Explicit Plethystic Formulas
for
Macdonald q,t-Kostka Coefficients

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Abstract. For a partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_k > 0)$ set $B_\mu(q,t) = \sum_{i=1}^{k} t^{i-1}\,(1+\cdots+q^{\mu_i-1})$. In [8] Garsia-Tesler proved that if γ is a partition of k and $\lambda = (n-k,\gamma)$ is a partition of n, then there is a unique symmetric polynomial $k_\gamma(x;q,t)$ of degree $\leq k$ with the property that $\tilde{K}_{\lambda\mu}(q,t) = k_\gamma[B_\mu(q,t);q,t]$ holds true for all partitions μ. It was shown there that these polynomials have Schur function expansions of the form $k_\gamma(x;q,t) = \sum_{|\rho|\leq|\gamma|} S_\lambda(x)\; k_{\rho,\gamma}(q,t)$ where the $k_{\rho,\gamma}(q,t)$ are polynomials in $q, t, 1/q, 1/t$ with integer coefficients. This result yielded the first proof of the Macdonald polynomiality conjecture. It also was used in a proof [7] of the positivity conjecture for the $\tilde{K}_{\lambda\mu}(q,t)$ for any λ of the form $\lambda = (r,2,1^m)$ and arbitrary μ. In this paper we show that the polynomials $k_\gamma(x;q,t)$ may be given a very simple explicit expression in terms of the operator ∇ studied in [2]. In particular we also obtain a new proof of the polynomiality of the coefficients $\tilde{K}_{\lambda\mu}(q,t)$. Further byproducts of these developments are a new explicit formula for the polynomial $\tilde{H}_\mu[X;q,t] = \sum_\lambda S_\lambda[X]\tilde{K}_{\lambda\mu}(q,t)$ and a new derivation of the symmetric function results of Sahi [16] and Knop [11], [12].

Introduction

To state our results we need to review some notation and recall some basic facts. We work with the algebra Λ of symmetric functions in a formal infinite alphabet $X = x_1, x_2, \ldots$, with coefficients in the field of rational functions $\mathbf{Q}(q,t)$. We also denote by $\Lambda_{Z[q,t]}$ the algebra of symmetric functions in X with coefficients in $Z[q,t]$. We write $\Lambda^{=d}$ for the space of symmetric functions homogeneous of degree d. The spaces $\Lambda^{\leq d}$ and $\Lambda^{>d}$ are analogously defined. We shall make extensive use here of "*plethystic*" notation. This is a notational device which simplifies manipulation of symmetric function identities. It can be easly defined and programmed in *MATHEMATICA* or *MAPLE* if we view symmetric functions as formal power series in the power symmetric functions p_k. To begin with, if $E = E[t_1,t_2,t_3,\ldots]$ is a formal Laurent series in the variables $t_1,t_2,t_3,\ldots$ (which may include the parameters q,t) we set

$$p_k[E] \;=\; E[t_1^k,t_2^k,t_3^k,\ldots]\;.$$

More generally, if a certain symmetric function F is expressed as the formal power series

$$F \;=\; Q[p_1,p_2,p_3,\ldots]$$

then we simply let

$$F[E] \;=\; Q[p_1,p_2,p_3,\ldots]\Big|_{p_k\to E[t_1^k,t_2^k,t_3^k,\ldots]}\;. \tag{I.1}$$

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and refer to it as "*plethystic substitution*" of E into the symmetric function F.

We make the convention that inside the plethystic brackets "[ ]", X and X_n respectively stand for $x_1 + x_2 + x_3 + \cdots$ and $x_1 + x_2 + \cdots + x_n$. In particular, one sees immediately from this definition that if $f(x_1, x_2, \ldots, x_n)$ is a symmetric function then $f[X_n] = f(x_1, x_2, \ldots, x_n)$. We shall also make use of the symbol $\Omega(x)$ to represent the symmetric function

$$\Omega(x) \;=\; \prod_{i\geq 1} \frac{1}{1-x_i} \;.$$

It is easily seen that in terms of it the Cauchy, Hall-Littlewood and Macdonald kernels may be respectively be given the compact forms

$$\Omega[X_nY_m] \;\;,\quad \Omega[X_nY_m(1-t)] \quad \text{and} \quad \Omega[X_nY_m\tfrac{1-t}{1-q}] \;.$$

Indeed, since we may write

$$\Omega \;=\; \exp\Big(\sum_{k\geq 1} \frac{p_k}{k}\Big) \;,$$

we see that the definition in I.2 gives

$$\Omega[X_nY_m] = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1}{1-x_iy_j} \quad , \qquad \Omega[X_nY_m(1-t)] = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1-t\,x_iy_j}{1-x_iy_j}$$

and

$$\Omega[X_nY_m\tfrac{1-t}{1-q}] \;=\; \prod_{i=1}^{n}\prod_{j=1}^{m}\prod_{k=0}^{\infty} \frac{1-t\,q^kx_iy_j}{1-q^kx_iy_j} \;.$$

In using plethystic notation we are forced to distinguish between two different minus signs. Indeed note that the definition in I.1 yields that we have

$$p_k[-X_n] \;=\; p_k[-x_1-x_2-\cdots-x_n] \;=\; -x_1^k-x_2^k-\cdots-x_n^k \;=\; -p_k[X_n] \;.$$

On the other hand, on using the ordinary meaning of the minus sign, we would obtain

$$p_k[X_n]\,\big|_{x_i\to -x_i} \;=\; (-1)^k\,p_k[X_n] \;.$$

Since both operations will necessarily occur in our formulas, we shall adopt the convention that when a certain variable has to be replaced by its negative, in the ordinary sense, then that variable will be prepended by a superscripted minus sign. For example, note that the ω involution, which is customarily defined as the map which interchanges the elementary and homogeneous bases, may also be defined by setting

$$\omega\, p_k \;=\; (-1)^{k-1}\, p_k \;.$$

However, note that by the above conventions we obtain that

$$p_k[-^{-}X_n] \;=\; (-1)^{k-1}\, p_k[X_n] \;.$$

In particular, for any symmetric polynomial P of degree $\leq n$, we may write

$$\omega\, P[X_n] \;=\; P[-^{-}X_n] \;. \qquad \text{I.2}$$

Sometimes it will be convenient to use the symbol "ϵ" to represent $^{-}1$. The idea is that we should treat ϵ as any of the other variables in carrying out plethystic operations and only at the end replace ϵ by -1 in the ordinary sense.

A partition μ will be represented and identified with its Ferrers diagram. We shall use the French convention here and, given that the parts of μ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, we let the corresponding Ferrers diagram have μ_i lattice cells in the i^{th} row (counting from the bottom up). It will be convenient to let $|\mu|$ and $l(\mu)$ denote respectively the sum of the parts and the number of nonzero parts of μ. In this case $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_k$ and $l(\mu) = k$. As customary the symbol "$\mu \vdash n$" will be used to indicate that $|\mu| = n$. Following Macdonald, the *arm, leg, coarm* and *coleg* of a lattice square s are the parameters $a_\mu(s), l_\mu(s), a'_\mu(s)$ and $l'_\mu(s)$ giving the number of cells of μ that are respectively *strictly* EAST, NORTH, WEST and SOUTH of s in μ.

This given, here and after, for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ we set

$$n(\mu) = \sum_{i=1}^{k} (i-1)\mu_i \;=\; \sum_{s\in\mu} l'_\mu(s) \;=\; \sum_{s\in\mu} l_\mu(s) \;.$$

If s is a cell of μ we shall refer to the monomial $w(s) = q^{a'_\mu(s)} t^{l'_\mu(s)}$ as the *weight* of s. The sum of the weights of the cells of μ will be denoted by $B_\mu(q,t)$ and will be called the *biexponent generator of* μ. Note that we have

$$B_\mu(q,t) \;=\; \sum_{s\in\mu} q^{a'_\mu(s)} t^{l'_\mu(s)} \;=\; \sum_{i\geq 1} t^{i-1}\, \frac{1-q^{\mu_i}}{1-q} \;. \qquad \text{I.3}$$

If $\gamma \vdash k$ and $n-k \geq \max(\gamma)$, the partition of n obtained by prepending a part $n-k$ to γ will be denoted by $(n-k,\gamma)$. It will also be convenient to set

$$T_\mu \;=\; t^{n(\mu)} q^{n(\mu')} \;=\; \prod_{s\in\mu} q^{a'_\mu(s)} t^{l'_\mu(s)} \quad \text{and} \quad D_\mu \;=\; (1-t)(1-q) B_\mu(q,t) - 1 \,. \qquad \text{I.4}$$

We shall work here with the symmetric polynomial $\tilde{H}_\mu[X;q,t]$ with Schur function expansion

$$\tilde{H}_\mu[X;q,t] \;=\; \sum_\lambda S_\lambda[X]\,\tilde{K}_{\lambda\mu}(q,t) \;, \qquad \text{I.5}$$

where the coefficients $\tilde{K}_{\lambda\mu}(q,t)$ are obtained from the Macdonald q,t-Kostka coefficients by setting

$$\tilde{K}_{\lambda\mu}(q,t) \;=\; t^{n(\mu)}\,K_{\lambda\mu}(q,1/t) \;. \qquad \text{I.6}$$

As we shall see, most of the properties of $\tilde{H}_\mu[X;,q,t]$ we will need here can be routinely derived from the corresponding properties of the Macdonald's integral form $J_\mu[X;q,t]$ (†) , via the identity

$$\tilde{H}_\mu[X;q,t] = t^{n(\mu)} J_\mu[\tfrac{X}{1-1/t}\,;q,1/t\,] \;. \qquad \text{I.7}$$

This polynomial occurs naturally in our previous work, where it is conjectured to give a representation theoretical interpretation to the coefficients $\tilde{K}_{\lambda\mu}(q,t)$. Another important ingredient in the present developments is the linear operator ∇ defined, in term of the basis $\{\tilde{H}_\mu[X;q,t]\}_\mu$, by setting

$$\nabla\,\tilde{H}_\mu[X;q,t] \;=\; T_\mu\,\tilde{H}_\mu[X;q,t] \;. \qquad \text{I.8}$$

This operator also plays a crucial role in the developments relating Macdonald polynomials to symmetric group representation theory [1], [3], [4], [5], [6] and to geometry [9]. Computer experimentation with ∇ revealed that it has some truly remarkable properties. The reader is referred to [2] for a collection of results and conjectures about ∇ that have emerged in the few years since its discovery.

It was shown in [8] that for any given $\gamma \vdash k$, there is a unique symmetric polynomial $\mathbf{k}_\gamma(x;q,t)$ of degree $\le k$ yielding

$$\tilde{K}_{(n-k,\gamma),\mu}(q,t) \;=\; \mathbf{k}_\gamma[\,B_\mu(q,t)\,;q,t] \qquad (\;\forall \quad \mu \vdash n \ge k+\max(\gamma)\;) \;. \qquad \text{I.9}$$

Although a formula for $\mathbf{k}_\gamma(x;q,t)$ could be extracted from the original proof of this results (see [8] Th. 4.1), it was of such complexity that it yielded very little information about the true nature of this polynomial. All that could be derived there is that $\mathbf{k}_\gamma(x;q,t)$ has a Schur function expansion of the form

$$\mathbf{k}_\gamma(x;q,t) \;=\; \sum_{|\rho|\le k} S_\rho\,\mathbf{k}_{\rho\gamma}(q,t) \qquad \text{I.10}$$

(†) [15] Ch. VI, (8.3)

with each $\mathbf{k}_{\rho\gamma}(q,t)$ a Laurent polynomial in q,t with integer coefficients. This result was sufficient to prove the integral polynomiality of the $K_{\lambda\mu}(q,t)$. Moreover, a relatively small number of these polynomials already permitted the computation of extensive tables of the polynomials $\tilde{H}_\mu[X;q,t]$.

The remarkable development here is that, in terms of ∇, the polynomial $\mathbf{k}_\gamma(x;q,t)$ may be given a surprisingly simple expression.

Theorem I.1

For each $\gamma \vdash k$ let

$$\mathbf{k}'_\gamma(x;q,t) \;=\; \nabla^{-1}\, S_\gamma[\tfrac{1-\quad -X}{(1-t)(1-q)} - 1] \;. \qquad (\dagger) \qquad \text{I.11}$$

Then

$$\tilde{K}_{(n-k,\gamma),\mu}(q,t) \;=\; \mathbf{k}'_\gamma[\,D_\mu(q,t)\,;q,t] \qquad (\,\forall\;\mu \vdash n \geq k + \max(\gamma)\,)\,. \quad \text{I.12}$$

In particular the symmetric polynomial uniquely characterized by I.9 and I.10 is given by the formula

$$\mathbf{k}_\gamma[X] \;=\; \mathbf{k}'_\gamma[(1-t)(1-q)X - 1] \qquad \text{I.13}$$

Let us recall that the Hall scalar product for symmetric functions is defined by setting for the power basis $\{p_\rho\}_\rho$

$$\langle p_{\rho^{(1)}}\;,\;p_{\rho^{(2)}}\rangle \;=\; \begin{cases} z_\rho & \text{if } \rho^{(1)} = \rho^{(2)} = \rho \\ 0 & \text{otherwise} \end{cases}$$

where for a partition $\rho = 1^{\alpha_1}, 2^{\alpha_2}, 3^{\alpha_3}, \cdots$ we set as customary

$$z_\rho = 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \cdots \alpha_1! \alpha_2! \alpha_3! \cdots \;.$$

We shall also need here the scalar product $\langle\;,\;\rangle_*$ defined by setting

$$\langle p_{\rho^{(1)}}\;,\;p_{\rho^{(2)}}\rangle_* = \begin{cases} (-1)^{|\rho|-l(\rho)}\, z_\rho\, p_\rho[(1-t)(1-q)] & \text{if } \rho^{(1)} = \rho^{(2)} = \rho\,, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{I.14}$$

It will be convenient, here and in the following, to set for every $F[X] \in \Lambda$,

$$F^*[X] \;=\; F[\tfrac{X}{(1-t)(1-q)}] \;.$$

Our main object here is the following very general result which has a variety of important consequences including our formula I.11:

($\dagger$) Here and in the following plethysms are to be carried out before operator actions.

Theorem I.2

For each symmetric polynomial f set

$$\mathbf{\Pi}'_f[X;q,t] \;=\; \nabla^{-1} f[X -^{-} 1] \qquad \text{I.15}$$

Then for all μ we have

$$\mathbf{\Pi}'_f[D_\mu;q,t] \;=\; \langle f \,,\, \tilde{H}_\mu[X+1]\rangle_* \qquad \text{I.16}$$

Alternatively, if f is homogeneous of degree k and we also set

$$\mathbf{\Pi}_f[X;q,t] \;=\; \nabla^{-1} f[\tfrac{1-^{-}X}{(1-t)(1-q)}] \;, \qquad \text{I.17}$$

then for all $\mu \vdash n \geq k$ we have

$$\begin{aligned} a) \quad \langle e^*_{n-k}\, f \,, \tilde{H}_\mu \rangle_* \;&=\; \mathbf{\Pi}'_f[D_\mu;q,t] \\ b) \quad \langle h_{n-k}\, f \,, \tilde{H}_\mu \rangle \;&=\; \mathbf{\Pi}_f[D_\mu;q,t] \;. \end{aligned} \qquad \text{I.18}$$

We can define a skew version $\tilde{H}_{\mu/\nu}$ of the symmetric polynomial $\tilde{H}_\mu$ yielding the addition formula

$$\tilde{H}_\mu[X+Y;q,t] \;=\; \sum_{\nu\subseteq\mu} \tilde{H}_\nu[X;q,t]\,\tilde{H}_{\mu/\nu}[Y;q,t] \;. \qquad \text{I.19}$$

This can be derived from the analogous result for the Macdonald polynomial $Q_\lambda[X;q,t\,]$ (see Ch. VI (7.9)). Now it develops that the identity in I.16 (with $\mathbf{\Pi}'_f$ given by I.15) is equivalent to the following truly remarkable formula yielding the polynomial $\tilde{H}_\mu$.

Theorem I.3

$$\tilde{H}_\mu[X+1\,;q,t\,] \;=\; \Omega[\tfrac{X}{M}]\,\nabla^{-1}\omega\,\Omega[\tfrac{X\,D_\mu}{M}] \qquad \text{I.20}$$

with

$$M \;=\; (1-t)(1-q) \qquad \text{I.21}$$

Another corollary of Theorem I.2 may be stated as follows.

Theorem I.4

For a partition μ set

$$\delta_\mu[X;q,t\,] \;=\; \frac{\nabla^{-1}\tilde{H}_\mu[X -^{-} 1]}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \qquad \text{I.22}$$

with

$$\tilde{h}_\mu(q,t) = \prod_{s\in\mu}(q^{a_\mu(s)} - t^{l_\mu(s)+1}) \quad , \quad \tilde{h}'_\mu(q,t) = \prod_{s\in\mu}(t^{l_\mu(s)} - q^{a_\mu(s)+1}) \;. \qquad \text{I.23}$$

Then

$$\delta_\mu[D_\lambda\,;q,t\,] \;=\; \begin{cases} \tilde H_{\lambda/\mu}[1;q,t\,] & \textit{if } \mu\subseteq\lambda \\ \\ 0 & \textit{otherwise.}\end{cases} \qquad \text{I.24}$$

We shall see that the identity in I.24 constitutes a new derivation and sharpening of the symmetric functions results of Sahi and Knop.

