SUPERNOMIAL COEFFICIENTS, BAILEY'S LEMMA AND ROGERS–RAMANUJAN-TYPE IDENTITIES.

A SURVEY OF RESULTS AND OPEN PROBLEMS

S. OLE WARNAAR

Abstract. An elementary introduction to the recently introduced A_2 Bailey lemma for supernomial coefficients is presented. As illustration, some A_2 q-series identities are (re)derived which are natural analogues of the classical (A_1) Rogers–Ramanujan identities and their generalizations of Andrews and Bressoud. The intimately related, but unsolved problems of supernomial inversion, A_{n-1} and higher level extensions are also discussed. This yields new results and conjectures involving A_{n-1} basic hypergeometric series, string functions and cylindric partitions.

1. Introduction

The purpose of this paper is twofold. Firstly, it intends to provide an easy introduction to recent results by Andrews, Schilling and the author [9] concerning an A_2 Bailey lemma for supernomial coefficients. The fact that the theorems of [9] have led to the discovery of A_2 analogues of the famous Rogers–Ramanujan identities, (hopefully) justifies such an introduction. Secondly, we hope to attract some interest in the numerous unsolved problems directly related to the results of ref. [9].

In the first part of this paper, comprising of sections 2–4, we review the A_1 Bailey lemma and its A_2 supernomial generalization, and show how this provides a natural framework for proving and deriving identities of the Rogers–Ramanujan type. To make this part of the paper as accessible as possible we have omitted all proofs and have removed all the usual Bailey miscellanea. Also, we have chosen to cover only the simplest possible cases that can be extracted from the general Bailey machinery (see e.g., [10, 5, 6, 30, 1, 12, 31, 9]).

In the second part of the paper (sections 5-7) we discuss various questions that have arisen in relation to our A_2 Bailey lemma. Most importantly there is the problem of generalizing the results of [9] to A_{n-1}, but also questions concerning supernomial inversion, higher-level Bailey lemmas and some related issues will be surveyed.

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We should remark here that this paper does not in any way discuss the A_{n-1} Bailey lemma of Milne and Lilly [28, 29], nor the A_{n-1} Rogers–Ramanujan identities of Milne [26, 27]. It is our current believe that the A_2 Bailey lemma for supernomials and the A_2 case of Milne and Lilly's lemma are generalizations of the classical A_1 Bailey lemma, which, in a sense, are orthogonal. Also the A_2 Rogers–Ramanujan identity of this paper appears to be unrelated to the A_2 case of Milne's A_{n-1} Rogers–Ramanujan identity.

2. A_1 Rogers–Ramanujan-type identities

The Gaussian polynomial or q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & \text{for } 0 \le m \le n \\ 0 & \text{otherwise,} \end{cases}$$

where $(a;q)_\infty = (a)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$ and

$$(a;q)_n = (a)_n = \frac{(a)_\infty}{(aq^n)_\infty}, \qquad n \in \mathbb{Z}.$$

In particular, $(q)_0 = 1$, $(q)_n = (1-q)\dots(1-q^n)$ and $1/(q)_{-n} = 0$ for $n \ge 1$. We will often use a shifted and normalized q-binomial coefficient, defined as

$$S(L,k) = \frac{1}{(q)_{2L}} \begin{bmatrix} 2L \\ L-k \end{bmatrix} = \begin{cases} \dfrac{1}{(q)_{L-k}(q)_{L+k}} & \text{for } -L \le k \le L \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

By the q-Chu–Vandermonde summation (equation (3.3.10) of [3]), it follows that the modified q-binomial satisfies the following "invariance property" (equation (13) of [31] with $c_k = \delta_{k,r}$) ,

$$\sum_{L=0}^{M} \frac{q^{L^2} S(L,r)}{(q)_{M-L}} = q^{r^2} S(M,r). \tag{2.2}$$

That is, the modified q-binomial is (up to an overall factor) invariant under multiplication by $q^{L^2}/(q)_{M-L}$ followed by a sum over L. It is this property of the q-binomial that we shall try to generalize to other q-functions.

First, however, let us demonstrate the effectiveness of the result (2.2) in deriving identities of the Rogers–Ramanujan type. To obtain identities for odd moduli our starting point is the following specialization of the the q-binomial formula (equation (II.4) of [15]),

$$\sum_{r=-L}^{L} (-1)^r q^{\binom{r}{2}} S(L,r) = \delta_{L,0}, \tag{2.3}$$

with $\binom{r}{2} = r(r-1)/2$ for $r \in \mathbb{Z}$. Applying (2.2) k times (that is, multiplying (2.3) by $q^{L^2}/(q)_{M-L}$, summing over L using (2.2) and replacing M by L, and iterating this k times) yields

$$\sum_{r=-L}^{L} (-1)^r q^{\binom{r}{2}+kr^2} S(L,r) = \sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+\dots+N_{k-1}^2}}{(q)_{L-N_1}(q)_{n_1}\cdots(q)_{n_{k-1}}},$$

where $N_j = n_j + \dots + n_{k-1}$. If we let L tend to infinity and use Jacobi's triple product identity

$$\sum_{j=-\infty}^{\infty} (-z)^j q^{\binom{j}{2}} = (z, q/z, q)_\infty, \tag{2.4}$$

where $(a_1,\dots,a_k;q)_n = (a_1,\dots,a_k)_n = (a_1)_n \dots (a_k)_n$, the following result is obtained.

Theorem 2.1. *For $k \ge 2$, $|q|<1$, and $N_j = n_j + \dots + n_{k-1}$,*

$$\sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+\dots+N_{k-1}^2}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} = \frac{(q^k, q^{k+1}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}. \tag{2.5}$$

For $k=2$ this is the (first) Rogers–Ramanujan identity [32, 33, 38, 34]

$$\sum_{n\ge 0} \frac{q^{n^2}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}. \tag{2.6}$$

For general k equation (2.5) is (a particular case) of Andrews' analytic counterpart [2] of Gordon's partition theorem [18].

To obtain a similar result for even moduli, we start with the simple identity (equation (40) of [30]),

$$\sum_{r=-L}^{L} (-1)^r q^{r^2} S(L,r) = \frac{1}{(q^2;q^2)_L}. \tag{2.7}$$

Applying (2.2) $k-1$ times yields

$$\sum_{r=-L}^{L} (-1)^r q^{kr^2} S(L,r) = \sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+\dots+N_{k-1}^2}}{(q)_{L-N_1}(q)_{n_1}\cdots(q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}}.$$

When L tends to infinity one can again apply the triple product (2.4), resulting in our next theorem.

Theorem 2.2. *For $k \ge 2$, $|q|<1$, and $N_j = n_j + \dots + n_{k-1}$,*

$$\sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+\dots+N_{k-1}^2}}{(q)_{n_1}\cdots(q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}} = \frac{(q^k, q^k, q^{2k}; q^{2k})_\infty}{(q)_\infty} \tag{2.8}$$

For $k=2$ this identity is due to Euler. For general k the above result was first obtained by Bressoud [11].

3. Supernomial coefficients

In this section we introduce A_{n-1} generalizations of the q-binomial coefficients and show how, in the case of A_2, the invariance property (2.2) can be generalized. This is then used to derive A_2 Rogers–Ramanujan-type identities for all moduli. First, however, to serve as a guide for subsequent generalizations, some of the equations of the previous section are rewritten in manifest A_1 form.

3.1. **$\mathbf{A}_1$ again.** As is often convenient when dealing with root systems of type A_{n-1}, we introduce n variables $k_1,\dots,k_n$ constrained to the hyperplane $k_1+\dots+k_n=0$. We denote $k=(k_1,\dots,k_n)$, $\rho=(1,\dots,n)$, and for arbitrary $v\in\mathbb{Z}^p$ we set $|v|=\sum_{i=1}^p v_i$, so that, in particular, $|k|=0$. The Cartan matrix of A_{n-1} will be denoted by C, i.e., $C_{i,j}=2\delta_{i,j}-\delta_{|i-j|,1}$, $i,j=1,\dots,n-1$.

