 13, 15, 16)$. Then $\mathbf{s} = (6, 6, 5, 2, 2, 2, 0, 0, 0, 0)$. Figure 3 shows a typical family for this choice of $\mathbf{i}$ and $\mathbf{j}$.

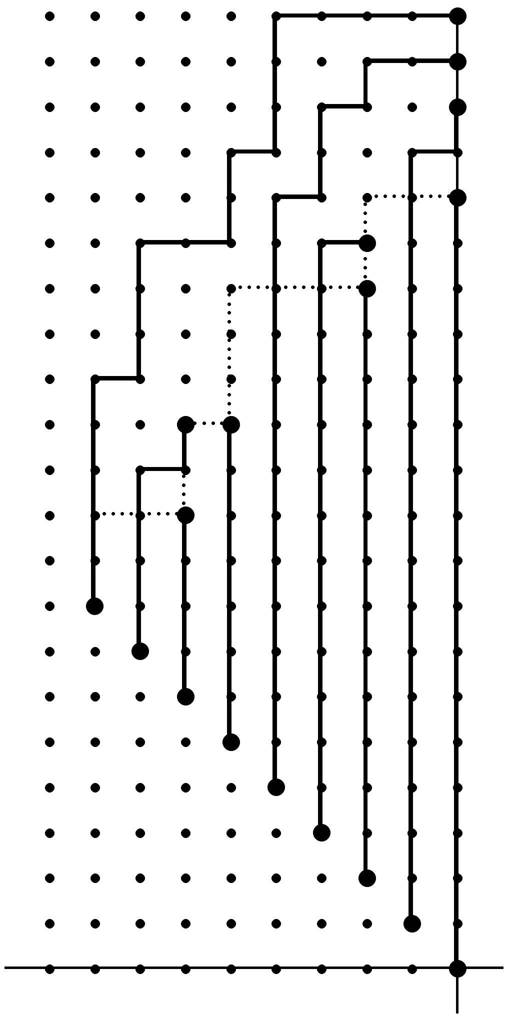


Figure 3

Again, large parts of the initial vertical portions of the paths are forced. To be precise, let $\{s_1, s_2, \dots, s_d\} = \{v_1, v_2, \dots, v_r\}$, $v_1 < v_2 < \cdots < v_r$. (I.e., the v_ℓ's are the distinct values that are attained by the s_ℓ's.) For each ℓ, consider the bottom-most end point $(-s_q, s_q + i_q)$ with $s_q = v_\ell$. Then we may cut off the vertical portions below height $s_q + i_q$ of the paths with starting points $(-v_{\ell+1} + 1, v_{\ell+1} - 1), \dots,$ $(-v_\ell - 1, v_\ell + 1)$, $(-v_\ell, v_\ell)$. These cuts are indicated by the dotted line in Figure 3. (I.e., portions below dotted lines can be omitted.) The result, after the cuts, is shown in Figure 4.

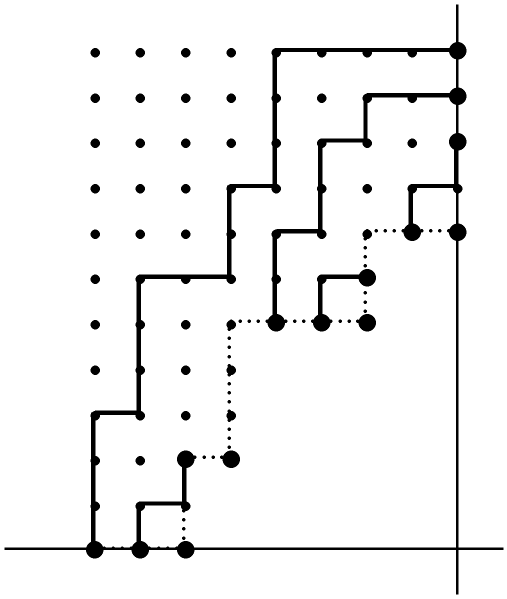


Figure 4

Now we may write down a Lindström–Gessel–Viennot determinant for these (new) starting points and (old) end points. The result would again be a determinant of binomials. However, it seems that it would require a considerable amount of notation to explicitely express what the new starting points are.

Moreover, this would not fully correspond to the Lakshmibai-Weyman formula because, to obtain the Lakshmibai-Weyman formula, one had to also drop the paths of length zero. If we would do that in the above picture, then we would have to restrict the paths explicitly to the indicated ladder-shaped region. If we would not do that then, after removal of the starting and end points corresponding to the zero length paths (which opens "holes"), there are now more possibilities to connect the (reduced set of) starting points to the (reduced set of) end points by nonintersecting lattice paths, and the corresponding permutation σ would not be unique anymore. So, we could still write down a Lindström–Gessel–Viennot determinant (that would now correspond to the Lakshibai-Weyman formula), however the entries would not be binomials anymore, they would count paths with given starting and end points *that stay in this ladder-shaped region*, for which no closed formula is available.