In summary, the apparently simple identity in I.16 has astonishing consequences. Several important results in the Theory of Macdonald polynomials may be derived from it. Namely,

(1) We recover the plethystic formulas for the Macdonald coefficients $K_{\lambda\mu}(q,t)$, in a simpler and more effective form than in [7] and [8];
(2) We obtain a new and simple proof of the theorem [7], [8], [10], [11], [12], [13], [16] that the $K_{\lambda\mu}(q,t)$ are polynomials with integer coefficients.
(3) We recover the vanishing theorem of Knop [11], [12] and Sahi [16] in a strong "extended" vanishing form, with an exact formula for their vanishing polynomials and a natural interpretation for their values at the points where they do not vanish.
(4) Finally we shall see that the curious and remarkable Koornwinder-Macdonald reciprocity formula [15] (VI (6.6)) is but a simple specialization of I.20.

As we shall see the derivation of all these results is not difficult and uses no machinery other than well-known symmetric function theory. It does however depend on the discovery of certain plethystic operator identities that do provide a powerful insight into Macdonald Theory.

This paper is divided into 4 sections. In the first section we introduce our basic tools which consist of plethystic forms of familiar symmetric function operations and certain new plethystic operators which naturally emerge in computations involving the polynomials $\tilde H_\mu$. The identities we prove there should have independent interest and have been shown to have further important applications (see [2]). In Section 2 we prove Theorems I.1 – I.4. In Section 3 we give our applications including our derivation of the Sahi-Knop symmetric function results and the reciprocity formula. Our developments rely on a number of identities for the polynomials $\tilde H_\mu[X;q,t]$ that may be derived from corresponding identities for the Macdonald polynomials $P_\lambda[x;q,t]$. The derivations that are less accessible will be carried out in Section 4, the others will be referred to the appropriate sources.

1. The basic tools

We shall start by reviewing a few facts about Schur functions we will need in our presentation. Recall that the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ occur in the expansion

$$S_\mu \, S_\nu \;=\; \sum_\lambda c^{\lambda}_{\mu\nu} \, S_\lambda \ , \tag{1.1}$$

and in the addition formula

$$S_\lambda[X+Y] \;=\; \sum_\mu \sum_\nu c^{\lambda}_{\mu\nu} \, S_\mu[X] \, S_\nu[Y] \ . \tag{1.2}$$

The same coefficients are used to define the "*skew*" Schur function $S_{\lambda/\mu}$ by setting

$$\partial_{S_\mu} S_\lambda \;=\; S_{\lambda/\mu} \;=\; \sum_\nu c^{\lambda}_{\mu\nu} \, S_\nu \ . \tag{1.3}$$

In the present context we shall interpret 1.1 and 1.3 as expressing the action, on the Schur basis, of the two operators "$\underline{S}_\mu$" and "∂_{S_μ}" respectively representing "*multiplication*" and "*skewing*" by S_μ. Note that since the orthogonality of Schur functions with respect to the Hall scalar product gives

$$\langle S_\mu \, S_\nu \, , \, S_\lambda \rangle \;=\; c^{\lambda}_{\mu\nu} \;=\; \langle S_\nu \, , \, S_{\lambda/\mu} \rangle \ , \tag{1.4}$$

we see that 1.4 may be viewed as expressing that ∂_{S_μ} is the Hall scalar product adjoint of $\underline{S}_\mu$.

In the same vein we can define two more general "*multiplication*" and "*translation*" operators "$\mathcal{P}_Y$" and "$\mathcal{T}_Y$" by setting for any given "*alphabet*" Y (†) , and any symmetric function $Q[X] \in \Lambda$

$$\begin{aligned} a) \quad \mathcal{T}_Y \; Q[X] &= Q[X+Y] \\ b) \quad \mathcal{P}_Y \; Q[X] &= \Omega[XY] Q[X] \ . \end{aligned} \tag{1.5}$$

These operators have the following useful "Schur function"expansions:

Theorem 1.1

$$\begin{aligned} a) \quad \mathcal{T}_Y &= \sum_\mu \; S_\mu[Y] \, \partial_{S_\mu} \ . \\ b) \quad \mathcal{P}_Y &= \sum_\mu \; S_\mu[Y] \, \underline{S}_\mu \ . \end{aligned} \tag{1.6}$$

(†) We use the word "alphabet" here in a very general manner, since Y itself may be any algebraic expression that can be plethystically substituted into a symmetric function. For example see formulas 1.6 a) and b) below.

In particular we see that when Y consists of a single variable u, we have

$$\mathcal{T}_u \;=\; \sum_{m\geq 0} u^m\, \partial_{S_m} \;. \qquad 1.7$$

Proof

Note that in view of 1.3, formula 1.2 may be written in the form

$$S_\lambda[X+Y] \;=\; \sum_\nu S_\nu[Y]\, S_{\lambda/\nu}[X] \;.$$

In other words we have

$$\mathcal{T}_Y\, S_\lambda[X] \;=\; \sum_\nu S_\nu[Y]\, \partial_{S_\nu}\, S_\lambda[X] \;.$$

This proves 1.6 a) when $\mathcal{T}_Y$ acts on the Schur basis. Thus the result must hold true for all symmetric functions. To prove 1.6 b) we simply observe that from the Cauchy identity we derive that for $P[X]\in\Lambda$

$$\mathcal{P}_Y\, P[X] = \Omega[XY]\, P[X] = \sum_\rho S_\rho[Y]\, S_\rho[X]\, P[X] = \Big(\sum_\rho S_\rho[Y]\, \underline{S}_\rho[X]\Big)\, P[X] \;.$$

Finally, we see that 1.6 a) reduces to 1.7 when $Y=\{u\}$, because $S_\rho[u]$ fails to vanish identically only when $\rho=\{m\}$. This completes our proof.

Our developments crucially depend on the operators D_k and D_k^* defined by setting for every $F\in\Lambda$:

$$\begin{aligned} &a)\;\; D_k\, F[X] \;=\; F[X+\tfrac{M}{z}]\,\Omega[-z\,X]\,\big|_{z^k} \\ &b)\;\; D_k^*\, F[X] \;=\; F[X-\tfrac{\tilde M}{z}]\,\Omega[z\,X]\,\big|_{z^k} \end{aligned} \qquad (\dagger)\quad \text{for } -\infty<k<+\infty\,, \qquad 1.8$$

where for convenience here and after we let

$$M \;=\; (1-t)(1-q) \quad , \qquad \tilde M \;=\; (1-1/t)(1-1/q). \qquad 1.9$$

We should note that an expression such as "$F[X+\frac{M}{z}]$" is easily implemented on the computer once F is expanded in the power basis. In fact if $F=Q[p_1,p_2,p_3,\ldots]$ then

$$F[X+\tfrac{M}{z}] \;=\; Q[p_1,p_2,p_3,\ldots]\,\Big|_{p_k\to p_k+\frac{(1-t^k)(1-q^k)}{z^k}} \;.$$

It is also easily seen that the generating functions of D_k and D_k^* have the following simple expressions in terms of the multiplication and translation operators:

$$\begin{aligned} &a)\;\; D(z) = \sum_{-\infty}^{\infty} z^k\, D_k = \mathcal{P}_{-z}\,\mathcal{T}_{M/z} \quad , \\ &b)\;\; D^*(z) = \sum_{-\infty}^{\infty} z^k\, D_k^* = \mathcal{P}_{z}\,\mathcal{T}_{-\tilde M/z} \;. \end{aligned} \qquad 1.10$$

($\dagger$) The symbol "$|_{z^k}$" denotes taking the coefficient of z^k in the preceding expression.

The importance of these operators in the study of the polynomials $\tilde{H}_\mu[X;q,t]$ derives from the following basic result.

Theorem 1.2

For $\mu \vdash n$ we have

$$\begin{aligned} &a)\ \ D_0\, \tilde{H}_\mu[X;q,t] \ = -D_\mu(q,t)\ \tilde{H}_\mu[X;q,t]\ \ , \\ &b)\ \ D_0^*\, \tilde{H}_\mu[X;q,t] \ = -D_\mu(1/q,1/t)\ \tilde{H}_\mu[X;q,t]\ \ . \end{aligned} \tag{1.11}$$

In particular $\tilde{H}_\mu[X;q,t]$ is uniquely characterized by either one of a) or b) above and the normalization

$$\tilde{H}_\mu[X;q,t]\ |_{S_n} \ = \ 1\ .\quad (\dagger) \tag{1.12}$$

The proof of this will be found in Section 4.

There are a number of identities, involving various combinations of these operators, which we will need in our developments. Since they are of independent interest, we will give them as a series of propositions.

For $F[X;q,t] \in \Lambda$ let us set

$$\downarrow F[X;q,t\,] \ = \ \omega\, F[X;1/q,1/t\,] \ = \ F[-{}^{-}X;1/q,1/t\,]\ . \tag{1.13}$$

It is easily seen that the operator "$\downarrow$" is an involution. It also has the following useful properties:

Proposition 1.1

Using $\epsilon = {}^{-}1$ we have

$$\begin{aligned} &a) \quad \downarrow \mathcal{T}_1 \downarrow \ = \ \mathcal{T}_\epsilon^{-1} \\ &b) \quad \downarrow \nabla \downarrow \ = \ \nabla^{-1} \\ &c) \quad \downarrow D_k \downarrow \ = \ (-1)^k\, D_k^*\ \ . \end{aligned} \tag{1.14}$$

Proof

For any $P[X;q,t] \in \Lambda$ we have

$$\begin{aligned} \downarrow\mathcal{T}_1\downarrow P[X;q,t] \ &= \ \downarrow\mathcal{T}_1 P[-{}^{-}X;1/q,1/t] \\ &= \ \downarrow P[-{}^{-}(X+1);1/q,1/t] \ = \ P[X-\epsilon;q,t]\ \ . \end{aligned}$$

This proves 1.14 a). Next, we shall show in Section 4 that we have

$$T_\mu\, \omega \tilde{H}_\mu[X;1/q,1/t\,] \ = \ \tilde{H}_\mu[X;q,t\,]\ \ . \tag{1.15}$$

($\dagger$) The symbol " $|_{S_n}$" represents taking the coefficient of the Schur function $S_n[X]$ in the Schur function expansion of the preceding expression.

Now this may be rewritten as

$$\downarrow \tilde{H}_\mu \;=\; \frac{1}{T_\mu}\,\tilde{H}_\mu \;. \qquad 1.16$$

Thus from the definition in I.8 we derive that

$$\downarrow \nabla \downarrow \tilde{H}_\mu \;=\; \downarrow \nabla \frac{1}{T_\mu}\,\tilde{H}_\mu \;=\; \downarrow \frac{T_\mu}{T_\mu}\,\tilde{H}_\mu \;=\; \frac{1}{T_\mu}\,\tilde{H}_\mu \;=\; \nabla^{-1}\,\tilde{H}_\mu \;.$$

This proves 1.14 b) since the $\tilde{H}_\mu\,'s$ are a basis for Λ. To prove 1.14 c) we note that for any $F[X;q,t]\in\Lambda$ the definitions in 1.6 a) and 1.13 give

$$\begin{aligned}
\downarrow D_k \downarrow F[X;q,t\,] &= \downarrow D_k\, F[-{}^{-}X;1/q,1/t\,]\\
&= \downarrow F[-{}^{-}(X+M/z);1/q,1/t\,]\Omega[-zX]\,\big|_{z^k}\\
&= \downarrow F[-{}^{-}X-M/\,{}^{-}z\,;1/q,1/t\,]\,\Omega[-zX]\,\big|_{z^k}\\
&= F[X-\tilde{M}/\,{}^{-}z\,;q,t\,]\,\Omega[\,{}^{-}zX]\,\big|_{z^k}\\
&= (-1)^k\,F[X-\tilde{M}/z\,;q,t\,]\,\Omega[zX]\,\big|_{z^k} \qquad \textbf{Q.E.D.}
\end{aligned}$$

Remark 1.1

The identities in 1.14 can be used to systematically derive results for the $D_k^*\,'s$ from corresponding results for the $D_k\,'s$. For instance note that to prove Theorem 1.2 we need only establish 1.11 a). Indeed 1.14 c), 1.11 a) and 1.16 give

$$D_0^*\,\tilde{H}_\mu \;=\downarrow D_0 \downarrow \tilde{H}_\mu \;=\downarrow D_0\,\frac{1}{T_\mu}\,\tilde{H}_\mu \;=\downarrow \frac{-D_\mu(q,t)}{T_\mu}\,\tilde{H}_\mu \;=\; -D_\mu(1/q,1/t)\,\tilde{H}_\mu\,.$$

Let us now set

$$\tilde{\Omega}[X] \;=\; \omega\Omega[X] \;=\; \Omega[-{}^{-}X] \;=\; \prod_i(1+x_i) \;=\; \exp\Big[\sum_{k\ge 1}\frac{(-1)^{k-1}p_k}{k}\Big]\;. \qquad 1.17$$

This given, we have the following basic expansions.

Theorem 1.3

$$\begin{aligned}
&a)\quad \tilde{\Omega}\big[\tfrac{XY}{(1-q)(1-t)}\big] \;=\; \sum_\rho \frac{p_\rho[X]\,p_\rho[Y]}{(-1)^{|\rho|-l(\rho)}\,z_\rho\,p_\rho[(1-t)(1-q)]}\,,\\
&b)\quad \tilde{\Omega}\big[\tfrac{XY}{(1-q)(1-t)}\big] \;=\; \sum_\lambda S_\lambda\big[\tfrac{X}{(1-q)(1-t)}\big]\,S_{\lambda'}[Y] \;=\; \sum_\lambda S^*_\lambda[X]\,S_{\lambda'}[Y]\,, \qquad 1.18\\
&c)\quad \tilde{\Omega}\big[\tfrac{XY}{(1-q)(1-t)}\big] \;=\; \sum_\mu \frac{\tilde{H}_\mu[X;q,t\,]\,\tilde{H}_\mu[Y;q,t\,]}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}\,.
\end{aligned}$$

Proof

The identity in 1.18 a) is an immediate consequence of the definition in 1.17. Note that if we make the plethystic substitution $X \to X/M$ in the classical expansion (†)

$$p_\rho[X] \;=\; \sum_\lambda \chi_\rho^\lambda \, S_\lambda[X] \;.$$

and substitute the result in 1.18 a) we obtain

$$\tilde\Omega[\tfrac{XY}{(1-q)(1-t)}] \;=\; \sum_\rho \frac{p_\rho[Y]}{(-1)^{|\rho|-l(\rho)}\, z_\rho} \sum_\lambda \chi_\rho^\lambda \, S_\lambda[\tfrac{X}{(1-q)(1-t)}] \;.$$

and 1.18 b) follows by interchanging the order of summation and using the identity

$$S_{\lambda'}[Y] \;=\; \sum_\rho \chi_\rho^\lambda \, \frac{(-1)^{|\rho|-l(\rho)} p_\rho[Y]}{z_\rho} \;.$$

Formula 1.18 c) is another way of stating the "Cauchy" formula for Macdonald polynomials. The details of this derivation can be found in Section 4.

Corollary 1.4

The following three pairs are dual bases with respect to the $*$-scalar product:

$$\begin{array}{lll} a) & \big\{p_\rho[X]\big\}_\rho \quad \& \quad \big\{(-1)^{|\rho|-l(\rho)} p_\rho[X]/z_\rho\big\}_\rho \\ b) & \big\{S_\lambda^*[X]\big\}_\lambda \quad \& \quad \big\{S_{\lambda'}[X]\big\}_\lambda \\ c) & \big\{\tilde H_\mu[X;q,t]\big\}_\mu \quad \& \quad \big\{\tilde H_\mu[X;q,t]/\tilde h_\mu \tilde h_{\mu'}\big\}_\mu \end{array} \qquad 1.19$$

Proof

The definition in I.14 asserts that the pair of bases in 1.19 a) are $*$-dual. We thus derive from 1.18 a) that $\tilde\Omega[\frac{XY}{(1-t)(1-q)}]$ is the reproducing kernel of the $*$-scalar product. That is to say, for all $F[X] \in \Lambda$ we have

$$F[Y] \;=\; \langle F[X]\,,\, \tilde\Omega[\tfrac{XY}{(1-t)(1-q)}]\rangle_* \;. \qquad 1.20$$

Using 1.18 b) and c) 1.20 yields the two expansions

$$F[Y] \;=\; \sum_\lambda \langle F[X]\,,\, S_\lambda^*[X]\rangle_* \, S_{\lambda'}[Y] \qquad 1.21$$

and

$$F[Y] \;=\; \sum_\mu \langle F[X]\,,\, \tilde H_\mu[X;q,t]\rangle_* \, \frac{\tilde H_\mu[Y;q,t]}{\tilde h_\mu \, \tilde h_{\mu'}} \qquad 1.22$$

(†) χ_ρ^λ denotes the irreducible S_n character indexed by λ at permutations of cycle structure ρ.

which are equivalent to the $*$-duality of the pairs in 1.19 b) and c).