Now assume $n=2$. Then $k=(k_1,k_2)=(k_1,-k_1)$, $\rho=(1,2)$ and $C=(2)$. We can then rewrite equation (2.1) in vector notation as

$$S(L,k)=\begin{cases}\dfrac{1}{(q)_{L_1-k_1}(q)_{L_1-k_2}} & \text{for } k_1,k_2\le L_1\\ 0 & \text{otherwise.}\end{cases} \tag{3.1}$$

where $L=(L_1)$.

Similarly, the invariance property (2.2) becomes

$$\sum_{L=0}^{M}\frac{q^{\frac12 LCL}S(L,k)}{(q)_{M-L}}=q^{\frac12(k_1^2+k_2^2)}S(M,k), \tag{3.2}$$

where, generally, for $v,w\in\mathbb{Z}^p$, $vAv=\sum_{i,j=1}^p v_iA_{i,j}v_j$, $(a)_v=(a)_{v_1}\dots(a)_{v_p}$ and $\sum_{v=0}^{w}=\sum_{v_1=0}^{w_1}\cdots\sum_{v_p=0}^{w_p}$.

The equations (2.3) and (2.7) that served as input in the derivation of the Rogers–Ramanujan-type identities become in the new notation

$$\sum_{|k|=0}\sum_{\sigma\in S_2}\epsilon(\sigma)q^{\sum_{i=1}^2(k_i-\sigma_i)k_i}S(L,2k-\sigma+\rho)=\delta_{L_1,0}. \tag{3.3}$$

and

$$\sum_{|k|=0}\sum_{\sigma\in S_2}\epsilon(\sigma)q^{\frac12\sum_{i=1}^2(2k_i-\sigma_i+i)^2}S(L,2k-\sigma+\rho)=\frac{1}{(q^2;q^2)_{L_1}}, \tag{3.4}$$

where S_n is the permutation group on $1,2,\dots,n$ and $\epsilon(\sigma)$ is the sign of the permutation σ.

3.2. **Completely antisymmetric A_{n-1} supernomials.** The generalization of the q-binomial coefficient that is needed here is a multivariable extension of the "n-multinomial coefficient"

$$\frac{(q)_{\lambda_1+\cdots+\lambda_n}}{(q)_{\lambda_1}\dots(q)_{\lambda_n}}, \tag{3.5}$$

defined as follows [19].

Definition 3.1. *Let $L\in\mathbb{Z}_+^{n-1}$, $\lambda\in\mathbb{Z}_+^n$ and let $\nu^{(n)}$ denote the conjugate of the partition $(1^{L_1}\dots(n-1)^{L_{n-1}})$, i.e., $\nu_j^{(n)}=L_j+\cdots+L_{n-1}$. Then, for $|\lambda|=|\nu^{(n)}|(=\sum_{a=1}^{n-1}aL_a)$,*

$$\begin{bmatrix}L\\ \lambda\end{bmatrix}=\sum_{\nu}\prod_{a=1}^{n-1}\prod_{j=1}^{a}\begin{bmatrix}\nu_j^{(a+1)}-\nu_{j+1}^{(a+1)}\\ \nu_j^{(a)}-\nu_{j+1}^{(a+1)}\end{bmatrix}, \tag{3.6}$$

where the sum over ν denotes a sum over sequences $\emptyset=\nu^{(0)}\subset\nu^{(1)}\subset\cdots\subset\nu^{(n)}$ of Young diagrams such that each skew diagram $\nu^{(a)}-\nu^{(a-1)}$ is a horizontal λ_a-strip[1].

Copying the example of ref. [19], we find that for $n=3$, $L=(1,3)$ and $\lambda=(3,2,2)$, the contributions to the above sum correspond to $\nu=(\emptyset,(3),(3,2),(4,3))$ and $\nu=(\emptyset,(3),(4,1),(4,3))$, yielding $\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]+\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]=2+3q+4q^2+2q^3+q^4$. When $L=(|\lambda|,0,\dots,0)$ the only term in the sum is $\nu=(\emptyset,\lambda_1,\lambda_1+\lambda_2,\dots,|\lambda|)$, yielding $\prod_{a=1}^{n}\left[\begin{smallmatrix}\lambda_1+\cdots+\lambda_{a+1}\\ \lambda_1+\cdots+\lambda_a\end{smallmatrix}\right]$ which is the multinomial (3.5). Perhaps not immediately evident are the symmetries [37]

$$\begin{bmatrix}L\\ \lambda\end{bmatrix}=\begin{bmatrix}L'\\ |L|(1^n)-\lambda\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}L\\ \lambda\end{bmatrix}=\begin{bmatrix}L\\ \sigma(\lambda)\end{bmatrix},$$

where $L'=(L_{n-1},\dots,L_1)$ and $\sigma\in S_n$.

The (completely antisymmetric) A_{n-1} supernomials have several interesting interpretations. In ref. [19] they were defined as

$$\begin{bmatrix}L\\ \lambda\end{bmatrix}=\sum_{\eta\vdash|\lambda|}K_{\eta\lambda}K_{\eta'\mu}(q),$$

where $\mu=(1^{L_1}\dots(n-1)^{L_{n-1}})$, $\lambda\in\mathbb{Z}^n$ a composition such that $|\lambda|=|\mu|$, and $K_{\lambda\mu}(q)$ and $K_{\lambda\mu}$ the Kostka polynomial and Kostka number, respectively [24]. In [21] this was shown to imply that the supernomials are connection coefficients between the elementary symmetric functions e_λ and the Hall–Littlewood

[1]Viewing the $\nu^{(a)}$'s as partitions, this means that $|\nu^{(a)}|-|\nu^{(a-1)}|=\lambda_a$ and $\nu_i^{(a)}\le\nu_{i-1}^{(a-1)}$ (i.e., the ith part of $\nu^{(a)}$ does not exceed the $(i-1)$th part of $\nu^{(a-1)}$).

polynomials P_λ in n variables [24], thanks to

$$e_\lambda(x_1,\dots,x_n)=\sum_{\mu\vdash|\lambda|}\Bigl(\sum_{\eta\vdash|\lambda|}K_{\eta\lambda}K_{\eta'\mu}(q)\Bigr)P_\mu(x_1,\dots,x_n;q).$$

In [37] the supernomials were introduced from a combinatorial point of view as the generating functions of inhomogeneous lattice paths, generalizing the fact that the multinomial coefficient (3.5) is the major index generating function on words over the alphabet $\{1,\dots,n\}$.

If we now restrict (3.6) to $n=3$, and set $\nu^{(1)}=\lambda_1$, $\nu^{(2)}=(\lambda_1+m,\lambda_2-m)$ and $\nu^{(3)}=(L_1+L_2,L_2)$ we get

$$\begin{bmatrix}L\\ \lambda\end{bmatrix}=\sum_m\begin{bmatrix}\lambda_1-\lambda_2+2m\\ m\end{bmatrix}\begin{bmatrix}L_1\\ \lambda_1-L_2+m\end{bmatrix}\begin{bmatrix}L_2\\ \lambda_2-m\end{bmatrix}$$

for $\lambda_1+\lambda_2+\lambda_3=L_1+2L_2$ and zero otherwise. The following, more symmetric, representation may be derived using the q-Chu–Vandermonde sum,

$$\begin{bmatrix}L\\ \lambda\end{bmatrix}=\sum_r\frac{q^{r_1r_{23}}(q)_{L_1}(q)_{L_2}}{(q)_{r_1}(q)_{r_2}(q)_{r_3}(q)_{r_{12}}(q)_{r_{13}}(q)_{r_{23}}},$$

where the summation over r denotes a sum over $r_1,\dots,r_{23}$ such that

$$r_1+r_{12}+r_{13}=\lambda_1,\quad r_2+r_{12}+r_{23}=\lambda_2,\quad r_3+r_{13}+r_{23}=\lambda_3$$

and

$$r_1+r_2+r_3=L_1,\quad r_{12}+r_{13}+r_{23}=L_2.$$

3.3. **An A_2 invariance property.** We now show how, in the case of A_2, the supernomials may be used the generalize the q-binomial invariance (3.2). The first step is, of course, to again shift and normalize the supernomials, and for general rank we define in analogy with (2.1) and (3.1),

$$S(L,k)=\frac{1}{(q)_{CL}}\begin{bmatrix}CL\\ L_{n-1}(1^n)-k\end{bmatrix}\tag{3.7}$$

with $L\in\mathbb{Z}_+^{n-1}$ and $k\in\mathbb{Z}^n$ such that $|k|=0$. Observe that $\sum_{i=1}^n(L_{n-1}-k_i)=nL_{n-1}=\sum_{a=1}^{n-1}a(CL)_a$ so that the condition $|\lambda|=|\nu^{(n)}|$ in definition (3.1) is automatically satisfied.