On the other hand, as announced, we may now introduce what Gessel and Viennot [3, Sec. 4] call "dual paths," see Figure 5. We mark starting points $(d - \ell, i_d)$, $\ell = 1, 2, \dots, d$. (They are indicated by circles in Figure 5.) Then, for each of these points, we move vertically downwards, unless we hit one of the existing paths. If the

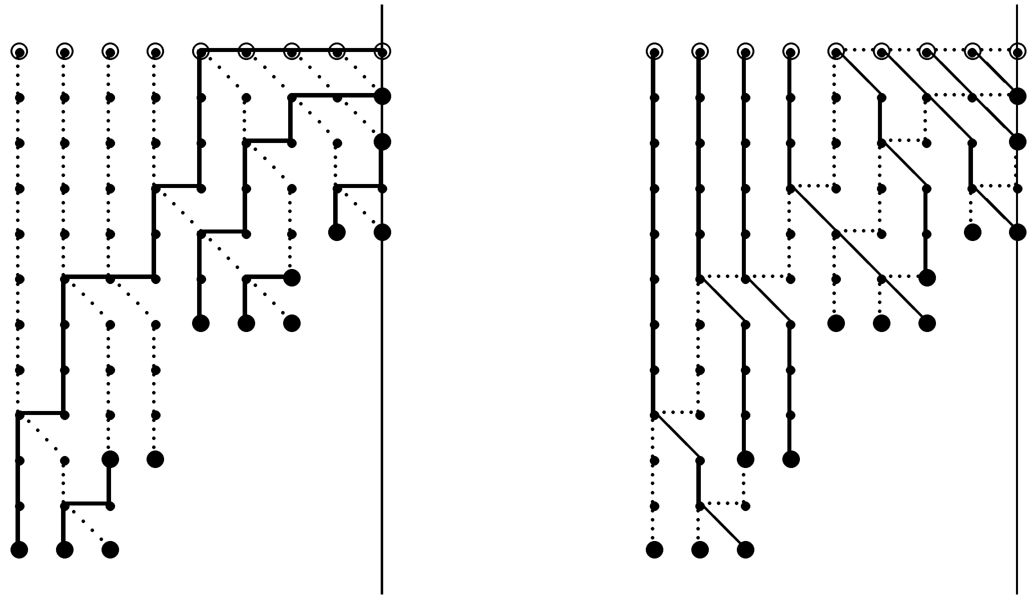


Figure 5

latter happens, then we continue by a diagonal step $(1,-1)$, etc. It is not difficult to see that in that manner we connect $(d-\ell, i_\ell)$ with $(-s_\ell, s_\ell+i_\ell)$, $\ell=1,2,\dots,d$. The resulting paths in our running example are indicated by dotted lines in the left picture in Figure 5. In the right picture the roles of the two families of paths are just interchanged.

Now we deform the lattice slightly, so that the newly introduced paths become orthogonal paths, see Figure 6.

So, what we obtain finally is a family $(R_1,R_2,\dots,R_d)$ of nonintersecting lattice paths, where the path R_ℓ is a path consisting of horizontal unit steps in the positive direction and vertical unit steps in the negative direction and runs from $(-d+\ell, i_d+\ell)$, to $(-s_\ell, i_\ell+d)$, $\ell=1,2,\dots,d$. Hence, again by Corollary 4, we obtain that the number of these families of nonintersecting lattice paths is equal to the determinant

$$\det_{1\le p,q\le d}\left(\binom{i_d-i_p-s_q}{d-p-s_q}\right).$$

This proves the following alternative to the result by Rosenthal and Zelevinsky.

Theorem 5. *The multiplicity $M_{\mathbf{j}}(\mathbf{i})$ of a point $x\in X_{\mathbf{j}}^\circ\subset X_{\mathbf{i}}$ is given by*

$$\det_{1\le p,q\le d}\left(\binom{i_d-i_p-s_q}{d-p-s_q}\right). \tag{5}$$

where, again, $s_q=|\{\ell: i_q<j_\ell\}|$.

Remark. In fact, this determinant could be restricted to $1\le p,q\le d-1$, because $\binom{i_d-i_d-s_q}{d-d-s_q}=\binom{-s_q}{-s_q}=\delta_{q,d}$ (with $\delta_{q,d}$ the Kronecker delta). This corresponds to the combinatorial fact that path R_d is a zero length path. See the top-right of Figure 6.