Note next that the operators $\mathcal{T}$ and $\mathcal{P}$ commute in the following manner:

Proposition 1.2

For any two alphabets Z and Y we have

$$\mathcal{T}_Y\, \mathcal{P}_Z \;=\; \Omega[ZY]\, \mathcal{P}_Z\, \mathcal{T}_Y \;. \tag{1.23}$$

Proof

For $Q \in \Lambda$ we obtain

$$\begin{aligned} \mathcal{T}_Y\, \mathcal{P}_Z\, Q[X] \;&=\; \mathcal{T}_Y\, \Omega[XZ]\, Q[X] \;=\; \Omega[(X+Y)Z]\, Q[X+Y] \\ &=\; \Omega[YZ]\, \Omega[XZ]\, Q[X+Y] \\ &=\; \Omega[YZ]\, \mathcal{P}_Z\, \mathcal{T}_Y\, Q[X] \qquad\qquad \mathbf{Q.E.D.} \end{aligned}$$

Proposition 1.3

$$\begin{aligned} &a)\quad D_k\, \partial_{S_m} \;-\; \partial_{S_m}\, D_k \;=\; D_{k-1}\, \partial_{S_{m-1}} \;, \\ &b)\quad D_k^*\, \partial_{S_m} \;-\; \partial_{S_m}\, D_k^* \;=\; -D_{k-1}^*\, \partial_{S_{m-1}} \;. \end{aligned} \tag{1.24}$$

In particular we also have

$$\begin{aligned} &a)\quad D_k\, \partial_{S_1} \;-\; \partial_{S_1}\, D_k \;=\; D_{k-1} \;, \\ &b)\quad D_k^*\, \partial_{S_1} \;-\; \partial_{S_1}\, D_k^* \;=\; -D_{k-1}^* \;. \end{aligned} \tag{1.25}$$

Proof

We may view the identity in 1.7 as expressing that the operator $\mathcal{T}_u$ is the generating function of the operators ∂_{S_m}. Note then that we may write

$$\partial_{S_m}\, D_k \;=\; \mathcal{T}_u\, D(z)\,|_{u^m z^k} \;.$$

This given, using 1.10 a) and 1.23 we get

$$\begin{aligned} \mathcal{T}_u\, D(z) \;&=\; \mathcal{T}_u\, \mathcal{P}_{-z}\, \mathcal{T}_{M/z} \;=\; \Omega[-zu]\, \mathcal{P}_{-z}\, \mathcal{T}_u\, \mathcal{T}_{M/z} \\ &=\; \Omega[-zu]\, \mathcal{P}_{-z}\, \mathcal{T}_{M/z}\, \mathcal{T}_u \\ &=\; (1-u\,z)\;\, D(z)\, \mathcal{T}_u \;, \end{aligned} \tag{1.26}$$

and 1.24 a) is obtained by equating coefficients of $u^m z^k$ on both sides. We also clearly see that equating coefficients of uz^k yields the special case in 1.25 a). This given, 1.24 b) and 1.25 b) may be obtained by means of 1.14 c).

Remark 1.2

Since the operator ∂_{S_1} will occur in many of our identities, it will be convenient to simply denote it by ∂_1. Note also that in this particular case, ∂_1 reduces to differentiation with respect to the power function p_1. More precisely, if $F = Q(p_1, p_2, p_3, \ldots)$ is a symmetric function expressed in the power basis, then

$$\partial_1 F \;=\; \partial_{p_1} Q(p_1, p_2, p_3, \ldots)\,.$$

Note also that iterations of the identities in 1.25 yield

$$\begin{aligned} D_{-k} &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \; \partial_1^i \, D_0 \, \partial_1^{k-i} \;\;, \\ D^*_{-k} &= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \; \partial_1^i \, D_0^* \, \partial_1^{k-i} \;\;. \end{aligned} \qquad (\;\forall \quad k \geq 1\;) \qquad 1.27$$

The relations in 1.25 and 1.27 have the following degree-raising counterparts:

Proposition 1.4

For all $k \in (-\infty, +\infty)$:

$$\begin{aligned} &a) \quad D_k \, \underline{e}_1 \;-\; \underline{e}_1 \, D_k \;=\; M \, D_{k+1} \\ &b) \quad D_k^* \, \underline{e}_1 \;-\; \underline{e}_1 \, D_k^* \;=\; -\tilde{M} \, D_{k+1}^* \;\;, \end{aligned} \qquad . \qquad 1.28$$

and by iteration we deduce that we must have

$$\begin{aligned} &a) \quad D_k \;=\; \frac{1}{M^k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \; \underline{e}_1^i \, D_0 \, \underline{e}_1^{k-i} \\ &b) \quad D_k^* \;=\; \frac{1}{\tilde{M}^k} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \; \underline{e}_1^i \, D_0^* \, \underline{e}_1^i \end{aligned} \qquad (\;\forall \quad k \geq 1\;) \qquad 1.29$$

Proof

Note that the definition in 1.8 a) gives that for any $F \in \Lambda$ we have

$$\begin{aligned} D_k \;\; \underline{e}_1 \, F[X] &= (e_1 \;+\; \tfrac{M}{z}) \, F[X + \tfrac{M}{z}] \, \Omega[-zX] \, |_{z^k} \\ &= \underline{e}_1 \, D_k \, F[X] \;+\; M \, F[X + \tfrac{M}{z}] \, \Omega[-zX] \, |_{z^{k+1}} \\ &= \underline{e}_1 \, D_k \, F[X] \;+\; M \, D_{k+1} \, F[X] \end{aligned}$$

This given, 1.28 b) follows from 1.14 c).

The operators D_k and D_k^* are tied to ∇ via the following basic relations

Proposition 1.5

$$a)\ D_0\,\partial_1\ -\ \partial_1 D_0\ =\ M\,\nabla^{-1}\partial_1\,\nabla\ ,\qquad a^*)\ D_0^*\,\partial_1\ -\ \partial_1 D_0^*\ =\ \tilde{M}\,\nabla\,\partial_1\,\nabla^{-1}\ ,$$
$$b)\ D_0\,\underline{e}_1\ -\ \underline{e}_1 D_0\ =\ -M\,\nabla\,\underline{e}_1\,\nabla^{-1}\ ,\quad b^*)\ D_0^*\,\underline{e}_1\ -\ \underline{e}_1 D_0^*\ =\ -\tilde{M}\,\nabla^{-1}\underline{e}_1\,\nabla\ . \tag{1.30}$$

Proof

It follows from the Macdonald Pieri rules (see [4] Proposition 1.3) that there are certain coefficients $c_{\mu\nu}(q,t)$ and $d_{\mu\nu}(q,t)$ giving

$$a)\ \partial_1\tilde{H}_\mu\ =\ \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,\tilde{H}_\nu\ \ ,\qquad b)\ \underline{e}_1\tilde{H}_\nu\ =\ \sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\,\tilde{H}_\mu \tag{1.31}$$

where the symbol "$\nu\to\mu$" means that ν is obtained by removing a corner of μ . Combining 1.31 a) with 1.11 a) gives

$$\begin{aligned} D_0\,\partial_1\,\tilde{H}_\mu\ &=\ \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,(\,-\,D_\nu(q,t))\,\tilde{H}_\nu\ \ ,\\ \partial_1\,D_0\,\tilde{H}_\mu\ &=\ \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,(\,-\,D_\mu(q,t))\,\tilde{H}_\nu\ \ . \end{aligned}$$

Subtracting and using I.4 then gives

$$(D_0\,\partial_1\ -\ \partial_1\,D_0)\,\tilde{H}_\mu\ =\ M\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,(B_\mu(q,t)-B_\nu(q,t))\,\tilde{H}_\nu\ \ . \tag{1.32}$$

On the other hand, from the definition I.8 we get that

$$M\,\nabla^{-1}\,\partial_1\,\nabla\,\tilde{H}_\mu\ =\ M\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,(T_\mu/T_\nu)\,\tilde{H}_\nu\ \ .$$

Comparing this with 1.32 we see that 1.31 a) will hold true if and only if

$$B_\mu(q,t)-B_\nu(q,t)\ =\ T_\mu/T_\nu\ \ . \tag{1.33}$$

But this is a simple consequence of the fact that the monomial T_μ/T_ν is precisely the weight of the cell we must add to ν to get μ.

Similarly, from 1.31 b) we derive that

$$\begin{aligned} (D_0\,\underline{e}_1\ -\ \underline{e}_1 D_0)\,\tilde{H}_\nu\ &=\ M\sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\,(\,-\,B_\mu(q,t)+B_\nu(q,t))\,\tilde{H}_\mu\\ &=\ -M\sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\ \ T_\mu/T_\nu\ \tilde{H}_\mu \qquad\qquad (1.34)\\ &=\ -M\,\nabla\,\underline{e}_1\nabla^{-1}\,\tilde{H}_\mu\ \ . \end{aligned}$$

This proves 1.30 b). The remaining relations may now be derived from 1.14 c). This completes our proof.

Proposition 1.6

$$a)\ \mathcal{P}_{1/M}\, D_k\, \mathcal{P}_{-1/M} \;=\; D_k - D_{k+1} \quad , \quad b)\ \mathcal{P}_{-1/\tilde M}\, D_k^*\, \mathcal{P}_{1/\tilde M} \;=\; D_k^* - D_{k+1}^* \qquad 1.35$$

Proof

From 1.10 a) and 1.23 we get that

$$\begin{aligned}\mathcal{P}_{1/M}\, D(z)\, \mathcal{P}_{-1/M} &= \mathcal{P}_{1/M}\, \mathcal{P}_{-z}\, \mathcal{T}_{M/z} \mathcal{P}_{-1/M}\\ &= \mathcal{P}_{1/M}\, \mathcal{P}_{-z}\, \Omega[-1/z]\, \mathcal{P}_{-1/M}\, \mathcal{T}_{M/z}\\ &= \mathcal{P}_{-z}\, (1-1/z)\, \mathcal{T}_{M/z} \;=\; (1-1/z)\, D(z)\ ,\end{aligned}$$

and 1.35 a) follows by equating coefficients of z^k on both sides. Similarly, from 1.10 b) we get

$$\begin{aligned}\mathcal{P}_{-1/\tilde M}\, D^*(z)\, \mathcal{P}_{1/\tilde M} &= \mathcal{P}_{-1/\tilde M}\, \mathcal{P}_{z}\, \mathcal{T}_{-\tilde M/z} \mathcal{P}_{1/\tilde M}\\ &= \mathcal{P}_{-1/\tilde M}\, \mathcal{P}_{z}\, \Omega[-1/z]\, \mathcal{P}_{1/\tilde M}\, \mathcal{T}_{-\tilde M/z}\\ &= (1-1/z)\, D^*(z)\ ,\end{aligned}$$

and 1.35 b) follows again by equating coefficients of z^k.

Proposition 1.7

Again with $\epsilon = {}^-1$ *we have*

$$\begin{aligned} &a)\ \mathcal{T}_1\, D_k\, \mathcal{T}_1^{-1} = D_k - D_{k-1}\,, \quad && a^*)\ \mathcal{T}_\epsilon^{-1}\, D_k^*\, \mathcal{T}_\epsilon = D_k^* + D_{k-1}^*\,,\\ &b)\ \mathcal{T}_\epsilon\, D_k\, \mathcal{T}_\epsilon^{-1} = D_k + D_{k-1}\,, \quad && b^*)\ \mathcal{T}_1^{-1}\, D_k^*\, \mathcal{T}_1 = D_k^* - D_{k-1}^*\,. \end{aligned} \qquad 1.36$$

Proof

Equating coefficients of z^k in 1.26 we get

$$\mathcal{T}_u\, D_k \;=\; (D_k - u\, D_{k-1})\, \mathcal{T}_u\ .$$

Now $u = 1$ gives 1.36 a) and $u = \epsilon$ gives 1.36 b). This given, 1.36 a*) and b*) follow by applications of 1.14 a) and c).

To carry out our proofs we need a few properties of the $*$-scalar product and its relations to our operators. We shall start with its relation to the ordinary Hall scalar product:

Proposition 1.8

For all symmetric functions P *and* Q *we have*

$$\langle P\,, Q\rangle_* \;=\; \langle \phi\, \omega P\,, Q\rangle \;=\; \langle \omega\, \phi\, P\,, Q\rangle \qquad 1.37$$

where ϕ *is the operator defined by the plethysm*

$$\phi\, P[X] \;=\; P[MX] \;=\; P[(1-t)(1-q)M] \qquad 1.38$$

Proof

Note first that since by I.2 we have

$$\omega\,\phi^{-1}\,P[X] \;=\; P[\tfrac{-^{-}X}{M}] \;=\; P[-^{-}(\tfrac{X}{M})] \;=\; \phi^{-1}\,\omega\,P[X]\;, \qquad 1.39$$

we see that the two operators ω and ϕ do commute with each other, and therefore the last equality in 1.37 must hold true.

To prove the first equality, we set $P=\phi^{-1}\,\omega\,p_{\rho^{(1)}}$ and $Q=p_{\rho^{(2)}}$ and observe that the definition in I.14 gives that for $\rho^{(1)}=\rho^{(2)}=\rho$, we have

$$\langle \phi^{-1}\,\omega\,p_{\rho^{(1)}}\;,\;p_{\rho^{(2)}}\rangle_* \;=\; ((-1)^{|\rho|-l(\rho)})^2 z_\rho\,\frac{\prod_i(1-q^{\rho_i})(1-t^{\rho_i})}{p_\rho[(1-t)(1-q)]} \;=\; z_\rho\;.$$

Since for $\rho^{(1)}\neq\rho^{(2)}$ we get

$$\langle \phi^{-1}\,\omega\,p_{\rho^{(1)}}\;,\;p_{\rho^{(2)}}\rangle_* \;=\; 0 \;=\; \langle p_{\rho^{(1)}}\;,\;p_{\rho^{(2)}}\rangle\;,$$

it follows that the identity

$$\langle \phi^{-1}\omega\,P\,,Q\rangle_* \;=\; \langle P\,,Q\rangle \qquad 1.40$$

must hold true for all pairs of symmetric functions P and Q. However, this is just another way of stating 1.37.

Proposition 1.9

The operators D_0, D_0^* and ∇ are all self-adjoint with respect to the $*$-scalar product. Moreover, for any pair of symmetric functions P and Q we have

$$\langle e_1^*\,P\,,Q\rangle_* \;=\; \langle P\,,\partial_1\,Q\rangle_*\;. \qquad 1.41$$

Proof

The identity in 1.18 c) and the definition I.8 give that

$$\nabla^x\,\tilde\Omega[\tfrac{XY}{(1-t)(1-q)}] \;=\; \sum_\mu \frac{T_\mu\,\tilde H_\mu(x;q,t)\,\tilde H_\mu(y;q,t)}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} \;=\; \nabla^y\,\tilde\Omega[\tfrac{XY}{(1-t)(1-q)}]\;. \qquad 1.42$$

where ∇^x and ∇^y denote ∇ acting on symmetric function in the alphabets X and Y respectively. However, since $\tilde\Omega[\frac{XY}{(1-t)(1-q)}]$ is the reproducing kernel of the $*$-scalar product, the relation in 1.42 is equivalent to the identity

$$\langle \nabla P\,,Q\rangle_* \;=\; \langle P\,,\nabla\,Q\rangle_*\;. \qquad 1.43$$

Entirely analogous arguments based on 1.11 a) and b) yield the identities

$$\langle D_0\,P\,,Q\rangle_* = \langle P\,,D_0\,Q\rangle_*\,,\qquad \langle D_0^*\,P\,,Q\rangle_* = \langle P\,,D_0^*\,Q\rangle_*\;.$$

Finally, recalling that ∂_1 is the Hall scalar product adjoint of multiplication by h_1 (or e_1), we see that 1.37 gives

$$\langle e_1^* P\,,\,Q\rangle_* = \langle \phi\omega(e_1^* P)\,,\,Q\rangle \;=\; \langle e_1\,\phi\omega P\,,\,Q\rangle \;=\; \langle \phi\omega P\,,\,\partial_1 Q\rangle \;=\; \langle P\,,\,\partial_1 Q\rangle_*\,.$$

Q.E.D.

Proposition 1.10

For $k \geq 1$, the operators D_k and D_k^* are $*$-adjoint to $(-1)^k D_{-k}$ and $(-qt)^k D_{-k}^*$ respectively.