Considering A_2 again, we would like to show that the following invariance property holds (compare with (3.2))

$$\sum_{L=0}^{M}\frac{q^{\frac12 LCL}S(L,k)}{(q)_{M-L}}=q^{\frac12(k_1^2+k_2^2+k_3^2)}S(M,k).\tag{3.8}$$

The analogy with A_1 breaks down, however, and a somewhat unexpected (to us at least) result arises as follows (theorem 4.3 of [9] with $a=1$).

Theorem 3.2. *Let $L, M \in \mathbb{Z}_+^2$, $k \in \mathbb{Z}^3$, such that $|k| = 0$ and let $S(L,k)$ be the A_2 supernomial defined in* (3.7) *and $T(L,k)$ be defined in* (3.11) *below. Then*

$$\sum_{L=0}^{M} \frac{q^{\frac{1}{2}LCL} S(L,k)}{(q)_{M-L}} = q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)} (q)_{|M|} T(M,k) \tag{3.9}$$

and

$$\sum_{L=0}^{M} \frac{q^{\frac{1}{2}LCL} T(L,k)}{(q)_{M-L}} = q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)} T(M,k). \tag{3.10}$$

So we do find an invariance property, but only after summing the supernomial S to a new q-function T, given in the following definition.

Definition 3.3. *For $L \in \mathbb{Z}_+^2$ and $k \in \mathbb{Z}^3$ such that $|k| = 0$,*

$$T(L,k) = \frac{1}{(q)_{L_1+L_2}^2} \prod_{i=1}^{3} \begin{bmatrix} L_1 + L_2 \\ L_1 + k_i \end{bmatrix}. \tag{3.11}$$

The fact that (3.8) is not correct and has to be replaced by the non-trivial theorem 3.2 is the main obstacle for treating the general rank case. Indeed, for arbitrary A_{n-1} we find that ($L, M \in \mathbb{Z}_+^{n-1}$, $k \in \mathbb{Z}^n$ such that $|k| = 0$)

$$\sum_{L=0}^{M} \frac{q^{\frac{1}{2}LCL} S(L,k)}{(q)_{M-L}} = q^{\frac{1}{2}(k_1^2+\cdots+k_n^2)} S(M,k)$$

is invalid for any $n \geq 3$. How to correct this, in a way similar to Theorem 3.2, is unclear to us at present. A partial result on A_{n-1} is given in proposition 6.1 of section 6.

4. A_2 Rogers–Ramanujan-type identities

We now use the two summations of theorem 3.2 to obtain A_2 analogues of theorems 2.1 and 2.2. Were there 2 cases to consider for A_1, corresponding to odd and even modulus, this time we have to consider moduli in the residue classes of 3.

First we need the A_2 generalization of (3.3) (proposition 5.1 of [9] with $\ell = 0$).

Proposition 4.1. *For $L \in \mathbb{Z}^2$ such that $CL \in \mathbb{Z}_+^2$,*

$$\sum_{|k|=0} \sum_{\sigma \in S_3} \epsilon(\sigma) q^{\frac{1}{2}\sum_{i=1}^{3}(3k_i - 2\sigma_i)k_i} S(L, 3k - \sigma + \rho) = \delta_{L_1,0}\delta_{L_2,0}. \tag{4.1}$$

We now invoke theorem 3.2. First this gives, thanks to (3.9), a doubly bounded version of the A_2 Euler identity,

$$\sum_{|k|=0}\sum_{\sigma\in S_3}\epsilon(\sigma)q^{\frac{1}{2}\sum_{i=1}^{3}(3k_i-2\sigma_i)k_i+(3k_i-\sigma_i+i)^2}T(L,3k-\sigma+\rho)=\frac{1}{(q)_L(q)_{|L|}}. \tag{4.2}$$

Next we can apply (3.10) and after an $(\ell-1)$-fold iteration we arrive at

$$\sum_{|k|=0}\sum_{\sigma\in S_3}\epsilon(\sigma)q^{\frac{1}{2}\sum_{i=1}^{3}(3k_i-2\sigma_i)k_i+\ell(3k_i-\sigma_i+i)^2}T(L,3k-\sigma+\rho)$$
$$=\sum_{n_1,\dots,n_{\ell-1}\in\mathbb{Z}_+^2}\frac{q^{\frac{1}{2}\sum_{j=1}^{\ell-1}N_jCN_j}}{(q)_{L-N_1}(q)_{n_1}\cdots(q)_{n_{\ell-1}}(q)_{|n_{\ell-1}|}},$$

where $N_j=n_j+\cdots+n_{\ell-1}\in\mathbb{Z}_+^2$. To transform this into identities of the Rogers–Ramanujan type we let L_1,L_2 tend to infinity and apply the A_2 Macdonald identity [23] (in a representation of [25])

$$\sum_{|k|=0}\sum_{\sigma\in S_3}\epsilon(\sigma)\prod_{i=1}^{3}q^{\frac{3}{2}k_i^2+\sigma_ik_i}x_i^{3k_i+\sigma_i-i}=(q)_\infty^2\prod_{1\le i<j\le 3}(x_ix_j^{-1},qx_jx_i^{-1})_\infty. \tag{4.3}$$

This leads to the following A_2 analogue of the identities (2.5) (theorem 5.1 of [9] with $i=k$).

Theorem 4.2. *For $|q|<1$, $k\ge 2$ and $N_j=n_j+\cdots+n_{k-1}$,*

$$\sum_{n_1,\dots,n_{k-1}\in\mathbb{Z}_+^2}\frac{q^{\frac{1}{2}\sum_{j=1}^{k-1}N_jCN_j}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q)_{|n_{k-1}|}}$$
$$=\frac{(q^k,q^k,q^{k+1},q^{2k},q^{2k+1},q^{2k+1},q^{3k+1},q^{3k+1};q^{3k+1})_\infty}{(q)_\infty^3}.$$

Although this theorem is indeed very much akin to theorem (2.1), there is a striking (as well as annoying) difference. This is the fact that on the right-hand side we have a $(q)_\infty^3$ in the denominator where one would have liked to see a $(q)_\infty^2$. Indeed, to interpret the right-hand side combinatorially, we have to first multiply with $(q)_\infty$. Then the right-hand side becomes the generating function of pairs of partitions (λ_1,λ_2) such that λ_1 has no parts congruent to $0,\pm k,\pm 2k \pmod{3k+1}$ and λ_2 has no parts congruent to $0,\pm k \pmod{3k+1}$. Also, the right-hand side can (again after multiplication by $(q)_\infty$) be identified with a character of the W_3 algebra. The conclusion clearly is that the left-hand side, when multiplied with $(q)_\infty$ is a series with only positive integer coefficients. How to make this manifest is unclear to us. Only when $k=2$ we have succeeded (section 5.4 of [9]) in rewriting the above identity when multiplied with $(q)_\infty$ such that both sides are manifestly positive series. The price for this, however, is that the $\mathbb{Z}_2$ symmetry of the summand is broken.

Theorem 4.3 (A_2 Rogers–Ramanujan identity). *For* $|q|<1$,

$$(q)_\infty \sum_{r_1,r_2\geq 0} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q)_{r_1}(q)_{r_2}(q)_{r_1+r_2}} = \sum_{r_1,r_2\geq 0} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q)_{r_1}} \begin{bmatrix} 2r_1 \\ r_2 \end{bmatrix}$$
$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{7n+1})^2(1-q^{7n+3})(1-q^{7n+4})(1-q^{7n+6})^2}.$$

It seems a worthwhile exercise to find a partition theoretic interpretation for the middle term of this A_2 Rogers–Ramanujan identity, or, even better, to rewrite the left-hand side into a series that is manifestly of A_2-type as well as manifestly positive, and to then find a combinatorial interpretation.