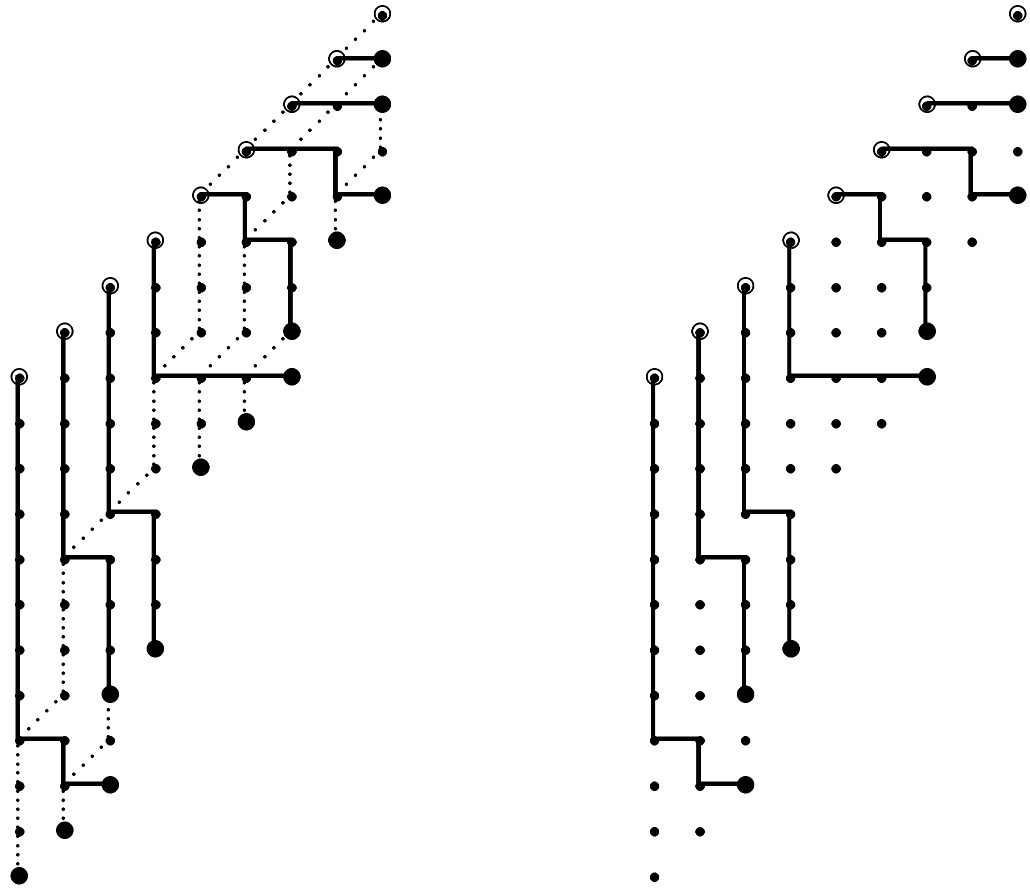
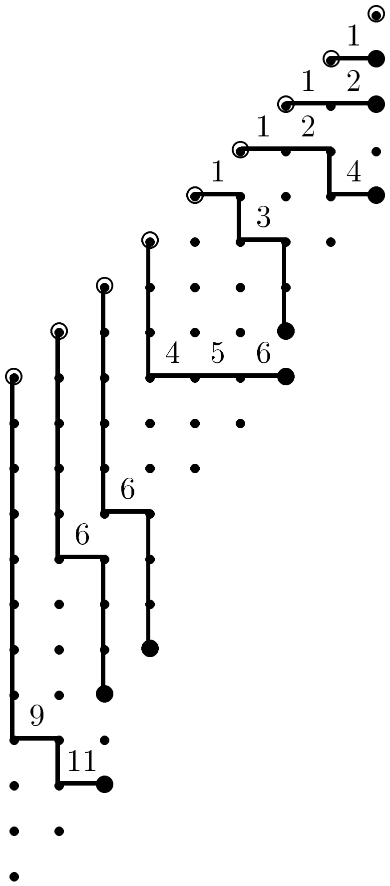


Figure 6

As a bonus, we are now able to derive a combinatorial interpretation of the multiplicities $M_{\mathbf{j}}(\mathbf{i})$ in terms of certain semistandard tableaux. If we label horizontal steps along diagonals by 1, 2, ... , respectively, as indicated in Figure 7 (i.e., along the first diagonal of horizontal steps immediately to the right of the starting points, the latter being indicated by circles, horizontal steps are labelled by 1, along the next diagonal horizontal steps are labelled by 2, etc.), and then read the labels along each path and form columns out of it, then we obtain an array, as shown in Figure 7 in our example, which has the following properties:

1. The length of column ℓ is $\ell - s_{d-\ell}$, $\ell = 1, 2, \dots, d-1$.
2. The entries along rows are weakly increasing.
3. The entries along columns are strictly increasing.
4. If at the bottom of column $d-\ell$ we write $i_d - i_\ell - s_\ell + 1$, $\ell = 1, 2, \dots, d-1$ (see Figure 8 for the corresponding extended array in our running example), then rows are still weakly increasing and columns are still strictly increasing.

Corollary 6. *The multiplicity $M_{\mathbf{j}}(\mathbf{i})$ of a point $x \in X_{\mathbf{j}}^{\circ} \subseteq X_{\mathbf{i}}$ is equal to the number of arrays of positive integers satisfying* (1)–(4) *above.*



1	1	1	1	4	6	6	9
	2	2	3	5			11
		4		6			

Figure 7

1	1	1	1	4	6	6	9
2	2	2	3	5	**10**	**10**	11
	3	4	**6**	6			**12**
		5		**7**			

Figure 8

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