Proof

We need only show this for one of the pairs since the other pair can be dealt with in exactly the same way. Now, the statement that D_k^* and $(-qt)^k D_{-k}^*$ are $*$-adjoint is equivalent to the identity

$$ {}^{x}D_k^*\,\tilde\Omega\Big[\frac{X\,Y}{M}\Big] \;=\; (-qt)^k\;{}^{y}D_{-k}^*\,\tilde\Omega\Big[\frac{X\,Y}{M}\Big] \qquad 1.44$$

where " ${}^{x}D_k^*$" and " ${}^{y}D_{-k}^*$" represent these operators acting on the X and Y alphabets respectively. However, 1.8 b) gives

$$\begin{aligned} {}^{x}D_k^*\,\tilde\Omega\Big[\frac{X\,Y}{M}\Big] &= \tilde\Omega\Big[\frac{(X-\frac{\tilde M}{z})\,Y}{M}\Big]\Omega\big[zX\big]\,\Big|_{z^k} \;=\; \tilde\Omega\Big[\frac{X\,Y}{M}\Big]\tilde\Omega\Big[\frac{-\frac{\tilde M}{z}\,Y}{M}\Big]\Omega\big[zX\big]\,\Big|_{z^k} \\ &= \tilde\Omega\Big[\frac{X\,Y}{M}\Big]\tilde\Omega\Big[-\frac{Y}{z\,t\,q}\Big]\Omega\big[zX\big]\,\Big|_{z^k} \;=\; \tilde\Omega\Big[\frac{X\,Y}{M}\Big]\Omega\Big[\frac{Y}{{}^{-}z\,t\,q}\Big]\Omega\big[zX\big]\,\Big|_{z^k} \end{aligned}$$

and similarly

$$\begin{aligned} {}^{y}D_{-k}^*\,\tilde\Omega\Big[\frac{X\,Y}{M}\Big] &= \tilde\Omega\Big[\frac{X\,(Y-\frac{\tilde M}{z})}{M}\Big]\Omega\big[zY\big]\,\Big|_{z^{-k}} = \tilde\Omega\Big[\frac{X\,Y}{M}\Big]\tilde\Omega\Big[\frac{-\frac{\tilde M}{z}X}{M}\Big]\Omega\big[zY\big]\,\Big|_{z^{-k}} \\ &= \tilde\Omega\Big[\frac{X\,Y}{M}\Big]\tilde\Omega\Big[-\frac{X}{z\,t\,q}\Big]\Omega\big[zY\big]\,\Big|_{z^{-k}} = \tilde\Omega\Big[\frac{X\,Y}{M}\Big]\Omega\Big[\frac{X}{{}^{-}z\,t\,q}\Big]\Omega\big[zY\big]\,\Big|_{z^{-k}}\;. \end{aligned}$$

Then 1.44 follows since for any two formal power series $\Phi(z)$ and $\Psi(z)$ we have

$$\Phi\Big(\frac{1}{{}^{-}z\,q\,t}\Big)\,\Psi(z)\Big|_{z^k} \;=\; \Big(-\frac{1}{q\,t}\Big)^k\,\Phi(z)\Psi\Big(\frac{1}{{}^{-}z\,q\,t}\Big)\,\Big|_{z^{-k}}\;.$$

The expansion in 1.7 has the following surprising corollary.

Proposition 1.11

If P and Q are homogeneous polynomials of degrees k and $n-k$ respectively we have

$$\begin{aligned} &a) \quad \langle h_{n-k}\,P\,,Q\rangle \;=\; \langle P\,,\mathcal{T}_1\,Q\rangle \\ &b) \quad \langle e_{n-k}^*\,P\,,Q\rangle_* \;=\; \langle P\,,\mathcal{T}_1\,Q\rangle_* \end{aligned} \qquad 1.45$$

Proof

From 1.7 with $u=1$ and the Hall adjointness of $\underline{S}_m$ and ∂_{S_m} we get

$$\langle P\,,\mathcal{T}_1\,Q\rangle \;=\; \sum_{m\geq 0}\langle\, P\,,\partial_{S_m}Q\rangle \;=\; \sum_{m\geq 0}\langle h_m\,P\,,Q\rangle\ .$$

However this reduces to 1.45 a) since $\langle h_m\,P\,,Q\rangle\neq 0$ only when $\deg(h_m\,P)=\deg(Q)$, and that is when $m=n-k$.

To prove 1.45 b) note that 1.37, 1.45 a) and 1.37 give

$$\begin{aligned}\langle P\,,\mathcal{T}_1\,Q\rangle_* &= \langle\phi\,\omega P\,,\mathcal{T}_1\,Q\rangle\\ &= \langle h_{n-k}\,\phi\,\omega\,P\,,Q\rangle\\ &= \langle\phi\,\omega(e^*_{n-k}\,P)\,,Q\rangle\\ &= \langle e^*_{n-k}\,P\,,Q\rangle_*\ .\end{aligned}$$

This completes our proof.

The last item we need to deal with here is the definition of the "*skewed*" version of the polynomials $\tilde{H}_\mu(x;q,t)$. To this end we need the following auxiliary result:

Proposition 1.12

There are rational functions $d^\lambda_{\mu\nu}(q,t)$ such that

$$\tilde{H}_\mu\,\tilde{H}_\nu \;=\; \sum_{\lambda\supseteq\mu,\nu} d^\lambda_{\mu\nu}(q,t)\,\tilde{H}_\lambda\ . \tag{1.46}$$

Proof

The $*$-duality of the bases $\{\tilde{H}_\lambda\}_\lambda$ and $\{\tilde{H}_\lambda/\tilde{h}_\lambda\tilde{h}'_\lambda\}_\lambda$ gives that these coefficients are given by the formula

$$d^\lambda_{\mu\nu}(q,t) \;=\; \langle\tilde{H}_\mu\,\tilde{H}_\nu\,,\tilde{H}_\lambda/\tilde{h}_\lambda\tilde{h}'_\lambda\rangle_*\ , \tag{1.47}$$

from which the rationality easily follows. The fact that the sum in 1.46 runs only over pairs partitions λ which contain both μ and ν is an immediate consequence of the Macdonald Pieri formulas (see [15] Ch VI (7.1') and (7.4)).

We have the following immediate consequence of 1.46.

Theorem 1.3

For any two alphabets X and Y we have

$$\tilde{H}_\lambda[X+Y;q,t] \;=\; \sum_{\mu,\nu\subseteq\lambda}\tilde{H}_\mu[X;q,t]\,\tilde{H}_\nu[Y;q,t]\,c^\lambda_{\mu,\nu}(q,t) \tag{1.48}$$

with

$$c^\lambda_{\mu,\nu} \;=\; \frac{d^\lambda_{\mu,\nu}\,\tilde{h}_\lambda\tilde{h}'_\lambda}{\tilde{h}_\mu\tilde{h}'_\mu\,\tilde{h}_\nu\tilde{h}'_\nu}\ . \tag{1.49}$$

Proof

Note that if Z is an additional auxiliary alphabet, and we make the replacements $X \to X + Y$, $Y \to Z$ in 1.18 c), we obtain

$$\tilde{\Omega}[\tfrac{XZ}{M}]\,\tilde{\Omega}[\tfrac{YZ}{M}] \;=\; \tilde{\Omega}[\tfrac{(X+Y)Z}{M}] \;=\; \sum_{\lambda} \frac{\tilde{H}_\lambda[X+Y;q,t]\,\tilde{H}_\lambda[Z;q,t]}{\tilde{h}_\lambda \tilde{h}'_\lambda} \;. \qquad 1.50$$

On the other hand again from 1.18 c) we get

$$\tilde{\Omega}[\tfrac{XZ}{M}]\,\tilde{\Omega}[\tfrac{YZ}{M}] \;=\; \sum_{\mu,\nu} \frac{\tilde{H}_\mu[X]\,\tilde{H}_\nu[Y]}{\tilde{h}_\mu \tilde{h}'_\mu\, \tilde{h}_\nu \tilde{h}'_\nu}\; \tilde{H}_\mu[Z;q,t]\,\tilde{H}_\nu[Z;q,t] \;. \qquad 1.51$$

Combining 1.50 and 1.51 and using 1.46, we finally obtain that

$$\sum_{\lambda} \frac{\tilde{H}_\lambda[X+Y;q,t]\,\tilde{H}_\lambda[Z;q,t]}{\tilde{h}_\lambda \tilde{h}'_\lambda} = \sum_{\mu,\nu} \frac{\tilde{H}_\mu[X]\,\tilde{H}_\nu[Y]}{\tilde{h}_\mu \tilde{h}'_\mu\, \tilde{h}_\nu \tilde{h}'_\nu} \sum_{\lambda \supseteq \mu,\nu} d^\lambda_{\mu\nu}(q,t)\, \tilde{H}_\lambda[Z;q,t]$$

$$= \sum_{\lambda} \tilde{H}_\lambda[Z;q,t] \sum_{\mu,\nu \subseteq \lambda} d^\lambda_{\mu\nu}(q,t)\, \frac{\tilde{H}_\mu[X]\,\tilde{H}_\nu[Y]}{\tilde{h}_\mu \tilde{h}'_\mu\, \tilde{h}_\nu \tilde{h}'_\nu}$$

and 1.48 (with 1.49) follows by equating coefficients of $\tilde{H}_\lambda[Z;q,t]$.

In analogy with the Schur function case (as well as definition 7.5, p. 344 of [15]) we shall here and after set, for any alphabet Y,

$$\tilde{H}_{\lambda/\mu}[Y;q,t] \;=\; \sum_{\nu \subseteq \lambda} c^\lambda_{\mu\,\nu}(q,t)\,\tilde{H}_\nu[Y;q,t] \;. \qquad 1.52$$

This permits us to write the addition formula 1.31 in the form

$$\tilde{H}_\lambda[X+Y;q,t] \;=\; \sum_{\mu \subseteq \lambda} \tilde{H}_\mu[X;q,t]\,\tilde{H}_{\lambda/\mu}[Y;q,t] \;. \qquad 1.53$$

Remark 1.1

An easy calculation yields that $\tilde{H}_{11/1} = (1+t)S_1$ and $\tilde{H}_{21/2} = \frac{t^2-q}{t-q}S_1$. This given, word of caution should be added here concerning the subscript λ/μ appearing in the left-hand side of 1.52. We have used this notation mainly as a reminder that $\tilde{H}_{\lambda/\mu}$ is defined by 1.52 only for $\mu \subseteq \lambda$. This should not be taken to mean that this polynomial depends only on the diagram of the skew partition λ/μ. The best way to interpret the meaning of our definition is that $\tilde{H}_{\lambda/\mu}$ is simply an abbreviation for the right-hand side of 1.52 when $\mu \subseteq \lambda$ and is equal to 0 when $\mu \not\subseteq \lambda$.

Remark 1.2

Note that since the definitions in 1.46 and 1.49 give

$$\Big\langle \frac{\tilde{H}_\mu}{\tilde{h}_\mu \tilde{h}'_\mu}\,\tilde{H}_\nu\,,\tilde{H}_\lambda \Big\rangle_* \;=\; \frac{\tilde{h}_\lambda \tilde{h}'_\lambda}{\tilde{h}_\mu \tilde{h}'_\mu}\; d^\lambda_{\mu,\nu} \;=\; c^\lambda_{\mu,\nu}\;\tilde{h}_\nu \tilde{h}'_\nu \;=\; \langle \tilde{H}_\nu\,,\tilde{H}_{\lambda/\mu}\rangle_* \;,$$

we see that the linear extension of the map

$$\tilde{H}_\mu \to \tilde{H}_{\lambda/\mu} \tag{1.54}$$

may be viewed as the $*$-scalar product adjoint of multiplication by $\tilde{H}_\mu/\tilde{h}_\mu\tilde{h}'_\mu$.

2. Proofs of the main results.

Our arguments here hinge on the following fundamental fact:

Theorem 2.1

Every symmetric polynomial P, homogeneous of degree $k \geq 1$, may be written in the form

$$P \;=\; D_1 A \;+\; \underline{e}_1 B \tag{2.1}$$

with A, B homogeneous symmetric polynomials of degree $k-1$. Moreover, if $P \in \Lambda_{Z[q,t]}$ then 2.1 holds true with $A = R[X;q,t]/M^{k-1}$ and $B = S[X;q,t]/M^{k-1}$, with R and S polynomials in $\Lambda_{Z[q,t]}$. Of course, the same result holds true with D_1 replaced by D_1^* in 2.1.

Proof

It is sufficient to work with D_1 since the result for D_1^* immediately follows by an application of 1.14 c). For convenience, we shall write

$$U[X;q,t] \;\equiv_{\underline{e}_1}\; V[X;q,t] \tag{2.2}$$

to indicate that $U[X;q,t] - V[X;q,t] = e_1 S[X;q,t]$ with $S[X;q,t] \in \Lambda_{Z[q,t]}$. This given, we shall show that for every elementary basis element $e_\alpha = e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_m}$ with $\alpha \vdash k$, we have an identity of the form

$$M^{k-1}\, e_\alpha = M^{k-1}\, e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_m} \;\equiv_{\underline{e}_1}\; D_1\, R[X;q,t] \tag{2.3}$$

(with $R[X;q,t] \in \Lambda_{Z[q,t]}$) .

We shall prove 2.3 by a process which was first used in [7]. The idea is to proceed by an induction which "descends" on the largest part of α. To begin with note that we have

$$\begin{aligned} D_1\, e_1^{k-1} \;&=\; (e_1[X+\tfrac{M}{z}])^{k-1}\Omega[-zX]\Big|_z \\ &=\; (e_1[X]+\tfrac{M}{z})^{k-1}\Omega[-zX]\Big|_z \\ &\equiv_{\underline{e}_1}\; \frac{M^{k-1}}{z^{k-1}}\,\Omega[-zX]\Big|_z \;=\; (-1)^k\, M^{k-1}e_k[X]\ . \end{aligned}$$

In other words

$$M^{k-1}\, e_k[X] \;\equiv_{\underline{e}_1}\; (-1)^k\, D_1\, e_1^{k-1}\ . \tag{2.4}$$

This proves 2.3 when the largest part of α is as large as possible. Let us then assume that we have

$$M^{k-1}\, e_{\beta_1}e_{\beta_2}\cdots e_{\beta_m} \;\equiv_{\underline{e}_1}\; D_1\, R_\beta[X;q,t] \quad (\text{with } R_\beta[X;q,t] \in \Lambda_{Z[q,t]}) \tag{2.5}$$

when $\beta_1 > a$. Our goal is to use this to prove 2.3 for

$$\alpha = (a \geq \alpha_2 \geq \cdots \geq \alpha_k \geq 0) \ .$$

To this end we note that

$$\begin{aligned} D_1\, e_1^{a-1}\, e_{\alpha_2} \cdots e_{\alpha_k} \;&=\; (e_1 + \tfrac{M}{z})^{a-1} \left(\prod_{i=2}^{k} \Big(\sum_{r_i=0}^{\alpha_i} e_{\alpha_i - r_i}[X]\, \tfrac{e_{r_i}[M]}{z^{r_i}} \Big) \right) \Omega[-zX] \Big|_{z} \\ &\equiv_{\,\underline{e}_1} \; \frac{M^{a-1}}{z^a} \left(\prod_{i=2}^{k} \Big(\sum_{r_i=0}^{\alpha_i} e_{\alpha_i - r_i}[X]\, \tfrac{e_{r_i}[M]}{z^{r_i}} \Big) \right) \Omega[-zX] \Big|_{z^0} \ . \end{aligned}$$

Multiplying both sides by M^{k-a}, expanding the product and arranging the resulting terms according to increasing powers of z gives

$$\begin{aligned} &M^{k-a}\, D_1\, e_1^{a-1}\, e_{\alpha_2} \cdots e_{\alpha_k} \\ &\quad \equiv_{\,\underline{e}_1} \; M^{k-1} \Big(\frac{e_{\alpha_2} \cdots e_{\alpha_k}}{z^a} + \sum_{b>a} \sum_{\beta_2 \geq \cdots \geq \beta_k} c_{b,\beta_2,\cdots,\beta_k}[M]\, \frac{e_{\beta_2} \cdots e_{\beta_k}}{z^b} \Big) \Omega[-zX] \Big|_{z^0} \\ &\quad \equiv_{\,\underline{e}_1} \; M^{k-1} \Big((-1)^a\, e_a e_{\alpha_2} \cdots e_{\alpha_k} + \sum_{b>a} (-1)^b \sum_{\beta_2 \geq \cdots \geq \beta_k} c_{b,\beta_2,\cdots,\beta_k}[M]\, e_b\, e_{\beta_2} \cdots e_{\beta_k} \Big) \end{aligned}$$

with $c_{b,\beta_2,\cdots,\beta_k}[M]$ an elementary basis element plethystically evaluated at M. In other words we obtain that

$$\begin{aligned} M^{k-1} e_a e_{\alpha_2} \cdots e_{\alpha_k} \equiv_{\,\underline{e}_1} & \ (-1)^a M^{k-a} D_1\, e_1^{a-1}\, e_{\alpha_2} \cdots e_{\alpha_k} \\ & - \sum_{b>a} (-1)^{b-a} \sum_{\beta_2 \geq \cdots \geq \beta_k} c_{b,\beta_2,\cdots,\beta_k}[M]\, M^{k-1} e_b e_{\beta_2} \cdots e_{\beta_k} \end{aligned}$$

and the induction hypothesis in 2.5 yields 2.3 as desired.