To obtain identities of the Rogers–Ramanujan type for moduli congruent to 2 modulo 3, we replace q by $1/q$ in the A_2 Euler identity (4.2). Using

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{1/q} = q^{-mn} \begin{bmatrix} m+n \\ n \end{bmatrix}$$

and definition (3.11) of T this yields

$$\sum_{|k|=0} \sum_{\sigma\in S_3} \epsilon(\sigma) q^{\frac{1}{2}\sum_{i=1}^3 (3k_i-\sigma_i+i)^2-3(k_i-2\sigma_i)k_i} T(L, 3k-\sigma+\rho) = \frac{q^{2L_1L_2}}{(q)_L(q)_{|L|}}.$$

Iterating this $\ell-1$ times using equation (3.10) leads to

$$\sum_{|k|=0} \sum_{\sigma\in S_3} \epsilon(\sigma) q^{\frac{1}{2}\sum_{i=1}^3 \ell(3k_i-\sigma_i+i)^2-3(k_i-2\sigma_i)k_i} T(L, 3k-\sigma+\rho)$$
$$= \sum_{n_1,\dots,n_{\ell-1}\in\mathbb{Z}_+^2} \frac{q^{\frac{1}{2}\sum_{j=1}^{\ell-2} N_jCN_j+\frac{1}{2}N_{\ell-1}BN_{\ell-1}}}{(q)_{L-N_1}(q)_{n_1}\cdots(q)_{n_{\ell-1}}(q)_{|n_{\ell-1}|}},$$

where $B_{i,j}=2\delta_{i,j}+\delta_{|i-j|,1}$, $i,j=1,2$. Letting L_1, L_2 go to infinity and using the Macdonald identity (4.3) gives the following theorem (theorem 5.3 of [9] with $i=k$).

Theorem 4.4. *For* $|q|<1$, $k\geq 2$ *and* $N_j = n_j+\cdots+n_{k-1}$,

$$\sum_{n_1,\dots,n_{k-1}\in\mathbb{Z}_+^2} \frac{q^{\frac{1}{2}\sum_{j=1}^{k-2} N_jCN_j+\frac{1}{2}N_{k-1}BN_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q)_{|n_{k-1}|}}$$
$$= \frac{(q^{k-1},q^k,q^k,q^{2k-1},q^{2k-1},q^{2k},q^{3k-1},q^{3k-1};q^{3k-1})_\infty}{(q)_\infty^3}.$$

For $k=2$ the above identity is the first Rogers–Ramanujan identity (2.6) in disguise [9].

It remains to find identities for moduli congruent to 0 (mod 3). What is needed now is the A_2 analogue of identity (3.4) provided by Gessel and Krattenthaler (equation (6.18) of [17]).

Proposition 4.5. *For $L \in \mathbb{Z}_+^2$,*

$$\sum_{|k|=0}\sum_{\sigma\in S_3}\epsilon(\sigma)q^{\frac{1}{2}\sum_{i=1}^{3}(3k_i-\sigma_i+i)^2}T(L,3k-\sigma+\rho)=\frac{(q^3;q^3)_{|L|}}{(q^3;q^3)_L(q)_{|L|}^2}. \tag{4.4}$$

Applying theorem 3.2 this readily gives

$$\sum_{|k|=0}\sum_{\sigma\in S_3}\epsilon(\sigma)q^{\frac{\ell}{2}\sum_{i=1}^{3}(3k_i-\sigma_i+i)^2}T(L,3k-\sigma+\rho)$$
$$=\sum_{n_1,\dots,n_{\ell-1}\in\mathbb{Z}_+^2}\frac{q^{\frac{1}{2}\sum_{j=1}^{\ell-1}N_jCN_j}(q^3;q^3)_{|n_{\ell-1}|}}{(q)_{L-N_1}(q)_{n_1}\cdots(q)_{n_{\ell-2}}(q^3;q^3)_{n_{\ell-1}}(q)_{|n_{\ell-1}|}^2}.$$

When L_1, L_2 approach infinity this yields our final Rogers–Ramanujan-type theorem (theorem 5.4 of [9] with $i=k$).

Theorem 4.6. *For $|q|<1$, $k\geq 2$ and $N_j=n_j+\dots+n_{k-1}$,*

$$\sum_{n_1,\dots,n_{k-1}\in\mathbb{Z}_+^2}\frac{q^{\frac{1}{2}\sum_{j=1}^{k-1}N_jCN_j}(q^3;q^3)_{|n_{k-1}|}}{(q)_{n_1}\cdots(q)_{n_{k-2}}(q^3;q^3)_{n_{k-1}}(q)_{|n_{k-1}|}^2}$$
$$=\frac{(q^k,q^k,q^k,q^{2k},q^{2k},q^{2k},q^{3k},q^{3k};q^{3k})_\infty}{(q)_\infty^3}.$$

5. Reduction and inversion

So far, we have used the A_1 and A_2 summations (2.2), (3.9) and (3.10) to derive complicated identities out of simpler ones. Of, course, when given a complicated identity, it is of interest to know whether this identity is reducible to a simpler one. That is, whether, iteration of some yet unknown simpler identity produces the complicated identity. The answer to this question is easily given, and in the case of A_1 the following result holds [7] (which actually is the $q\to 1/q$ version of (2.2))

$$\sum_{L=0}^{M}\frac{(-1)^{M-L}q^{\binom{M-L}{2}-M^2}}{(q)_{M-L}}S(L,r)=q^{-r^2}S(M,r). \tag{5.1}$$

Iterating this identity using the invariance property (2.2) implies

$$\sum_{M=0}^{N}\frac{q^{M^2}}{(q)_{N-M}}\sum_{L=0}^{M}\frac{(-1)^{M-L}q^{\binom{M-L}{2}-M^2}}{(q)_{M-L}}S(L,r)=S(N,r).$$

Interchanging the two sums on the left-hand side and then shifting $M \to M+L$ gives

$$\sum_{L=0}^{N} \frac{S(L,r)}{(q)_{N-L}} \sum_{M=0}^{N-L} (-1)^M q^{\binom{M}{2}} \begin{bmatrix} N-L \\ M \end{bmatrix} = S(N,r),$$

which is indeed true since, by (2.3), the inner sum yields $\delta_{L,N}$. (Of course, applying (5.1) to (2.2), instead of (2.2) to (5.1) is consistent with the above.)

In exactly the same way one can establish that

$$\sum_{L=0}^{M} \Bigl(\prod_{i=1}^{2} \frac{(-1)^{M_i-L_i} q^{\binom{M_i-L_i}{2}}}{(q)_{M_i-L_i}}\Bigr) q^{-\frac{1}{2}MCM} (q)_{|L|} T(L,k) = q^{-\frac{1}{2}(k_1^2+k_2^2+k_3^2)} S(M,k)$$

and

$$\sum_{L=0}^{M} \Bigl(\prod_{i=1}^{2} \frac{(-1)^{M_i-L_i} q^{\binom{M_i-L_i}{2}}}{(q)_{M_i-L_i}}\Bigr) q^{-\frac{1}{2}MCM} T(L,k) = q^{-\frac{1}{2}(k_1^2+k_2^2+k_3^2)} T(M,k).$$

(The second equation can also be found by taking $q \to 1/q$ in (3.10).) Using this one can see that the two initial condition identities (4.1) and (4.4) are indeed "maximally reduced". To further reduce (4.4), for example, one would have to find "nice" representations for

$$\sum_{L=0}^{M} \Bigl(\prod_{i=1}^{2} \frac{(-1)^{M_i-L_i} q^{\binom{M_i-L_i}{2}}}{(q)_{M_i-L_i}}\Bigr) q^{-\frac{1}{2}MCM} \frac{(q^3;q^3)_{|L|}}{(q^3;q^3)_L (q)_{|L|}^{2-\tau}},$$

with τ either 0 or 1. This appears not to be possible. (Hence the analogy with A_1 breaks down in this case, since (2.7) can be reduced to $\sum_r (-1)^r S(L,r) = (-1)^L/(q^2;q^2)_L$ using (5.1) and the q-Chu–Vandermonde sum.)

Another question of interest is that of inversion. For A_1 it can be stated as follows: find a function $\bar{S}(r,L)$ such that

$$\sum_{L\geq 0} \bar{S}(r,L) S(L,s) = \delta_{r,s} \quad \text{and} \quad \sum_{r\geq 0} S(L,r)\bar{S}(r,M) = \delta_{L,M}. \tag{5.2}$$

In [4] Andrews provides the solution,

$$\bar{S}(r,L) = (-1)^{r-L} q^{\binom{r-L}{2}} (1-q^{2r}) \frac{(q)_{L+r-1}}{(q)_{r-L}},$$

with $\bar{S}(0,0)=1$. This can be used to find further identities of the type

$$\sum_{r\geq 0} \alpha_r S(L,r) = \beta_L, \tag{5.3}$$

since (5.2) and (5.3) imply

$$\alpha_r = \sum_{L\geq 0} \bar{S}(r,L)\beta_L.$$

For example, taking $\beta_L = \delta_{L,0}$ immediately yields $\alpha_r = \bar{S}(r,0) = (-1)^r q^{\binom{r}{2}}(1+q^r)$ and $\alpha_0 = 1$, which gives identity (2.3).

The A_2 analogue of the inversion formula (5.2) can be stated as follows:

$$\sum_{L_1,L_2\geq 0} \bar{S}(k,L)S(L,k') = \delta_{k,k'} \quad \text{and} \quad \sum_{\substack{k_1\geq k_2\geq k_3\\ |k|=0}} S(L,k)\bar{S}(k,L') = \delta_{L,L'}. \tag{5.4}$$

To see that the inverse supernomial $\bar{S}(L,k)$ exists we observe that $S(L,k)$ (for $k_1 \geq k_2 \geq k_3$ and $|k| = 0$) is non-zero if and only if $2L_1 \geq L_2$, $2L_2 \geq L_1$, $k_3 \geq -L_1$ and $k_1 \leq L_2$. Hence, if we define $K = (k_1 + k_2, k_1)$ and write $S(L,K)$ instead of $S(L,k)$ it follows that $S(L,K)$ is non-zero if and only if

$$K_1 \leq L_1, \quad K_2 \leq L_2 \tag{5.5}$$

and

$$2K_1 \geq K_2, \quad 2K_2 \geq K_1, \quad 2L_1 \geq L_2, \quad 2L_2 \geq L_1. \tag{5.6}$$

Consequently, if we view $S(L,K)$ as an entry of an infinite-dimensional matrix with L and K in the ranges given by (5.6), then equation (5.5) implies that S is invertible. We have computed $\bar{S}(K,L)$ for many different L and K, and despite the fact that we identified $\bar{S}(K,L)$ for all $K = (r, 2r-p)$ with $p = 0,\dots,3$ we failed to observe enough regularity to guess (and then prove, of course) a formula for arbitrary K. For those in for a challenge, here are the cases $p = 0$ and 1, ($\bar{S}((0,0),(0,0)) = 1$)

$$\bar{S}((r,2r),L) = (-1)^{L_2} q^{\binom{r-L_1}{2}+\binom{r+L_1-L_2}{2}}(1-q^{3r})\frac{(q)_{L_2+r-1}}{(q)_{r-L_1}}$$

and

$$\begin{aligned}\bar{S}((r,2r-1),L) &= (-1)^{L_2+1} q^{\binom{r-L_1}{2}+\binom{r+L_1-L_2-1}{2}}(1-q^{3r-2})\frac{(q)_{L_2+r-2}(q)_{L_1-L_2+r}}{(q)_{r-L_1}(q)_{L_1-L_2+r-1}}\\ &\quad - (-1)^{L_2+1} q^{\binom{r-L_1-1}{2}+\binom{r+L_1-L_2-1}{2}+3r-3}\frac{(q)_{L_2+r-2}(q)_1}{(q)_{r-L_1-1}},\end{aligned}$$

both for $r \geq 1$. (To correctly get $\bar{S}((1,1),(0,0))$, first set $L_1 = L_2 = 0$ and simplify to $\bar{S}((2,2r-1),(0,0)) = -q^{r(r-2)}(q+q^r+q^{2r})$. Then set $r = 1$.)

The situation for $T(L,k)$ is much simpler, in that an inverse does not exists. Indeed, assuming again that $k_1 \geq k_2 \geq k_3$ (and, of course, $|k| = 0$), and writing $T(L,K)$, we find that $T(L,K)$ is non-zero if and only if (5.5) and

$$2K_1 \geq K_2, \quad 2K_2 \geq K_1, \quad L_1 \geq 0, \quad L_2 \geq 0 \tag{5.7}$$

hold. Hence, viewing T as an infinite-dimensional matrix with rows indexed by L and columns by K, with ranges given by (5.7), T is no longer invertible, its rows and columns ranging over different (infinite) sets. (A right and/or a left inverse might of course exist, but the highly overdetermined set of equations does not admit a solution.)

6. Higher rank and level

Perhaps the two most important open questions are those of generalizing theorem (3.2) to A_{n-1} and to higher level.

First addressing the A_{n-1} problem, we would like to establish the arbitrary rank version of the A_1 sum (2.2) and the A_2 summations (3.9) and (3.10). As mentioned in section 3.3, the non-trivial nature of theorem 3.2, has so-far prevented us from making much progress in this direction. The only general result that we have established can be stated as follows.

Proposition 6.1. *Let $L\in\mathbb{Z}_+^{n-1}$ and $k\in\mathbb{Z}^n$ such that $|k|=0$ and let $S(L,k)$ be the A_{n-1} supernomial of equation* (3.7)*. Then, for $M_1,M_{n-1}\in\mathbb{Z}_+$,*

$$\sum_{L\in\mathbb{Z}_+^{n-1}}\frac{q^{\frac12 LCL}S(L,k)}{(q)_{M_1-L_1}(q)_{M_{n-1}-L_{n-1}}}=\frac{q^{\frac12\sum_{i=1}^n k_i^2}(q)^{n-1}_{M_1+M_{n-1}}}{\prod_{i=1}^n(q)_{M_1+k_i}(q)_{M_{n-1}-k_i}}. \tag{6.1}$$

The proof of this only requires the q-Chu–Vandermonde sum and will be presented in the appendix. Letting M_1,M_{n-1} tend to infinity, this result yields an A_{n-1} version of what is referred to in ref. [8] as the weak form of Bailey's lemma,

$$\sum_{L\in\mathbb{Z}_+}q^{\frac12 LCL}S(L,k)=\frac{q^{\frac12\sum_{i=1}^n k_i^2}}{(q)_\infty^{n-1}}. \tag{6.2}$$

Thus, given a supernomial identity one may derive a new q-series identity by the above summation, but one cannot iterate ad infinitum.

For A_3 we further have the following isolated result.

Proposition 6.2. *Let $L\in\mathbb{Z}_+^3$ and $k\in\mathbb{Z}^4$ such that $|k|=0$ and let $S(L,k)$ be the A_3 supernomial of equation* (3.7)*. Then, for $M_2\in\mathbb{Z}_+$,*

$$\sum_{L\in\mathbb{Z}_+^3}\frac{q^{\frac12 LCL}S(L,k)}{(q)_{M_2-L_2}}=\frac{q^{\frac12\sum_{i=1}^4 k_i^2}(q)^2_{2M_2}}{\prod_{1\le i<j\le 4}(q)_{M_2+k_i+k_j}}. \tag{6.3}$$

But even for A_3 we have not found a sufficiently simple expression for the more general

$$\sum_{L\in\mathbb{Z}_+^3}\frac{q^{\frac12 LCL}S(L,k)}{(q)_{M-L}}$$

(with $M \in \mathbb{Z}_+^3$) to suggest how (after taking out possible factors) to iterate further.