We are now in a position to give our

Proof of Theorem I.2

We shall begin by showing I.16. To this end it will be convenient to write our operator $\mathbf{\Pi}'$ in the form (see I.15)

$$\mathbf{\Pi}' \;=\; \nabla^{-1} \mathcal{T}_{\epsilon}^{-1} \qquad\qquad (\text{with } \epsilon =^{-} 1). \tag{2.6}$$

This given, we are to show that for any homogeneous polynomial f, of degree $d(f)$, we have

$$\langle f \ , \ \tilde{H}_\mu[X+1; q, t] \rangle_* \;=\; \mathbf{\Pi}'_f[D_\mu] \qquad\qquad (\text{for all } \mu). \tag{2.7}$$

We shall proceed by induction on $d(f)$. Since $\tilde{H}_\emptyset = 1$ we have

$$\nabla 1 \;=\; \nabla^{-1} 1 \;=\; 1 \ . \tag{2.8}$$

Thus 2.6 gives $\mathbf{\Pi}'_f \equiv 1$ for $f \equiv 1$. On the other hand, the expansion in I.5 and the normalization in 1.12 yield that

$$\langle 1\,,\ \tilde{H}_\mu[X+1;q,t]\rangle_* \;=\; \tilde{H}_\mu[1;q,t] \;=\; \tilde{K}_{n,\mu}(q,t) \;=\; \tilde{H}_\mu\,|_{S_n} \;=\; 1\ .$$

Thus 2.7 is trivially true when f is a constant, and we can start our induction at $d(f)=0$. Let us then assume that 2.7 is true for all $\mu \vdash n$ and for $d(f)<k$. Now, since both sides of 2.7 are linear in f, we can use Theorem 2.1 and complete the induction argument by a direct verification of 2.7 when $f=D_1\,A$ and $f=\underline{e}_1\,B$.

Case 1) $f=D_1\,A$ with A homogeneous of degree $k-1$

We start by noting that we have (using Propositions 1.10 & 1.9, 1.36 a) for $k=0$, and 1.11 a)):

$$\begin{aligned}\langle f\,,\ \tilde{H}_\mu[X+1;q,t]\rangle_* &= \langle D_1 A\,,\ \mathcal{T}_1\tilde{H}_\mu\rangle_* \;=\; -\langle A\,,\ D_{-1}\mathcal{T}_1\tilde{H}_\mu\rangle_*\\ &= \langle A\,,\ \mathcal{T}_1 D_0\tilde{H}_\mu\rangle_* \;-\; \langle A\,,\ D_0\mathcal{T}_1\tilde{H}_\mu\rangle_*\\ &= -D_\mu(q,t)\langle A\,,\ \mathcal{T}_1\tilde{H}_\mu\rangle_* \;-\; \langle D_0 A\,,\ \mathcal{T}_1\tilde{H}_\mu\rangle_*\end{aligned}$$

Since by assumption A is homogeneous of degree $k-1$ and D_0 preserves degree, we can use the induction hypothesis on A and D_0A and finally obtain that

$$\langle f\,,\ \tilde{H}_\mu[X+1;q,t]\rangle_* \;=\; -D_\mu(q,t)\,\mathbf{\Pi}'_A[D_\mu(q,t)] \;-\; \mathbf{\Pi}'_{D_0A}[D_\mu(q,t)]\ .$$

In conclusion, the validity of 2.7 in this case will be established if we can show that we have

$$\mathbf{\Pi}'_{D_1A}[D_\mu(q,t)] \;=\; -D_\mu(q,t)\,\mathbf{\Pi}'_A[D_\mu(q,t)] \;-\; \mathbf{\Pi}'_{D_0A}[D_\mu(q,t)]\,, \qquad (\text{for all } \mu\,)$$

or, equivalently, that

$$\mathbf{\Pi}'_{D_1A}[X] \;=\; -e_1[X]\,\mathbf{\Pi}'_A[X] \;-\; \mathbf{\Pi}'_{D_0A}[X]\ .$$

Recalling the definition of $\mathbf{\Pi}'$ in 2.6, we are brought to verify the operator identity

$$\nabla^{-1}\mathcal{T}_\epsilon^{-1}D_1 \;=\; -\underline{e}_1\nabla^{-1}\mathcal{T}_\epsilon^{-1} \;-\; \nabla^{-1}\mathcal{T}_\epsilon^{-1}D_0\ . \tag{2.10}$$

To prove this, note that equating the left hand side of 1.28 a) (with $k=0$) with the left hand side of 1.30 b) we derive that

$$D_1 \;=\; -\nabla\,\underline{e}_1\nabla^{-1}\ . \tag{2.11}$$

On the other hand, 1.36 b) gives

$$\mathcal{T}_\epsilon D_1\mathcal{T}_\epsilon^{-1} \;=\; D_0+D_1\ . \tag{2.12}$$

Combining these two identities yields

$$- \mathcal{T}_\epsilon \nabla \underline{e}_1 \nabla^{-1} \mathcal{T}_\epsilon^{-1} \;=\; \mathcal{T}_\epsilon D_1 \mathcal{T}_\epsilon^{-1} \;=\; D_0 + D_1$$

which is easily seen to be just another way of writing 2.10. This completes the proof of the first case.

Case 2) $f = e_1 B$ *with* B *homogeneous of degree* $k-1$

We start by noting that ∂_1 and $\mathcal{T}_1$ are commuting operators. This is easily verified by having them alternately act on any power basis element. This given, since $e_1 = Me_1^*$, the identity in 1.31 a) gives

$$\begin{aligned} \langle e_1 B \,,\, \tilde H_\mu[X+1;q,t\,]\rangle_* \;&=\; M\,\langle B \,,\, \partial_1 \mathcal{T}_1 \tilde H_\mu \rangle_* \\ &=\; M\,\langle B \,,\, \mathcal{T}_1 \partial_1 \tilde H_\mu \rangle_* \\ &=\; M \sum_{\nu \to \mu} c_{\mu\nu}(q,t)\,\langle B \,,\, \tilde H_\nu[X+1;q,t\,]\rangle_* \;. \end{aligned} \qquad 2.13$$

Now it was shown in [8] (Theorem 2.2) that the following identities hold true for every partition μ:

$$\sum_{\nu \to \mu} c_{\mu\nu}(q,t)(T_{\mu/\nu})^r \;=\; \begin{cases} \frac{tq}{M}\, h_{r+1}[D_\mu(q,t)/tq] & \text{if } r > 0\,, \\[2ex] B_\mu(q,t) & \text{if } r = 0\,. \end{cases} \qquad 2.14$$

This given, we see that 2.13 for $k = 1$ and $B = 1$ reduces to

$$\langle e_1 \,,\, \tilde H_\mu[X+1;q,t\,]\rangle_* \;=\; M \sum_{\nu\to\mu} c_{\mu\nu}(q,t) \;=\; MB_\mu(q,t) \;=\; D_\mu(q,t) + 1$$

Thus the validity of 2.7 for the case $f = e_1 \cdot 1$ requires that

$$\nabla^{-1} \mathcal{T}_\epsilon^{-1} e_1 \;=\; e_1 + 1 \;. \qquad 2.15$$

To verify this, we note that definition in 1.8 a), 2.8 and 2.11 give

$$-e_1 = \Omega[-zX]\big|_z \;=\; D_1 \cdot 1 \;=\; D_1 \nabla\, 1 \;=\; -\nabla e_1 \cdot 1 \;=\; -\nabla e_1 \;.$$

In other words

$$\nabla^{-1} e_1 \;=\; e_1 \;.$$

We have then

$$\nabla^{-1} \mathcal{T}_\epsilon^{-1} e_1 \;=\; \nabla^{-1}(e_1 - e_1[\epsilon]) \;=\; \nabla^{-1}(e_1 + 1) \;=\; e_1 + 1\,,$$

as desired.

This establishes Case 2) for $k=1$. Let us now deal with the case when B is of degree $k-1>0$. To this end, we start by using the induction hypothesis in 2.13 and get

$$\langle e_1 B\,,\ \tilde H_\mu[X+1;q,t\,]\rangle_* \;=\; M\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,\mathbf{\Pi}'_B[D_\nu(q,t)]\ . \qquad 2.16$$

Since $D_\nu = D_\mu - MT_{\mu/\nu}$, by I.4 and 1.33 we may write

$$\mathbf{\Pi}'_B[D_\nu(q,t)] \;=\; \sum_{r=0}^{k-1}\mathbf{\Pi}'_B[D_\mu - M/z]\big|_{z^{-r}}(T_{\mu/\nu})^r\ .$$

Substituting this back into 2.16 gives

$$\begin{aligned}\langle e_1 B\,, \tilde H_\mu[X+1;q,t]\rangle_* &= M\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\sum_{r=0}^{k-1}\mathbf{\Pi}'_B[D_\mu - M/z]\big|_{z^{-r}}(T_{\mu/\nu})^r\\ &= M\sum_{r=0}^{k-1}\mathbf{\Pi}'_B[D_\mu - M/z]\big|_{z^{-r}}\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,(T_{\mu/\nu})^r\ .\end{aligned} \qquad 2.17$$

We now use 2.14 and get

$$\begin{aligned}&\langle e_1 B\,,\ \tilde H_\mu[X+1;q,t\,]\rangle_*\\ &\quad= M\,\mathbf{\Pi}'_B[D_\mu]B_\mu(q,t)\\ &\qquad\quad+\ M\sum_{r=1}^{k-1}\mathbf{\Pi}'_B[D_\mu - M/z]\big|_{z^{-r}}\times\frac{t\,q}{M}\,h_{r+1}[D_\mu(q,t)/tq]\\ &\quad= M\,\mathbf{\Pi}'_B[D_\mu]B_\mu(q,t)\ -\ \mathbf{\Pi}'_B[D_\mu]D_\mu(q,t)\\ &\qquad\quad+\ \sum_{r=0}^{k-1}\mathbf{\Pi}'_B[D_\mu - M/z]\big|_{z^{-r}}\times t\,q\,\Omega[zD_\mu(q,t)/tq]\big|_{z^{r+1}}\\ &\quad= \mathbf{\Pi}'_B[D_\mu] + \sum_{r=0}^{k-1}\mathbf{\Pi}'_B[D_\mu - M/z]\big|_{z^{-r}}\times t\,q\,\Omega[zD_\mu(q,t)/tq]\big|_{z^{r+1}}\ .\end{aligned}$$

In other words we have

$$\langle e_1 B\,,\ \tilde H_\mu[X+1;q,t]\rangle_* = \mathbf{\Pi}'_B[D_\mu] + \mathbf{\Pi}'_B[D_\mu - M/tqz]\Omega[zD_\mu(q,t)]\Big|_z\,. \quad 2.18$$

This equality results from the fact that for any two formal power series $\Phi(z),\Psi(z)$, we have

$$\sum_{r\ge 0}\Phi(1/z)\Big|_{z^{-r}}\times q\,t\,\Psi(z/q\,t)\big|_{z^{r+1}} \;=\; q\,t\,\Phi(1/z)\Psi(z/q\,t)\big|_z \;=\; \Phi(1/tqz)\Psi(z)\big|_z\,.$$

Recalling the definition of D_1^* given in 1.8, we see that 2.18 is none other than

$$\langle e_1 B\,,\ \tilde H_\mu[X+1;q,t\,]\rangle_* \;=\; \mathbf{\Pi}'_B[D_\mu]\ +\ (D_1^*\mathbf{\Pi}'_B)[D_\mu]\ .$$

Thus the validity of 2.7 for $f = e_1 B$ is reduced to showing that we have

$$\mathbf{\Pi}'_{e_1 B} \;=\; \mathbf{\Pi}'_B \;+\; D_1^* \mathbf{\Pi}'_B \;. \qquad 2.19$$

For this to hold for all k we must have

$$\nabla^{-1} \mathcal{T}_\epsilon^{-1} \underline{e}_1 \;=\; \nabla^{-1} \mathcal{T}_\epsilon^{-1} \;+\; D_1^* \nabla^{-1} \mathcal{T}_\epsilon^{-1} \;,$$

or better, multiplying on the right by $\mathcal{T}_\epsilon \nabla$,

$$\nabla^{-1} \mathcal{T}_\epsilon^{-1} \underline{e}_1 \, \mathcal{T}_\epsilon \nabla \;=\; 1 \;+\; D_1^* \;. \qquad 2.20$$

Now to show this, note that for any polynomial $P[X]$ we have

$$\mathcal{T}_\epsilon^{-1} \underline{e}_1 \, \mathcal{T}_\epsilon \, P[X] \;=\; \mathcal{T}_\epsilon^{-1} \underline{e}_1 \, P[X+\epsilon] \;=\; (\underline{e}_1 - e_1[\epsilon]) P[X] \;=\; (\underline{e}_1 + 1) P[X] \;.$$

Thus 2.20 is equivalent to

$$\nabla^{-1} (\underline{e}_1 + 1) \nabla \;=\; 1 \;+\; D_1^* \;.$$

Namely

$$\nabla^{-1} \underline{e}_1 \nabla \;=\; D_1^* \;. \qquad 2.21$$

But this is none other than what we obtain by equating the right hand side of 1.28 b) to the right hand side of 1.30 b*). This completes the proof of I.16. Note then that when $d(f) = k \leq n$ we can use 1.45 b) and obtain that

$$\langle f \;,\; \tilde{H}_\mu[X+1; q,t] \rangle_* \;=\; \langle e_{n-k}^* f \;,\; \tilde{H}_\mu \rangle_* \qquad 2.22$$

which yields I.18 a). As for I.18 b), we note that 2.22 for $f \to f[\frac{-^-X}{M}] = \omega \phi^{-1} f$ gives

$$\begin{aligned} \langle \omega \phi^{-1} f \;,\; \tilde{H}_\mu[X+1; q,t] \rangle_* \;&=\; \langle (\omega \phi^{-1} h_{n-k}) \, \omega \phi^{-1} f \;,\; \tilde{H}_\mu \rangle_* \\ &=\; \langle \omega \phi^{-1} (h_{n-k} f) \;,\; \tilde{H}_\mu \rangle_* \;=\; \langle h_{n-k} \, f \;,\; \tilde{H}_\mu \rangle \;, \end{aligned}$$

and I.18 a) gives I.18 b) with

$$\mathbf{\Pi}_f[X; q,t] \;=\; \mathbf{\Pi}'_{\omega \phi^{-1} f}[X; q,t] \;=\; \nabla^{-1} \mathcal{T}_\epsilon^{-1} f[\tfrac{-^-X}{M}] \;=\; \nabla^{-1} f[\tfrac{1 - ^-X}{(1-t)(1-q)}]$$

completing the proof of Theorem I.2.

Proof of Theorem I.3

For convenience let us set

$$E_\mu[X; q,t] \;=\; \Omega[\tfrac{X}{M}] \, \nabla^{-1} \, \omega \, \Omega[\tfrac{D_\mu}{M} X] \;. \qquad 2.23$$

This given, the equality

$$E_\mu[X;q,t\,] = \tilde H_\mu[X+1;q,t\,] \qquad 2.24$$

is a consequence of the following remarkable sequence of equivalent expressions:

$$\begin{array}{rcll}
\mathbf{\Pi}'_f[D_\mu\,;q,t] &=& \langle \nabla^{-1} f\,[X -^{-} 1;q,t]\,,\Omega[D_\mu X]\rangle & (1)\\
(1.40)\rightarrow &=& \langle \nabla^{-1} f\,[X -^{-} 1;q,t]\,,\omega\,\Omega[\tfrac{D_\mu}{M}X]\rangle_* & (2)\\
(\text{Prop. } 1.9)\rightarrow &=& \langle f\,[X -^{-} 1;q,t]\,,\nabla^{-1}\omega\,\Omega[\tfrac{D_\mu}{M}X]\rangle_* & (3)\\
(\text{def. } 1.5\text{a})\ \&\ 1.37)\rightarrow &=& \langle \mathcal{T}_{-1} f\,[X;q,t]\,,\phi\,\omega\,\nabla^{-1}\omega\,\Omega[\tfrac{D_\mu}{M}X]\rangle & (4)\\
(\text{def. } 1.5\,\text{b})\,)\rightarrow &=& \langle f\,[X;q,t]\,,\mathcal{P}_{-1}\,\phi\,\omega\,\nabla^{-1}\omega\,\Omega[\tfrac{D_\mu}{M}X]\rangle & (5)\\
(\text{I}.2\ \&\ \text{def. } 1.38)\rightarrow &=& \Big\langle f\,[X;q,t]\,,\phi\,\omega\,\Big(\Omega[\tfrac{X}{M}]\,\nabla^{-1}\omega\,\Omega[\tfrac{D_\mu}{M}X]\Big)\Big\rangle & (6)\\
(1.37)\rightarrow &=& \langle f\,[X;q,t]\,,\Omega[\tfrac{X}{M}]\,\nabla^{-1}\omega\,\Omega[\tfrac{D_\mu}{M}X]\rangle_*\ . & (7)
\end{array} \qquad 2.25$$

In other words, we must have

$$\mathbf{\Pi}'_f[D_\mu\,;q,t] \;=\; \langle f\ ,E_\mu[X;q,t]\rangle_*\ .$$

Combining this with 2.7 we obtain that

$$\langle f\,,\,\tilde H_\mu[X+1]\,\rangle_* \;=\; \langle f\,,E_\mu[X;q,t]\rangle_* \qquad 2.26$$

must hold true for any symmetric polynomial f, forcing the equality in 2.24. This completes our proof of Theorem I.3.