Another important problem is to generalize (6.2) to higher levels. Here the observation is that $1/(q)_\infty^{n-1}$ (divided by $q^{(n-1)/24}$) is the level-1 $\mathrm{A}_{n-1}^{(1)}$ string function. It is thus natural to ask for a generalization of (6.2) involving level-N A_{n-1} string functions [20]. The simplest such functions admit the following representation

$$C_k(q) = \frac{1}{(q)_\infty^{n-1}} \sum_\eta \frac{q^{\frac{1}{2}\eta(C\otimes C^{-1})\eta}}{(q)_\eta},$$

for $k \in \mathbb{Z}^n$ such that $|k| = 0$. Here η is a vector in the tensor-product space $\mathbb{Z}^{n-1} \otimes \mathbb{Z}^{N-1}$ with entries $\eta_j^{(a)}$, $a = 1, \dots, n-1$, $j = 1, \dots, N-1$, $C \otimes C^{-1}$ denotes the tensor product of the A_{n-1} Cartan matrix and the inverse A_{N-1} Cartan matrix, and the sum is over η such that

$$\frac{\sum_{i=a}^{n-1} k_i}{N} - \sum_{j=1}^{N-1} C_{1,j}^{-1} \eta_j^{(a)} \in \mathbb{Z}, \qquad a = 1, \dots, n-1.$$

Multiplied by $q^{-(n^2-1)N/24(N+n)}$, C_k is the string function C_μ^Λ in the representation of Georgiev [16], with $\Lambda = N\Lambda_0$ and $\mu = \sum_{a=1}^{n-1}(k_a - k_{a-1})\bar{\Lambda}_a$ ($k_0 = k_n$). It was conjectured in [37] (equation (9.9) with $\ell = \lambda = \sigma = 0$) that the following identity holds.

Conjecture 6.3. *For $n \geq 2$, $N \geq 1$ and $k \in \mathbb{Z}^n$ such that $|k| = 0$,*

$$\sum_{L \in \mathbb{Z}_+^{n-1}} q^{\frac{1}{2N} LCL} S(L,k) \sum_\eta q^{\frac{1}{2}\eta(C\otimes C^{-1})\eta} \begin{bmatrix} \mu + \eta \\ \eta \end{bmatrix} = q^{\frac{1}{2N}(k_1^2 + \cdots + k_n^2)} C_k(q). \tag{6.4}$$

Here the following notation is employed on the left-hand side. The sum over $\eta \in \mathbb{Z}^{n-1} \otimes \mathbb{Z}^{N-1}$ denotes a sum such that

$$\frac{L_a}{N} - \sum_{j=1}^{N-1} C_{1,j}^{-1} \eta_j^{(a)} \in \mathbb{Z}, \qquad a = 1, \dots, n-1.$$

The vector μ is fixed by η through the equation

$$(C \otimes I)\eta + (I \otimes C)\mu = CL \otimes e_{N-1},$$

where I is the identity matrix and e_j is the jth standard unit vector. For $v, w \in \mathbb{Z}^p$, $\begin{bmatrix} v+w \\ v \end{bmatrix} = \prod_{i=1}^p \begin{bmatrix} v_i + w_i \\ v_i \end{bmatrix}$. A proof of conjecture 6.3 for $n = 2$ has been given in [36].

The simplest application of the previous propositions and conjecture requires the A_{n-1} form of equations (3.3) and (4.1).

Proposition 6.4. *For $L \in \mathbb{Z}^{n-1}$ such that $CL \in \mathbb{Z}_+^{n-1}$,*

$$\sum_{|k|=0} \sum_{\sigma \in S_n} \epsilon(\sigma) q^{\frac{1}{2}\sum_{i=1}^n (nk_i - 2\sigma_i)k_i} S(L, nk - \sigma + \rho) = \delta_{L_1,0} \dots \delta_{L_{n-1},0}. \tag{6.5}$$

A proof using crystal base theory has recently been obtained by Schilling and Shimozono (equation. (6.6) of [35]). If we apply proposition 6.1 to this identity we find a doubly bounded A_{n-1} Euler identity,

$$\sum_{|k|=0} q^{\binom{n+1}{2}\sum_{i=1}^n k_i^2 + (n+1)\sum_{i=1}^n ik_i} \det_{1\le i,j\le n} \left(q^{j(j-i-nk_i)} \begin{bmatrix} M+M' \\ M+i-j+nk_i \end{bmatrix} \right) = \begin{bmatrix} M+M' \\ M \end{bmatrix},$$

where we have traded the sum over S_n for a determinant. Using some results of [17] this can be rewritten in q-hypergeometric notation as

$$\sum_{|k|=S} q^{\binom{n+1}{2}\sum_{i=1}^n k_i^2 + \sum_{i=1}^n ik_i} \prod_{1\le i<j\le n} (1 - q^{nk_j - nk_i + j - i}) \times \prod_{i=1}^n \frac{(q)_{M+M'+i-1}}{(q)_{M+nk_i+i-1}(q)_{M'-nk_i-i+n}} = (-1)^{(n-1)S} q^{(n+1)\binom{S+1}{2}} \begin{bmatrix} M+M' \\ M+S \end{bmatrix}. \tag{6.6}$$

This generalizes Milne's theorem 1.9 of [26] (or theorem 6.1 of [27]), which is recovered when $M = 0$ and $M' \to \infty$. It also generalizes theorem 22 of Gessel and Krattenthaler [17] which corresponds to (6.6) with $M' \to \infty$. A proof of (6.6) based on Milne and Lilly's A_{n-1} analogue of Watson's q-Whipple transform [29] has recently been found by Krattenthaler [22].

If instead of (6.1) we use (6.3) it follows that

$$\sum_{|k|=S} q^{10\sum_{i=1}^4 k_i^2 + \sum_{i=1}^4 ik_i} \prod_{1\le i<j\le 4} \frac{1 - q^{4k_j - 4k_i + j - i}}{(q)_{M+4k_i+4k_j+i+j-3}} = \frac{(-1)^S q^{5\binom{S+1}{2}}}{(q)_{2M+4S+4}(q)_{2M+4S+2}(q)_{M+2S}} \tag{6.7}$$

which we have failed to recognize as a (generalization of a) known A_3 q-hypergeometric identity. In fact, (6.7) is very misleading in that it incorrectly hints at the possibility to sum

$$\sum_{|k|=S} q^{\binom{n+1}{2}\sum_{i=1}^n k_i^2 + \sum_{i=1}^n ik_i} \prod_{1\le i<j\le n} (1 - q^{nk_j - nk_i + j - i}) \times \prod_{1\le i_1<\dots<i_p\le n} \frac{1}{(q)_{M+nk_{i_1}+\dots+nk_{i_p}+i_1+\dots+i_p-\binom{p+1}{2}}}.$$

Computer experimentations reveal that only for $p = 1$ and $p = n-1$ (equation (6.6) with $M' \to \infty$ and (6.6) with $M \to \infty$) and for $n = 3$, $p = 2$ this is the case.

Finally we note that a level-N A_{n-1} Euler identity is obtained if we sum (6.5) by application of (6.4),

$$\sum_{|k|=0}\sum_{\sigma\in S_n}\epsilon(\sigma)q^{\frac{1}{2}\sum_{i=1}^n(nk_i-2\sigma_i)k_i+\frac{1}{N}(nk_i-\sigma_i+i)^2}C_{nk-\sigma+\rho}(q)=1. \tag{6.8}$$

7. Supernomial identities and beyond?

So far we have given two applications of theorem 3.2, based on the initial condition identities (4.1) and (4.4). Many more identities can however be derived. In ref. [37] an infinite hierarchy of A_2 supernomial identities was conjectured, of which (4.1) is the first instance. Taking this conjectured hierarchy as input to theorem 3.2 leads to a doubly-infinite family of A_2 q-series identities. We will not carry out this programme in full here, but shall instead make some intriguing observations concerning some of the identities that may be derived. In fact, since all will be conjectural, we shall present our speculations in a more general A_{n-1} setting.