Remark 2.1

We should note that, conversely, starting from 2.24 and carrying out the steps in 2.25 in the reverse order (7)$\rightarrow$(6)$\rightarrow\cdots\rightarrow$(2)$\rightarrow$(1) shows that I.20 and I.16 are simply equivalent statements.

Remark 2.2

We should also mention that the identity in I.16 contains the remarkable fact that for any homogeneous polynomial P of degree $d(P)$ we have

$$\mathbf{\Pi}'_P[D_\mu(q,t)\,] \;=\; 0 \qquad\qquad \text{for all } |\mu|<d(P) \qquad 2.27$$

Indeed, according to I.16, the right hand side of this equality should be given by $\langle P,\tilde H_\mu[X+1;q,t]\rangle_*$, but this vanishes simply because $\tilde H_\mu[X+1;q,t\,]$ has degree less than $d(P)$.

This brings us to the

Proof of Theorem I.4

Applying I.16 with the replacements $\mu \to \lambda$ and $f \to \tilde{H}_\mu$, and using the expansion in 1.53 gives

$$\begin{aligned} \mathbf{\Pi}'_{\tilde{H}_\mu}[D_\lambda(q,t)] &= \langle \tilde{H}_\mu, \tilde{H}_\lambda[X+1] \rangle_* \\ &= \sum_{\rho \subseteq \lambda} \langle \tilde{H}_\mu \ , \ \tilde{H}_\rho \rangle_* \, \tilde{H}_{\lambda/\rho}[1;q,t\,] \\ &= \tilde{h}_\mu \tilde{h}'_\mu \, \tilde{H}_{\lambda/\mu}[1;q,t\,] \end{aligned}$$

the last equality resulting from the $*$–duality of the pair of bases in 1.19 c). This proves the first part of I.24 when $\mu \subseteq \lambda$. The second part follows from the observations in Remark 1.1. In particular this shows that the polynomial $\delta_\mu[X;q,t\,]$ has much stronger vanishing properties than the more general polynomials $\mathbf{\Pi}'_P[X;q,t\,]$.

We terminate this section by establishing the main result of the paper:

Proof of Theorem I.1

Our point of departure is the identity

$$\tilde{K}_{(n-k,\gamma),\mu}(q,t) \;=\; \langle S^*_{(n-k,\gamma)'} \ , \ \tilde{H}_\mu \rangle_* \qquad 2.28$$

which is an immediate consequence of the expansion in I.5 and the $*$-duality of the pair of bases in 1.19 b). This given, our argument is based on the fact that $D_k^*|_{q=t=\infty}$ is the "creation" operator for Schur functions. More precisely, if we set for any $P \in \Lambda$

$$\mathbf{H}_k \, P[X] \;=\; P[X - \tfrac{1}{z}] \, \Omega[zX]\big|_{z^k} \qquad 2.29$$

then for $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ we have

$$S_\lambda[X] \;=\; \mathbf{H}_{\lambda_1} \mathbf{H}_{\lambda_2} \cdots \mathbf{H}_{\lambda_k} 1 \ . \qquad 2.30$$

This is a classical result whose proof may be found in Section 4. This given we see that when λ is of the form $(n-k,\gamma)$ with $\gamma \vdash k$, we can write

$$\begin{aligned} S_{(n-k,\gamma)}[X] &= \mathbf{H}_{n-k} \, S_\gamma[X] \;=\; S_\gamma[X - \tfrac{1}{z}] \, \Omega[zX]\big|_{z^{n-k}} \\ &= \sum_{i=0}^{|\gamma|} S_\gamma[X - \tfrac{1}{z}]\,\big|_{z^{-i}} \ \Omega[zX]\big|_{z^{n-k+i}} \\ &= \sum_{i=0}^{|\gamma|} h_{n-k+i}[X] \, S_\gamma[X - \tfrac{1}{z}]\,\big|_{z^{-i}} \ . \end{aligned}$$

In particular, we derive that

$$S^*_{(n-k,\gamma)'}[X] \;=\; \sum_{i=0}^{|\gamma|} e^*_{n-k+i}[X]\, S_\gamma[\tfrac{X}{M}-\tfrac{1}{z}]\,\big|_{z^{-i}} \;.$$

Substituting this in 2.28 gives

$$\tilde K_{(n-k,\gamma),\mu}(q,t) \;=\; \sum_{i=0}^{|\gamma|} \langle e^*_{n-k+i}[X]\, S_\gamma[\tfrac{X}{M}-\tfrac{1}{z}]\;,\; \tilde H_\mu\rangle_*\Big|_{z^{-i}} \;. \qquad 2.31$$

From I.18 a) we then get that

$$\langle e^*_{n-k+i}[X]\, S_\gamma[\tfrac{X}{M}-\tfrac{1}{z}]\;,\; \tilde H_\mu\rangle_*\Big|_{z^{-i}} \;=\; \mathbf{\Pi'}_{P_i}[D_\mu(q,t)]$$

with

$$P_i \;=\; S_\gamma[\tfrac{X}{M}-\tfrac{1}{z}]\Big|_{z^{-i}} \;.$$

Thus

$$\tilde K_{(n-k,\gamma),\mu}(q,t) \;=\; \sum_{i=0}^{|\gamma|} \mathbf{\Pi'}_{P_i}[D_\mu(q,t)] \;. \qquad 2.32$$

Now the definition in 2.6 gives

$$\mathbf{\Pi'}_{P_i} \;=\; \nabla^{-1} S_\gamma[\tfrac{X-^{-1}}{M}-\tfrac{1}{z}]\Big|_{z^{-i}} \;.$$

Substituting this in 2.32, we finally get

$$\begin{aligned}\tilde K_{(n-k,\gamma),\mu}(q,t) \;&=\; \sum_{i=0}^{|\gamma|} \nabla^{-1} S_\gamma[\tfrac{X-^{-1}}{M}-\tfrac{1}{z}]\Big|_{z^{-i}}\;\Big|_{X\to D_\mu(q,t)} \\ &=\; \nabla^{-1} S_\gamma[\tfrac{X-^{-1}}{M}-1]\;\Big|_{X\to D_\mu(q,t)} \;.\end{aligned}$$

This completes our proof of Theorem I.1.

Remark 2.3

It follows from Theorem I.3 that the expression

$$E_\mu[X;q,t] \;=\; \Omega[\tfrac{X}{M}]\,\nabla^{-1}\,\omega\,\Omega[\tfrac{D_\mu}{M}X] \qquad 2.33$$

defines a polynomial of degree $|\mu|$. However, there is a more illuminating way to see this. To begin with, note that from 1.18 c) and the definition in I.8, we get that

$$\nabla^{-1}\omega\Omega[\tfrac{D_\mu}{M}X] \;=\; \sum_\alpha \frac{\tilde H_\alpha[X;q,t]}{T_\alpha}\,\frac{\tilde H_\alpha[D_\mu]}{\tilde h_\alpha \tilde h'_\alpha} \;. \qquad 2.34$$

On the other hand, since we have

$$\Omega[\tfrac{X}{M}] \;=\; \tilde{\Omega}[\tfrac{-^{-}X}{M}] \;,$$

again from 1.18 c) we obtain that

$$\Omega[\tfrac{X}{M}] \;=\; \sum_{\beta} \tilde{H}_\beta[X;q,t\,]\, \frac{\tilde{H}_\beta[-^{-}1]}{\tilde{h}_\beta \tilde{h}'_\beta} \;. \tag{2.35}$$

Multiplying 2.35 and 2.34, the definition in 2.33 gives

$$E_\mu[X;q,t\,] \;=\; \sum_{\alpha}\sum_{\beta} \tilde{H}_\alpha[X]\,\tilde{H}_\beta[X]\, \frac{\tilde{H}_\alpha[D_\mu]}{T_\alpha\, \tilde{h}_\alpha \tilde{h}'_\alpha}\, \frac{\tilde{H}_\beta[-^{-}1]}{\tilde{h}_\beta \tilde{h}'_\beta} \;. \tag{2.36}$$

We can thus apply 1.46 and 1.49, and then 1.48 to obtain

$$\begin{aligned} E_\mu[X;q,t\,] &= \sum_{\alpha}\sum_{\beta}\Big(\sum_{\lambda\supseteq\alpha,\beta} d^{\lambda}_{\alpha,\beta}\,\tilde{H}_\lambda[X]\Big)\frac{\tilde{H}_\alpha[D_\mu]}{T_\alpha\,\tilde{h}_\alpha\tilde{h}'_\alpha}\,\frac{\tilde{H}_\beta[-^{-}1]}{\tilde{h}_\beta\tilde{h}'_\beta} \\ &= \sum_{\lambda}\tilde{H}_\lambda[X]\sum_{\alpha,\beta\subseteq\lambda} d^{\lambda}_{\alpha,\beta}\,\frac{\tilde{H}_\alpha[D_\mu]}{T_\alpha\,\tilde{h}_\alpha\tilde{h}'_\alpha}\,\frac{\tilde{H}_\beta[-^{-}1]}{\tilde{h}_\beta\tilde{h}'_\beta} \\ &= \sum_{\lambda}\frac{\tilde{H}_\lambda[X]}{\tilde{h}_\lambda\tilde{h}'_\lambda}\Big(\sum_{\alpha,\beta\subseteq\lambda}\frac{c^{\lambda}_{\alpha,\beta}}{T_\alpha}\,\tilde{H}_\alpha[D_\mu]\,\tilde{H}_\beta[-^{-}1]\Big) \\ &= \sum_{\lambda}\frac{\tilde{H}_\lambda[X]}{\tilde{h}_\lambda\tilde{h}'_\lambda}\Big(\sum_{\alpha,\beta\subseteq\lambda}\frac{c^{\lambda}_{\alpha,\beta}}{T_\alpha}\,\tilde{H}_\alpha[Y]\,\tilde{H}_\beta[-^{-}1]\Big)\Big|_{Y\to D_\mu} \\ &= \sum_{\lambda}\tilde{H}_\lambda[X]\;\frac{\nabla_y^{-1}\,\tilde{H}_\lambda[Y-^{-}1]}{\tilde{h}_\lambda\tilde{h}'_\lambda}\Big|_{Y\to D_\mu} \;=\; \sum_{\lambda}\tilde{H}_\lambda[X]\;\;\delta_\lambda[D_\mu] \;, \end{aligned}$$

which shows that the polynomiality of $E_\mu[X;q,t\,]$ is a direct consequence of the vanishing properties of δ_λ.

Remark 2.4

Soon after the original conjecture of formula 2.24, we discovered the following extremely simple "*proof*". First rewrite the formula as

$$\tilde{H}_\mu[X;q,t\,] \;=\; \mathcal{T}_{-1}\mathcal{P}_{1/M}\nabla^{-1}\omega\Omega[\tfrac{XD_\mu}{M}] \;. \tag{2.37}$$

Now $\tilde{H}_\mu$ is uniquely characterized up to a scalar factor as the eigenfunction of D_0 with eigenvalue D_μ, so we must verify that the right hand side of 2.37 has the same property. Now we have

$$\begin{aligned} D_0 &= \mathcal{T}_1^{-1}(D_0 - D_{-1})\mathcal{T}_1 && \text{(by 1.36 a))} \\ &= -\,\mathcal{T}_1^{-1}\mathcal{P}_{1/M}D_{-1}\mathcal{P}_{-1/M}\mathcal{T}_1 && \text{(by 1.35 a))} \\ &= -M\,\mathcal{T}_1^{-1}\mathcal{P}_{1/M}\nabla^{-1}\partial_1\nabla\mathcal{P}_{-1/M}\mathcal{T}_1 && \text{(by 1.24 a) \& 1.30 a))} \end{aligned}$$

and this immediately yields

$$\begin{aligned} & D_0\, \mathcal{T}_{-1}\mathcal{P}_{1/M}\nabla^{-1}\omega\Omega[\tfrac{XD_\mu}{M}] \\ & \qquad = \; -M\; \mathcal{T}_1^{-1}\mathcal{P}_{1/M}\nabla^{-1}\partial_1\nabla\mathcal{P}_{-1/M}\mathcal{T}_1\;\; \mathcal{T}_{-1}\mathcal{P}_{1/M}\nabla^{-1}\omega\Omega[\tfrac{XD_\mu}{M}] \\ & \qquad = \; -M\; \mathcal{T}_1^{-1}\mathcal{P}_{1/M}\nabla^{-1}\partial_1\omega\Omega[\tfrac{XD_\mu}{M}] \\ (\dagger) & \qquad = \; -M\; \mathcal{T}_1^{-1}\mathcal{P}_{1/M}\nabla^{-1}\frac{D_\mu}{M}\;\omega\Omega[\tfrac{XD_\mu}{M}] \\ & \qquad = \; -D_\mu\; \mathcal{T}_1^{-1}\mathcal{P}_{1/M}\nabla^{-1}\;\omega\Omega[\tfrac{XD_\mu}{M}] \end{aligned}$$

as desired. The missing scalar factor is easily shown to be 1 by setting $X = 0$ in (2.23). For this proof we don't need the deeper identity 2.14; we only need that $\partial_1 \tilde{H}_\mu = \sum_{\nu\to\mu} c_{\mu\nu}\tilde{H}_\nu$ without explicit knowledge of the coefficients $c_{\mu\nu}$, (that is only formula (6.7) p. 332 of [15]) .

On further reflection, however, it is clear that something must be wrong with this argument. Indeed, if A is any quantity whatsoever, it appears to show that the expression

$$\mathcal{T}_1^{-1}\mathcal{P}_{1/M}\nabla^{-1}\omega\Omega[\tfrac{XA}{M}]$$

is an eigenfunction of D_0 with eigenvalue A, a highly unlikely possibility!

The problem is that the right hand side of 2.33 is, *a priori*, a formal series, containing terms of unbounded degree. One cannot apply the operator $\mathcal{T}^{-1}$ to such a series, just as one cannot substitute $x-1$ for x in a formal power series in one variable.

It is possible to evade this difficulty to a certain extent by proving instead that $E_\mu[X;q,t\,]$ is an eigenfunction of the operator $\mathcal{T}_1 D_0 \mathcal{T}_1^{-1}$. This makes sense because the latter operator is equal to $D_0 - D_{-1}$ which **can** be applied to a formal series. The problem this causes is that if we admit formal series as eigenfunctions, the inhomogeneous operator $D_0 - D_{-1}$ no longer has a "discrete spectrum": it has in fact infinitely many independent eigenfunctions with any given eigenvalue A. All we can say is that $\tilde{H}_\mu[X+1;q,t\,]$ is its unique "*polynomial*" eigenvector with eigenvalue $-D_\mu$. Absent a separate ($\dagger\dagger$) demonstration that $E_\mu[X;q,t\,]$ is a polynomial, this particular would-be "*proof*" is incomplete.

3. Some applications

Formula I.11 yields yet one more path for establishing the integrality of the Macdonald q,t-Kostka coefficients. To see this we need a few preliminary observations. To begin with it follows from the Macdonald "duality" result ([15] (5.1) p. 327) that we have

$$\tilde{K}_{\lambda'\mu}(q,t) \;=\; t^{n(\mu)}q^{n(\mu')}\tilde{K}_{\lambda\mu}(1/q,1/t)\ . \qquad 3.1$$

($\dagger$) This is because for any A we have $\partial_1\omega\exp(p_1[XA]) = A\,\omega\exp(p_1[XA])$.

($\dagger\dagger$) One that does not use Theorem I.2 .