First we need the conjecture of [37] (equation (9.2) with $q\to 1/q$ and $N=1$). Using the tensor-product notation of the previous section we define for all integers $p\geq n$,

$$F_{p,L}(q)=\frac{q^{\frac{LCL}{2(p-n)}}}{(q)_{CL}}\sum_{\mu}q^{\frac{1}{2}\eta(C\otimes C^{-1})\eta}\begin{bmatrix}\mu+\eta\\ \mu\end{bmatrix}.$$

Here $L\in\mathbb{Z}_+^{n-1}$, $C\otimes C^{-1}$ is the tensor product of the A_{n-1} and inverse A_{p-n-1} Cartan matrices, and the sum is over $\mu\in\mathbb{Z}^{n-1}\otimes\mathbb{Z}^{p-n-1}$, with entries $\mu_j^{(a)}$, $a=1,\dots,n-1$, $j=1,\dots,p-n-1$, such that

$$(C^{-1}\otimes I)\mu\in\mathbb{Z}^{p-n-1}. \tag{7.1}$$

The vector η is determined by L and μ through the relation

$$\eta=L\otimes e_1-(C^{-1}\otimes C)\mu. \tag{7.2}$$

As special cases we have

$$F_{n,L}(q)=\delta_{L_1,0}\dots\delta_{L_{n-1},0}$$

and

$$F_{n+1,L}(q)=\frac{q^{\frac{1}{2}LCL}}{(q)_{CL}}.$$

Conjecture 7.1. *For $n\geq 2$, $p\geq n$, $L\in\mathbb{Z}_+^{n-1}$ and $k\in\mathbb{Z}^n$ such that $|k|=0$,*

$$\sum_{|k|=0}\sum_{\sigma\in S_n}\epsilon(\sigma)q^{\frac{1}{2}\sum_{i=1}^n(pk_i-2\sigma_i)k_i}S(L,pk-\sigma+\rho)=F_{p,L}(q). \tag{7.3}$$

For $p=n$ the conjecture becomes proposition 6.4 and for $p=n+1$ a proof has been found by Schilling and Shimozono (equation. (6.5) of [35]).

Now we apply proposition 6.1 to this conjecture and replace p by $p-1$. Extending definition (3.11) of T to A_{n-1} by

$$T(L,k)=\frac{1}{(q)^2_{L_1+L_2}}\prod_{i=1}^{n}\begin{bmatrix}L_1+L_2\\ L_1+k_i\end{bmatrix}$$

for $|k|=0$, this gives

$$\sum_{|k|=0}\sum_{\sigma\in S_n}\epsilon(\sigma)q^{\phi_{p,k,\sigma}}T(M,(p-1)k-\sigma+\rho)\\ =\frac{1}{(q)_{|M|}}\sum_{\mu}\frac{q^{\frac12\mu(C^{-1}\otimes C)\mu}}{(q)_{M_1-\sum_b C^{-1}_{1,b}\mu_1^{(b)}}(q)_{M_2-\sum_b C^{-1}_{n-1,b}\mu_1^{(b)}}}\begin{bmatrix}\mu+\eta\\ \mu\end{bmatrix}, \tag{7.4}$$

where the sum over μ is again restricted by (7.1), η is given by (7.2) with $L\to(\infty^{n-1})$ and

$$\phi_{p,k,\sigma}=\frac12\sum_{i=1}^{n}((p-1)k_i-2\sigma_i)k_i+((p-1)k_i-\sigma_i+i)^2.$$

Next we observe that a very similar identity can be derived by replacing $q\to 1/q$ in (7.3). Defining the reciprocal A_{n-1} supernomial T' as

$$T'(L,k)\sim S(L,k;1/q)$$

such that $T'(L,k)=\sum_{j\geq 0}a_jq^j$ with $a_0>0$, we find

$$\sum_{|k|=0}\sum_{\sigma\in S_n}\epsilon(\sigma)q^{\phi_{p,k,\sigma}}T'(L,pk-\sigma+\rho)=\sum_{\mu}\frac{q^{\frac12\mu(C^{-1}\otimes C)\mu}}{(q)_{CL}}\begin{bmatrix}\mu+\eta\\ \mu\end{bmatrix}, \tag{7.5}$$

where (7.1) and (7.2) again apply.

Still this is not the end of the story. In [17] Gessel and Krattenthaler generalized plane partitions to what they termed cylindric partitions. For a special class of these cylindric partitions they derived an expression for the generating function close to the left-hand sides of (7.4) and (7.5). Supported by computer checks, we turn this into the following conjecture. For $L,M\in\mathbb{Z}_+$ and $k,k'\in\mathbb{Z}^n$ such that $|k|=|k'|=0$, set

$$T''(L,M,k,k')=\prod_{i=1}^{n}\begin{bmatrix}M+L-k_i+k'_i\\ L-k_i\end{bmatrix}.$$

Then, for $L,M\in\mathbb{Z}_+$,

$$\sum_{|k|=0}\sum_{\sigma\in S_n}\epsilon(\sigma)q^{\phi_{p,k,\sigma}}T''(L,M,pk-\sigma+\rho,(p-1)k-\sigma+\rho)\\
=\sum_{\mu}q^{\frac{1}{2}\mu(C^{-1}\otimes C)\mu}\begin{bmatrix}M+nL-\sum_b C^{-1}_{1,b}\mu_1^{(b)}\\ nL\end{bmatrix}\begin{bmatrix}\mu+\eta\\ \mu\end{bmatrix}\tag{7.6}$$

with sum over η as in (7.1) and η given by

$$\eta=nL(C^{-1}e_1\otimes e_1)-(C^{-1}\otimes C)\mu.$$

The left-hand side of this identity coincides with the generating function in theorem 3 of [17] with $r\to n$, $\boldsymbol{\lambda}\to(L^n)$, $\boldsymbol{\mu}\to(0^n)$, $d\to p-n$, $\boldsymbol{\alpha}\to(0^{p-n})$, $\boldsymbol{\beta}\to(0,\dots,0,n-p+1)$, $a_i\to M$ and $b_i\to 0$. For $p=n+1$ the above is identity (6.2) of [17].

We note that (7.4), (7.5) and (7.6) are consistent. Specifically, (7.4) with $M_2\to\infty$ and (7.6) with $L\to\infty$ coincide (identifying M_1 with M), and (7.5) with $L=((n-1)L_1,\dots,2L_1,L_1)$ and (7.6) with $M\to\infty$ coincide (identifying L_1 and L).

The three above conjectures strongly suggest the existence of a unifying identity of the form

$$\sum_{|k|=0}\sum_{\sigma\in S_n}\epsilon(\sigma)q^{\phi_{p,k,\sigma}}\mathcal{T}(L,M,pk-\sigma+\rho,(p-1)k-\sigma+\rho)\\
=\sum_{\mu}q^{\frac{1}{2}\mu(C^{-1}\otimes C)\mu}\prod_{a=1}^{n-1}\Bigl(\begin{bmatrix}M_a+(CL)_a-\sum_b C^{-1}_{a,b}\mu_1^{(b)}\\ (CL)_a\end{bmatrix}\Bigr)\begin{bmatrix}\mu+\eta\\ \mu\end{bmatrix}\tag{7.7}$$

with sum over μ such that (7.1) holds, η given by (7.2) and $L,M\in\mathbb{Z}_+^{n-1}$. The generalized supernomial $\mathcal{T}(L,M,k,k')$ (where $L,M\in\mathbb{Z}^{n-1}$, $k,k'\in\mathbb{Z}^n$ and $|k|=|k'|=0$) must satisfy the following consistency conditions:

$$\lim_{CL\to(\infty^{n-1})}\mathcal{T}(L,(M_1,0^{n-3},M_2),k,k')=T((M_1,M_2),k)(q)_{M_1+M_2}$$

$$\lim_{M\to(\infty^{n-1})}\mathcal{T}(L,M,k,k')=T'(L,k)$$

$$\mathcal{T}(((n-1)L_1,\dots,2L_1,L_1),(M_1,0^{n-2}),k,k')=T''(L_1,M_1,k,k').$$

(The first condition applies when $n\geq 3$ only.) A further restriction on the possible form of $\mathcal{T}$ is obtained by observing that the right-hand side is, up to a factor q^{MCL}, invariant under the change $q\to 1/q$, so that

$$\frac{\mathcal{T}(L,M,k,k';1/q)}{\mathcal{T}(L,M,k,k';q)}=q^{-MCL+\sum_{i=1}^n k_ik_i'}.$$

Despite these strong restrictions on $\mathcal{T}$ (especially when $n=3$) we have not succeeded in finding a closed form expression when $n\geq 3$. For $n=2$ the third condition specifies $\mathcal{T}$ and (7.7) has been proven in [14] using the Burge transform [13].

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Appendix A. Proof of proposition 6.1

To prove the proposition we recall the definitions (3.1) and (3.7) of the A_{n-1} supernomials. Inserting these, the left-hand side of (6.1) corresponds to the following multiple sum

$$\sum_{L\in\mathbb{Z}_+^{n-1}}\sum_{\nu}\frac{q^{\frac{1}{2}LCL}}{(q)_{M_1-L_1}(q)_{M_{n-1}-L_{n-1}}(q)_{CL}}\prod_{a=1}^{n-1}\prod_{j=1}^{a}\begin{bmatrix}\nu_j^{(a+1)}-\nu_{j+1}^{(a+1)}\\ \nu_j^{(a)}-\nu_{j+1}^{(a+1)}\end{bmatrix}, \tag{A.1}$$

where the sum over ν denotes a sum over the variables $\nu_j^{(a)}$ for $1\le a\le n$ and $1\le j\le a$ such that

$$\nu_j^{(n)}=L_{n-1}+L_j-L_{j-1},\qquad j=1,\dots,n$$

(with $L_0=L_n=0$) and

$$\sum_{j=1}^{a}\nu_j^{(a)}-\sum_{j=1}^{a-1}\nu_j^{(a-1)}=L_{n-1}-k_a$$

Instead of working with the variables L and $\nu_j^{(a)}$ we find it more convenient to introduce new variables $\mu_j^{(a)}=\nu_j^{(a+1)}-\nu_j^{(a)}$ for $1\le a\le n-1$ and $1\le j\le a$. If we also define the quantities

$$A_j^{(a)}=k_{a+1}-k_j+\sum_{i=1}^{a-j}\mu_j^{(a-i)}-\sum_{i=1}^{j-1}\mu_i^{(j-1)},$$

(again for $1\le a\le n-1$ and $1\le j\le a$) then it is elementary to show the following string of relations:

$$\nu_j^{(a)}=\begin{cases}L_{n-1}-k_a+\mu_j^{(a-1)}+A_j^{(a-1)} & \text{for } j=1,\dots a-1,\\ L_{n-1}-k_a-\sum_{i=1}^{a-1}\mu_i^{(a-1)} & \text{for } j=a,\end{cases}$$

$$\sum_{j=1}^{a}A_j^{(a)}=ak_{a+1}-\sum_{j=1}^{a}k_j,$$

$$L_a=\sum_{j=1}^{a}(\mu_j^{(n-1)}+A_j^{(n-1)}-k_n),$$

and

$$\nu_j^{(n)}-\nu_{j+1}^{(n)}=(CL)_j=\begin{cases}\mu_j^{(n-1)}-\mu_{j+1}^{(n-1)}+A_j^{(n-1)}-A_{j+1}^{(n-1)} & j=1,\dots,n-2,\\ \mu_{n-1}^{(n-1)}+\sum_{i=1}^{n-1}\mu_i^{(n-1)}+A_{n-1}^{(n-1)} & j=n-1.\end{cases}$$

Using all of these, the expression (A.1) can be rewritten as

$$\begin{aligned}
&q^{\frac12\sum_{i=1}^n k_i^2}\sum_{\mu}\frac{q^{\sum_{a=1}^{n-1}\sum_{j=1}^{a}\mu_j^{(a)}(A_j^{(a)}+\sum_{i=1}^{j}\mu_i^{(a)})}}{(q)_{M_1+k_n-A_1^{(n-1)}-\mu_1^{(n-1)}}(q)_{M_{n-1}-k_n-\sum_{j=1}^{n-1}\mu_j^{(n-1)}}}\\
&\quad\times(q)^{-1}_{\mu_{n-1}^{(n-1)}+\sum_{j=1}^{n-1}\mu_j^{(n-1)}+A_{n-1}^{(n-1)}}\prod_{j=1}^{n-2}(q)^{-1}_{\mu_j^{(n-1)}-\mu_{j+1}^{(n-1)}+A_j^{(n-1)}-A_{j+1}^{(n-1)}}\\
&\quad\times\prod_{a=1}^{n-1}\left(\begin{bmatrix}\mu_a^{(a)}+\sum_{j=1}^{a}\mu_j^{(a)}+A_a^{(a)}\\ \mu_a^{(a)}\end{bmatrix}\prod_{j=1}^{a-1}\begin{bmatrix}\mu_j^{(a)}-\mu_{j+1}^{(a)}+A_j^{(a)}-A_{j+1}^{(a)}\\ \mu_j^{(a)}\end{bmatrix}\right).
\end{aligned}$$

Observing that $A_j^{(a)}$ depends on $\mu_k^{(b)}$ with $1\le b\le a-1$ only, we can now successively sum over $\mu^{(n-1)},\dots,\mu^{(1)}$ by repeatedly applying

$$\begin{aligned}
&\sum_{\mu^{(a)}}\frac{q^{\sum_{j=1}^{a}\mu_j^{(a)}(A_j^{(a)}+\sum_{i=1}^{j}\mu_i^{(a)})}}{(q)_{M_1+k_{a+1}-A_1^{(a)}-\mu_1^{(a)}}(q)_{M_{n-1}-k_{a+1}-\sum_{j=1}^{a}\mu_j^{(a)}}}\\
&\qquad\times\frac{1}{(q)_{\mu_a^{(a)}}(q)_{A_a^{(a)}+\sum_{j=1}^{a}\mu_j^{(a)}}\prod_{j=1}^{a-1}(q)_{\mu_j^{(a)}}(q)_{A_j^{(a)}-A_{j+1}^{(a)}-\mu_{j+1}^{(a)}}}\\
&=\frac{(q)_{M_1+M_{n-1}}}{(q)_{M_1+k_{a+1}}(q)_{M_{n-1}-k_{a+1}}(q)_{M_1+k_{a+1}-A_1^{(a)}}(q)_{M_{n-1}-k_{a+1}+A_a^{(a)}}\prod_{j=1}^{a-1}(q)_{A_j^{(a)}-A_{j+1}^{(a)}}}
\end{aligned}\tag{A.2}$$

and by rewriting the right-hand side as

$$\begin{aligned}
&\frac{(q)_{M_1+M_{n-1}}}{(q)_{M_1+k_{a+1}}(q)_{M_{n-1}-k_{a+1}}(q)_{M_1+k_a-A_1^{(a-1)}-\mu_1^{(a-1)}}(q)_{M_{n-1}-k_a-\sum_{j=1}^{a-1}\mu_j^{(a-1)}}}\\
&\qquad\times\frac{1}{(q)_{A_{a-1}^{(a-1)}+\mu_{a-1}^{(a-1)}+\sum_{j=1}^{a-1}\mu_j^{(a-1)}}}\prod_{j=1}^{a-2}\frac{1}{(q)_{A_j^{(a-1)}-A_{j+1}^{(a-1)}+\mu_j^{(a-1)}-\mu_{j+1}^{(a-1)}}}
\end{aligned}\tag{A.3}$$

using

$$A_j^{(a)}=\begin{cases}k_{a+1}-k_a+A_j^{(a-1)}+\mu_j^{(a-1)} & \text{for } j=1,\dots,a-1,\\ k_{a+1}-k_a-\sum_{i=1}^{a-1}\mu_i^{(a-1)} & \text{for } j=a.\end{cases}$$

The final sum over μ_1 then yields the right-hand side of (6.1). The proof of the key identity (A.2) follows by successively summing over $\mu_a^{(a)},\dots,\mu_1^{(a)}$ using the q-Chu–Vandermonde sum (equation (3.3.10) of [3]),

$$\sum_{j\geq 0} q^{j(j+a)} \begin{bmatrix} b \\ j \end{bmatrix} \begin{bmatrix} a+c \\ j+a \end{bmatrix} = \begin{bmatrix} a+b+c \\ c \end{bmatrix}.$$

Of course, in a complete proof, the statements that one can successively sum over the $\mu^{(n-1)},\dots,\mu^{(1)}$ using (A.2) and (A.3), and that (A.2) follows from successively summing over $\mu_a^{(a)},\dots \mu_1^{(a)}$ require a proof by induction. This takes (a lot of) space, but requires no intellectual effort other than that of avoiding typos. It is therefore omitted here.

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Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
E-mail address: `warnaar@wins.uva.nl`