In particular we see that if $\tilde{K}_{\lambda\mu}(q,t)$ is a polynomial then it must be of degrees $\leq n(\mu)$ in t and $\leq n(\mu')$ in q. Thus the definition $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,1/t)$ guarantees that

$$a) \quad \tilde{K}_{\lambda\mu}(q,t) \in \mathbf{Z}[q,t] \qquad \Longleftrightarrow \qquad b) \quad K_{\lambda\mu}(q,t) \in \mathbf{Z}[q,t] \ . \tag{3.2}$$

There are a number of algorithms for constructing the $K_{\lambda\mu}(q,t)$ that stem from the various identities established in Macdonald's original papers [14], [15]. All of these algorithms introduce denominators of one kind or another. The simplest and most remarkable of these algorithms is one discovered by Vinet-Lapointe [13]. They observed that the Macdonald "integral form"

$$J_\mu[X_n;q,t] = \sum_{\lambda \vdash n} S_\lambda[X_n(1-t)]K_{\lambda\mu}(q,t) \tag{3.3}$$

may be constructed "one column at a time" by applications of successive specializations of the Macdonald operator $D_n(X;q,t)$. More precisely they set

$$\mathbf{B_n}^{(r)} \ = \ D(-1/qt^{n-r-1};q,t)\,\underline{e}_r$$

and note that if ν is any partition with no more than $r \leq n$ parts, and μ is obtained by adding a column of length r to ν, then

$$J_\mu[X;q,t] \ = \ \frac{1}{\prod_{i=r+1}^{n}(1-q^{-1}t^{r-i+1})} \ \mathbf{B}_n^{(r)}\, J_\nu[X;q,t\,] \ . \tag{3.4}$$

Recalling that we have set

$$\tilde{H}_\mu[X;q,t] \ = \ t^{n(\mu)}J_\mu[\tfrac{X}{1-1/t};q,1/t\,] \ , \tag{3.5}$$

we can easily see that when $\tilde{H}_\mu[X;q,t]$ is constructed by combining 3.5 with the recurrence in 3.4, the $\tilde{K}_{\lambda\mu}(q,t)\,'s$ will necessarily come out as polynomials in $\mathbf{Z}[q,t]$ divided by factors of the form

$$1-t^r \quad , \qquad q-t^r \quad , \qquad t^rq^s \ .$$

Now another consequence of the Macdonald duality theorem is that we have

$$\tilde{K}_{\lambda\mu}(q,t) \ = \ \tilde{K}_{\lambda\mu'}(t,q) \ . \tag{3.6}$$

This shows that $\tilde{K}_{\lambda\mu}(q,t)$ itself may also be given an expression consisting of a polynomial in $\mathbf{Z}[q,t]$ divided by factors of the form

$$1-q^r \quad , \qquad t-q^r \quad , \qquad t^rq^s \ .$$

Comparing these two sets of factors we see that each of these two different expressions for $\tilde{K}_{\lambda\mu}(q,t)$ must in the end simplify to the point that the only remaining factors are of the form

$$1-t \quad , \qquad 1-q \quad , \qquad t-q \quad , \qquad t^r q^s \quad .$$

Now specializations at $t=1$ or at $q=1$ have been given by Macdonald (see ex. 7, p. 364 of [15]) yielding, for instance, that

$$\tilde{H}_\mu[X;q,1] \;=\; \prod_{i=1}^{l(\mu)} h_{\mu_i}[\tfrac{X}{1-q}](1-q)(1-q^2)\cdots(1-q^{\mu_i})$$

from which we can easily derive that $\tilde{K}_{\lambda\mu}(q,1) \in \mathbf{Z}[q,t]$. This excludes at once both $1-q$ and $1-t$ as possible denominator factors. Similarly, it is also shown in [15] ((8.12) p. 354) that $K_{\lambda\mu}(0,t)$ is none other than the "Kostka-Foulkes" coefficient. This, together with 3.6, eliminates at once both factors t^r and q^s, leaving only powers of

$$t-q \tag{3.7}$$

as possible denominators!

In conclusion, to complete the proof of 3.2 a) and b), we only need a result expressing $\tilde{K}_{\lambda\mu}(q,t)$ as a rational function with denominator factors coprime with $t-q$. Our formula I.11 provides precisely such an expression. In fact, the two sources of denominators in I.11 are the application of ∇^{-1} and the plethystic substitution of $1/(1-t)(1-q)$. However, it is easily seen from the definition in I.1 that the latter only introduces denominator factors of the form $(1-t^i)(1-q^i)$, and this is sufficient for our purposes here. As for ∇^{-1}, we can use the identities we have already collected in this section and derive that the only denominator factors it can possibly introduce are powers of t, q and M.

To see this, let us assume, by induction, that for all $g \in \Lambda_{\mathbf{Z}[q,t]}$, which are homogeneous of degree $k-1$, we have $\nabla^{-1} g \in \Lambda_{\mathbf{Z}[q,t,1/q,1/t,1/M]}$. By Theorem 2.1, we can complete the induction by proving that we also have $\nabla^{-1} f \in \Lambda_{\mathbf{Z}[q,t,1/q,1/t,1/M]}$ when $f = D_1 g$ or $f = \underline{e}_1 g$, with $g = S[X;q,t]/M^{k-1}$ and S homogeneous of degree $k-1$. Now in the first case the identity in 2.11 gives

$$\nabla^{-1} f \;=\; \nabla^{-1} D_1\, S[X;q,t]/M^{k-1} \;=\; -\underline{e}_1\, \nabla^{-1} S[X;q,t]/M^{k-1} \quad ,$$

and in the second case, we can apply 2.21 and derive that

$$\nabla^{-1} f \;=\; \nabla^{-1} \underline{e}_1\, S[X;q,t]/M^{k-1} \;=\; D_1^*\, \nabla^{-1} S[X;q,t]/M^{k-1} \quad .$$

Since $\underline{e}_1$ introduces no denominators and D_1^* (see 1.8 b), at the worst, introduces powers of qt in the denominator, the induction hypothesis yields that in both cases we must have $f \in \Lambda_{\mathbf{Z}[q,t,1/q,1/t,1/M]}$ as desired. This completes our proof that $\tilde{K}_{\lambda\mu}(q,t) \in \mathbf{Z}[q,t]$.

Remark 3.1

We should mention that a more refined argument (see [2]) proves that ∇ itself is "integral" and ∇^{-1} is "Laurent". More precisely we have

$$\nabla \Lambda_{\mathbf{Z}[q,t]} \subseteq \Lambda_{\mathbf{Z}[q,t]} \qquad \text{and} \qquad \nabla^{-1}\Lambda_{\mathbf{Z}[q,t]} \subseteq \Lambda_{\mathbf{Z}[q,t,\frac{1}{t},\frac{1}{q}]} \ ,$$

and this is best possible.

The next application is our derivation of the symmetric function results of Sahi [16] and Knop [11], [12]. Since these two works are very closely related we shall deal only with Sahi's case here.

The results we are concerned with here may be stated as follows:

Theorem 3.1 (Sahi)

For any $\mu \vdash n$ there is a unique polynomial $R_\mu[X;q,t\,] \in \Lambda_{\mathbf{Z}[q,t]}$ with the vanishing properties

$$R_\mu[\,\textstyle\sum_{i=1}^n t^{-n+i} q^{-\nu_i}; q,t\,] \;=\; 0 \qquad \textit{for all } |\nu| \leq |\mu| \;\&\; \nu \neq \mu \tag{3.8}$$

and the normalization

$$R_\mu[\,\textstyle\sum_{i=1}^n t^{-n+i} q^{-\mu_i}; q,t\,] \;=\; 1\,. \tag{3.9}$$

This polynomial can also be characterized, up to a scalar factor, by the difference equation

$$\tilde{D}_1\, R_\mu[X;q,t\,] \;=\; \Big(\tfrac{1-t^n}{1-t} \;-\; \textstyle\sum_{i=1}^n t^{n-i} q^{\mu_i}\Big) R_\mu[X;q,t\,]\,, \tag{3.10}$$

where $\tilde{D}_1$ is the non-homogeneous difference operator

$$\tilde{D}_1 \;=\; \sum_{i=1}^n \Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{tx_i - x_j}{x_i - x_j}\Big)(1-\tfrac{1}{x_i})(1-T_q^{(i)}) \tag{3.11}$$

and $T_q^{(i)}$ is the operator that changes x_i into qx_i in a polynomial in $x_1, x_2, \ldots, x_n$.

Our results not only explicitly identify $R_\mu[X;q,t\,]$ as an image of the Macdonald polynomial P_μ, but also determine the values taken by the left hand side of 3.8 for all the other choices of μ. Moreover we can show that the difference equation in 3.10 is itself the appropriately "shifted" image of the Macdonald difference operator.

To be precise we have:

Theorem 3.2

The polynomial $R_\mu[X;q,t\,]$ may be obtained by deforming the polynomial defined in I.22 according to the following equation:

$$R_\mu[X;q,t\,] \;=\; \delta_\mu[t^n(1-1/t)X-t^n;q^{-1},t\,]\;. \qquad 3.12$$

In particular we must also have

$$R_\mu\Big[\textstyle\sum_{i=1}^n t^{-n+i}q^{-\nu_i}\;;q\,,t\Big] \;=\; \begin{cases}\tilde H_{\nu/\mu}[1;q^{-1},t\,] & \textit{if } \mu\subseteq\nu\,,\\ 0 & \textit{otherwise.}\end{cases} \qquad 3.13$$

Moreover, the difference equation in 3.10 may be obtained by applying the corresponding deformation to the difference equation

$$D_0^*\,\delta_\mu[X;q,t\,] \;-\; \tfrac{M}{tq}\partial_1\,\delta_\mu[X;q,t\,] \;=\; -D_\mu(1/q,1/t)\,\delta_\mu[X;q,t\,] \qquad 3.14$$

which characterizes our polynomial $\delta_\mu[X;q,t\,]$.

Proof

For convenience, let us set as Sahi does

$$\overline{\mu}(q,t) \;=\; \sum_{i=1}^n t^{-n+i}q^{-\mu_i}\;.$$

The definition in I.3 can then be rewritten as

$$MB_\mu(q,t) \;=\; 1-t^n-t^{n-1}(1-t)\,\overline{\mu}(q^{-1},t)\;.$$

Thus from I.4 we obtain that

$$D_\nu(q^{-1},t) \;=\; t^n(1-1/t)\,\overline{\nu}(q,t)-t^n\;. \qquad 3.15$$

Making the replacements $q\to q^{-1}$ and $\lambda\!\to\!\nu$ in I.24, we get

$$\delta_\mu\Big[t^n(1-1/t)\;\overline{\nu}(q,t)-t^n\;;q^{-1},t\Big] \;=\; \begin{cases}\tilde H_{\nu/\mu}[1;q^{-1},t\,] & \text{if } \mu\subseteq\nu\,,\\ 0 & \text{otherwise.}\end{cases} \qquad 3.16$$

In particular, we obtain that the polynomial $\delta_\mu\Big[t^n(1-1/t)\;X-t^n\;;q^{-1},t\Big]$ satisfies the conditions in 3.8 and 3.9 that characterize $R_\mu[X;q,t\,]$. This proves the identity in 3.12 and thus 3.13 follows from 3.16 .

To prove 3.14 we start by noting that 1.36 b) gives

$$\mathcal{T}_\epsilon^{-1} D_0^* \;=\; D_0^* \mathcal{T}_\epsilon^{-1} \;+\; D_{-1}^* \mathcal{T}_\epsilon^{-1} \;.$$

Thus the Macdonald equation

$$D_0^* \tilde{H}_\mu[X;q,t] \;=\; -D_\mu(\tfrac{1}{q},\tfrac{1}{t})\, \tilde{H}_\mu[X;q,t]$$

may be converted to

$$D_0^* \mathcal{T}_\epsilon^{-1} \tilde{H}_\mu[X;q,t] \;+\; D_{-1}^* \mathcal{T}_\epsilon^{-1} \tilde{H}_\mu[X;q,t] \;=\; -D_\mu(\tfrac{1}{q},\tfrac{1}{t})\, \mathcal{T}_\epsilon^{-1} \tilde{H}_\mu[X;q,t] \;.$$

Applying ∇^{-1} to both sides and using the commutativity of D_0^* and ∇, we can write

$$\begin{aligned} D_0^* \nabla^{-1} \mathcal{T}_\epsilon^{-1} \tilde{H}_\mu[X;q,t] + \nabla^{-1} D_{-1}^* \nabla \nabla^{-1} \mathcal{T}_\epsilon^{-1} \tilde{H}_\mu[X;q,t] \\ = -D_\mu(\tfrac{1}{q},\tfrac{1}{t})\, \nabla^{-1} \mathcal{T}_\epsilon^{-1} \tilde{H}_\mu[X;q,t] \;. \end{aligned}$$

Thus the definition in I.22 gives

$$D_0^* \delta_\mu[X;q,t] \;+\; \nabla^{-1} D_{-1}^* \nabla\, \delta_\mu[X;q,t] \;=\; -D_\mu(\tfrac{1}{q},\tfrac{1}{t})\, \delta_\mu[X;q,t] \;. \qquad 3.17$$

On the other hand, 1.24 b) and 1.30 a*) give

$$-D_{-1}^* \;=\; D_0^* \partial_1 - \partial_1 D_0^* \;=\; \tfrac{M}{qt}\, \nabla \partial_1 \nabla^{-1}$$

reducing 3.17 to

$$D_0^* \delta_\mu[X;q,t] \;-\; \tfrac{M}{qt} \partial_1\, \delta_\mu[X;q,t] \;=\; -D_\mu(\tfrac{1}{q},\tfrac{1}{t})\, \delta_\mu[X;q,t] \qquad 3.18$$

as desired.

We are left to show that 3.10 is just another way of writing 3.18. To this end, we recall the definition in 1.8 b), and write 3.18 as

$$\delta_\mu[X - \tfrac{M}{zqt};q,t]\Omega[zX]\,\big|_{z^0} \;-\; \tfrac{M}{qt}\, \partial_1\, \delta_\mu[X;q,t] \;=\; -D_\mu(\tfrac{1}{q},\tfrac{1}{t}) \delta_\mu[X;q,t] \;.$$

Making the replacements $q \to 1/q$, $X \to t^n(1-1/t)X - t^n$ gives

$$\begin{aligned} &\delta_\mu[t^n(1-1/t)X - t^n - \tfrac{(1-1/t)(1-q)}{z};q^{-1},t]\Omega[z(t^n(1-1/t)X - t^n)]\,\big|_{z^0} \\ &\quad - \;(1-1/t)(1-q)\,(\partial_1\, \delta_\mu)[t^n(1-1/t)X - t^n;q^{-1},t] \qquad 3.19 \\ &\quad\quad = \; -D_\mu(q,\tfrac{1}{t}) \delta_\mu[t^n(1-1/t)X - t^n;q^{-1},t] \;. \end{aligned}$$

Since

$$(\partial_1\, \delta_\mu)[t^n(1-1/t)X - t^n;q^{-1},t] \;=\; \frac{1}{t^n(1-1/t)}\, \partial_1 \Big(\delta_\mu[t^n(1-1/t)X - t^n;q^{-1},t]\Big)$$

we can use 3.12 and rewrite 3.19 as

$$\begin{aligned} R_\mu[X - \tfrac{1-q}{t^n z}\,; q,t\,]\;\Omega[zt^n(1-1/t)X](1-t^n z)]\,\big|_{z^0} & \\ - \;\tfrac{1-q}{t^n}\,\partial_1\, R_\mu[X;q,t\,] \;\; = & \\ - \,D_\mu(q,\tfrac{1}{t})\, R_\mu[X;q,t\,]\,. \end{aligned}$$

Since we can make the replacement $zt^n \to z$ before taking the coefficient of z^0, this equation is equivalent to

$$\begin{aligned} R_\mu[X - \tfrac{1-q}{z}\,; q,t\,]\;\Omega[z(1-1/t)X](1-z)]\,\big|_{z^0} & \\ - \;\tfrac{1-q}{t^n}\,\partial_1\, R_\mu[X;q,t\,] \;\; = & \\ - \,D_\mu(q,\tfrac{1}{t})\, R_\mu[X;q,t\,]\,. \end{aligned} \tag{3.20}$$

Simple manipulations yield that

$$-t^n\, D_\mu(q,1/t) \quad = \quad 1-(1-t)\sum_{i=1}^{n} t^{n-i} q^{\mu_i} \quad .$$

This given, multiplying 3.20 by $t^n/(1-t)$, and adding $\frac{t^n}{t-1}R_\mu$ to both sides we finally obtain that

$$\begin{aligned} \tfrac{t^n}{t-1}\, R_\mu[X;q,t\,] & \\ + \;\tfrac{t^n}{1-t}\, R_\mu[X - \tfrac{1-q}{z}\,; q,t\,]\;\Omega[z(1-1/t)X](1-z)]\,\big|_{z^0} & \\ - \;\tfrac{1-q}{1-t}\,\partial_1\, R_\mu[X;q,t\,] \;\; = \;\; \left(\tfrac{1-t^n}{1-t} - \textstyle\sum_{i=1}^{n} t^{n-i} q^{\mu_i}\right)\; R_\mu[X;q,t\,] & \end{aligned} \tag{3.21}$$

Now we shall show in Section 4 that the Sahi operator $\tilde{D}_1$ may also be given the plethystic form

$$\tilde{D}_1 P[X] = \tfrac{t^n}{t-1}\, P[X] + \tfrac{t^n}{1-t}\, P[X - \tfrac{1-q}{z}]\;\Omega[z(1-1/t)X](1-z)\,\big|_{z^0} - \tfrac{1-q}{1-t}\,\partial_1\, P[X]\,. \tag{3.22}$$

Thus 3.20 reduces to 3.10 as desired, and our proof is complete.

Our final application is the Macdonald-Koornwinder reciprocity formula ([15] eq. (6.6) p. 332). Simple manipulations allow us to state this identity in the following plethystic form.

Theorem 3.3

For all pairs of partitions λ , μ we have

$$\tilde{H}_\mu[1+u\,D_\lambda(q,t);q,t]\,\Omega[uB_\mu(q,t)] \;=\; \tilde{H}_\lambda[1+u\,D_\mu(q,t);q,t]\,\Omega[uB_\lambda(q,t)]\,. \tag{3.23}$$

Proof

Since $\Omega[XY]$ is the reproducing kernel for the Hall scalar product, we have

$$\tilde{H}_\mu[1+u\,D_\lambda;q,t\,] \quad = \quad \langle \tilde{H}_\mu[X+1;q,t\,]\;,\;\Omega[uXD_\lambda]\rangle \quad .$$

Thus, expressing $\tilde H_\mu[X+1;q,t]$ by means of our formula I.20, we obtain the following remarkable sequence of equalities.

$$
\begin{aligned}
&\tilde H_\mu[1+u\,D_\lambda;q,t\,]\,\Omega[uB_\mu]\\
&\qquad = \Omega[uB_\mu]\left\langle \mathcal{P}_{1/M}\nabla^{-1}\omega\Omega[\tfrac{XD_\mu}{M}]\,,\,\Omega[uXD_\lambda]\right\rangle\\
&\qquad = \Omega[uB_\mu]\left\langle \nabla^{-1}\omega\Omega[\tfrac{XD_\mu}{M}]\,,\,\mathcal{T}_{1/M}\Omega[uXD_\lambda]\right\rangle\\
&\qquad = \Omega[uB_\mu]\,\Omega[\tfrac{uD_\lambda}{M}]\left\langle \nabla^{-1}\Omega[\tfrac{XD_\mu}{M}]\,,\,\Omega[uXD_\lambda]\right\rangle\\
&\qquad = \Omega[uB_\mu]\,\Omega[uB_\lambda]\,\Omega[-\tfrac{u}{M}]\left\langle \nabla^{-1}\omega\Omega[\tfrac{XD_\mu}{M}]\,,\,\Omega[uXD_\lambda]\right\rangle\\
(\text{by } 1.40)&\qquad = \Omega[uB_\mu]\,\Omega[uB_\lambda]\,\Omega[-\tfrac{u}{M}]\left\langle \nabla^{-1}\omega\Omega[\tfrac{XD_\mu}{M}]\,,\,\omega\Omega[\tfrac{XD_\lambda}{M}]\right\rangle_*
\end{aligned}
$$

and this proves 3.23 since the last expression is symmetric in μ and λ by virtue of the $*$-self-adjointness of the operator ∇.

As a corollary we immediately obtain our version of the Macdonald specialization ([15] (6.17) p. 338).

Theorem 3.4

$$\tilde H_\mu[1-u;q,t\,] \;=\; \prod_{s\in\mu}(1-u\,t^{l'_\mu(s)}q^{a'_\mu(s)}) \qquad 3.24$$

Proof

Simply set $\lambda=\emptyset$ in 3.23.

4. Auxiliary identities

We shall begin by converting some of the basic difference operators to plethystic form.

Theorem 4.1

For any $P\in\Lambda$ set

$$\mathbf{H}_m P]X] \;=\; P[X-\tfrac{1}{z}]\Omega[zX]\,\big|_{z^m}\ . \qquad 4.1$$

Then for $\lambda=(\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n\ge 0)$ we have

$$S_\lambda[X] \;=\; \mathbf{H}_{\lambda_1}\mathbf{H}_{\lambda_2}\cdots\mathbf{H}_{\lambda_n}\cdot 1\ . \qquad 4.2$$

Proof

The bideterminatal formula for Schur functions (†) may be written in the form

$$S_{\lambda_1,\ldots,\lambda_n}[X_n] \;=\; \sum_{\sigma\in S_{[1,n]}}\sigma\Big(\frac{x_1^{n-1+\lambda_1}x_2^{n-2+\lambda_2}\cdots x_n^{n-n+\lambda_n}}{\prod_{1\le i<j\le n}(x_i-x_j)}\Big). \qquad 4.3$$

(†) (3.1) p. 40 of [15]

Now, by means of the left coset decomposition

$$\sum_{\sigma\in S_{[1,n]}} \sigma \;=\; \sum_{i=1}^{n}(i,1)\sum_{\alpha\in S_{[2,n]}}\alpha \;,$$

we can readily transforms 4.3 into the recursion

$$S_{\lambda_1,\dots,\lambda_n}[X_n] \;=\; \sum_{i=1}^{n} \frac{x_i^{\lambda_1+n-1}}{\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)}\, S_{\lambda_2,\dots,\lambda_n}[X_n-x_i]\;. \qquad 4.4$$

Let us then set for $P\in\Lambda$

$$\mathbf{H}_m^{(n)}P[X_n] \;=\; \sum_{i=1}^{n} A_i(x)\,x_i^m \quad P[X_n-x_i]\;, \qquad 4.5$$

where for convenience we let

$$A_i(x) \;=\; \frac{x_i^{n-1}}{\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)}\;.$$

This given, to prove 4.2 we only need to show that $\mathbf{H}_m^{(n)}$ also has the plethystic form

$$\mathbf{H}_m^{(n)}\,P[X_n] \;=\; P[X_n-\tfrac{1}{z}]\Omega[zX_n]\,\big|_{z^m}\;.$$

To this end we note that we can write, for an arbitrary alphabet Y

$$\begin{aligned}
\frac{\mathbf{H}_m^{(n)}\,\Omega[X_nY]}{\Omega[X_nY]} &= \sum_{i=1}^{n} A_i(x)\,x_i^m\,\Omega[-x_iY]\\
&= \sum_{i=1}^{n} A_i(x)\,x_i^m \sum_{k\geq 0} x_i^k\,h_k[-Y] \qquad 4.6\\
&= \sum_{k\geq 0} h_k[-Y]\sum_{i=1}^{n} A_i(x)\,x_i^{m+k}\,.
\end{aligned}$$

Now from the partial fraction expansion

$$\Omega[zX_n] \;=\; \prod_{i=1}^{n}\frac{1}{1-zx_i} \;=\; \sum_{i=1}^{n} A_i(x)\,\frac{1}{1-zx_i}\;,$$

we derive that for all $m+k\geq 0$, we have

$$\sum_{i=1}^{n} A_i(x)\,x_i^{m+k} \;=\; \Omega[zX_n]\,\big|_{z^{m+k}}\;.$$

Substituting this in 4.6 gives

$$\begin{aligned}\frac{\mathbf{H}_m^{(n)}\,\Omega[X_nY]}{\Omega[X_nY]} &= \sum_{k\geq 0} h_k[-Y]\,\Omega[zX_n]\,\big|_{z^{m+k}} \\ &= \sum_{k\geq 0} \Omega[-\tfrac{1}{z}Y]\,\big|_{z^{-k}}\,\Omega[zX_n]\,\big|_{z^{m+k}} \\ &= \Omega[-\tfrac{1}{z}Y]\,\Omega[zX_n]\,\big|_{z^m}\ .\end{aligned}$$

That is

$$\mathbf{H}_m^{(n)}\,\Omega[X_nY] = \Omega[(X_n-\tfrac{1}{z})Y]\,\Omega[zX_n]\,\big|_{z^m}\ .$$

Equating coefficients of $S_\lambda[Y]$ on both sides of this equation yields that 4.5 is true for the Schur functions and therefore must be true for all $P\in\Lambda$ as desired, completing our proof

We shall prove next a similar result for the Macdonald and Sahi operators.

Theorem 4.2

For any $P\in\Lambda$ set

$$D_1^{(n)}P[X_n] = \sum_{i=1}^{n} A_i(x;t)\,T_q^{(i)} \qquad 4.7$$

and

$$\tilde{D}_1^{(n)}P[X_n] = \sum_{i=1}^{n} A_i(x;t)(1-\tfrac{1}{x_i})(1-T_q^{(i)}) \qquad 4.8$$

where

$$A_i(x;t) = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{tx_i-x_j}{x_1-x_j}\ . \qquad 4.9$$

Then for all $P\in\Lambda$ we have

$$D_1^{(n)}P[X_n] = \tfrac{1}{1-t}\,P[X_n] - \tfrac{t^n}{1-t}\,P[X_n-\tfrac{1-q}{z}]\,\Omega[z(1-1/t)X_n]\,\big|_{z^0}\ , \qquad 4.10$$

and

$$\begin{aligned}\tilde{D}_1^{(n)}P[X_n] = \tfrac{t^n}{t-1}\,P[X_n] + \tfrac{t^n}{1-t}\,P[X_n-\tfrac{1-q}{z}]\,\Omega[z(1-1/t)X_n](1-z)]\,\big|_{z^0} \\ -\tfrac{1-q}{1-t}\,\partial_1\,P[X_n]\ .\end{aligned} \qquad 4.11$$

Proof

The crucial ingredient here is the partial fraction expansion

$$\Omega[(1-1/t)zX_n] = \prod_{i=1}^{n}\frac{1-zx_i/t}{1-zx_i} = \frac{1}{t^n} + \frac{t-1}{t^n}\sum_{i=1}^{n}\frac{A_i(x;t)}{1-zx_i}\ , \qquad 4.12$$

which gives

$$\sum_{i=1}^{n} A_i(x;t)\, x_i^m \;=\; \frac{t^n}{t-1}\,\Omega[(1-1/t)zX_n]\Big|_{z^m} \qquad (\text{for all } m\geq 1)\,. \quad 4.13$$

We should also note that setting $z=0$ in 4.12 yields

$$\sum_{i=1}^{n} A_i(x;t) \;=\; \frac{t^n-1}{t-1}\;. \qquad 4.14$$

This given we have

$$\begin{aligned}
\frac{\tilde{D}_1^{(n)}\Omega[X_nY]}{\Omega[X_nY]} &= \sum_{i=1}^{n} A_i(x;t)(1-\tfrac{1}{x_i})(1-\Omega[(q-1)x_iY]) \\
&= \;-\;\sum_{i=1}^{n} A_i(x;t)(1-\tfrac{1}{x_i})\sum_{m\geq 1} h_m[(q-1)Y]\,x_i^m \qquad 4.15 \\
&= \;-\;\sum_{m\geq 1} h_m[(q-1)Y]\sum_{i=1}^{n} A_i(x;t)(1-\tfrac{1}{x_i})\,x_i^m\;.
\end{aligned}$$

Now using 4.13 we get that

$$\begin{aligned}
&\sum_{m\geq 1} h_m[(q-1)Y]\sum_{i=1}^{n} A_i(x;t)\,x_i^m \\
&\quad= \tfrac{t^n}{t-1}\sum_{m\geq 1} h_m[(q-1)Y]\,\Omega[(1-1/t)zX_n\,]\Big|_{z^m} \\
&\quad= \tfrac{t^n}{t-1}\sum_{m\geq 0} h_m[(q-1)Y]\,\Omega[(1-1/t)zX_n\,]\Big|_{z^m} - \tfrac{t^n}{t-1} \qquad 4.16 \\
&\quad= \tfrac{t^n}{t-1}\sum_{m\geq 0} \Omega[(q-1)Y/z]\Big|_{z^{-m}}\Omega[(1-1/t)zX_n\,]\Big|_{z^m} - \tfrac{t^n}{t-1} \\
&\quad= \tfrac{t^n}{t-1}\Omega[(q-1)Y/z]\,\Omega[(1-1/t)zX_n\,]\Big|_{z^0} - \tfrac{t^n}{t-1}
\end{aligned}$$

Similarly, using 4.13 and 4.14 we get

$$\begin{aligned}
&\sum_{m\geq 1} h_m[(q-1)Y]\sum_{i=1}^{n} A_i(x;t)\,x_i^{m-1} \\
&\quad= \tfrac{t^n}{t-1}\sum_{m\geq 2} h_m[(q-1)Y]\Omega[(1-1/t)zX_n\,]\Big|_{z^{m-1}} + \tfrac{(q-1)(t^n-1)}{t-1}e_1[Y] \\
&\quad= \tfrac{t^n}{t-1}\sum_{m\geq 1} h_m[(q-1)Y]\Omega[(1-1/t)zX_n\,]\Big|_{z^{m-1}} - \tfrac{q-1}{t-1}e_1[Y] \\
&\quad= \tfrac{t^n}{t-1}\Omega[(q-1)Y/z\,]\Omega[(1-1/t)zX_n\,]\Big|_{z^{-1}} - \tfrac{q-1}{t-1}e_1[Y]
\end{aligned} \qquad 4.17$$

Substituting 4.16 and 4.17 into 4.15 gives

$$\frac{\tilde D_1^{(n)}\Omega[X_nY]}{\Omega[X_nY]} \;=\; \tfrac{t^n}{t-1}+\tfrac{t^n}{1-t}\Omega[(q-1)Y/z\,]\Omega[(1-1/t)zX_n\,]\,(1-z)\Big|_{z^0}-\tfrac{q-1}{t-1}e_1[Y]\,.$$

In other words, we must have

$$\tilde D_1^{(n)}\Omega[X_nY]=\tfrac{t^n}{t-1}\Omega[X_nY]+\tfrac{t^n}{1-t}\,\Omega[(X-\tfrac{q-1}{z})Y\,]\Omega[(1-1/t)zX_n\,]\,(1-z)\Big|_{z^0}$$
$$-\tfrac{q-1}{t-1}\partial_1\Omega[X_nY]\,, \qquad 4.18$$

since

$$e_1[Y]\Omega[X_nY] \;=\; \partial_1\Omega[X_nY]\ .$$

Equating coefficients of $S_\lambda[Y]$ in 4.18 proves 4.11 for the Schur function basis and therefore establishes the validity of 4.11 for all symmetric polynomials.

To prove 4.10, note first that it follows immediately from the definitions in 4.7 and 4.8 that the Macdonald and Sahi operators are related by the identity

$${}^{top}\tilde D_1^{(n)} \;=\; \sum_{i=1}^n A_i\,-D_1^{(n)}\ .$$

Where the symbol "${}^{top}\tilde D_1^{(n)}$" is to represent the highest homogeneous component of $\tilde D_1^{(n)}$. Using 4.14 this can be written as

$$D_1^{(n)} \;=\; \tfrac{1-t^n}{1-t}\;-\;{}^{top}\tilde D_1^{(n)}\ . \qquad 4.19$$

Now from 4.11 we derive that

$${}^{top}\tilde D_1^{(n)}P[X_n] \;=\; \tfrac{t^n}{t-1}\,P[X_n]+\tfrac{t^n}{1-t}\,P[X_n-\tfrac{1-q}{z}]\;\Omega[z(1-1/t)X_n]\,|_{z^0}\ . \qquad 4.20$$

and 4.10 follows by combining 4.19 with 4.20. This completes our proof.

We can now complete our

Proof of Theorem 1.2

We start by recalling the Macdonald identity (see [15] (4.15) p. 324)

$$D_1^{(n)}P_\mu[X_n;q,t\,] \;=\; \Big(\sum_{i=1}^n t^{n-i}q^{\mu_i}\Big)\,P_\mu[X_n;q,t\,]\ . \qquad 4.21$$

Now the "integral form" $J_\mu[X;q,t\,]$ defined in p. 352 of [15], can be written as

$$J_\mu[X;q,t\,] \;=\; h_\mu(q,t)P_\mu[X_n;q,t\,] \;=\; h'_\mu(q,t)Q_\mu(X;q,t)\ , \qquad 4.22$$

with

$$h_\mu(q,t)=\prod_{s\in\mu}(1-q^{a_\mu(s)}t^{l_\mu(s)+1}) \quad\text{and}\quad h'_\mu(q,t)=\prod_{s\in\mu}(1-t^{l_\mu(s)}q^{a_\mu(s)+1})\,. \qquad 4.23$$

Combining, 4.22 with 4.21 and 4.10 we derive that

$$\begin{aligned}\tfrac{1}{1-t}\,J_\mu[X_n;q,t\,]-\tfrac{t^n}{1-t}\,J_\mu[X_n-\tfrac{1-q}{z};q,t\,]\,\Omega[z(1-1/t)X_n]\,\big|_{z^0}&\\ =\Big(\sum_{i=1}^n t^{n-i}q^{\mu_i}\Big)\,J_\mu[X_n;q,t\,]\,.&\end{aligned}\tag{4.24}$$

Making the replacements $t\to 1/t$ and $X_n\to X_n/(1-1/t)$ and multiplying both sides by t^{n-1} we can write this in the form

$$\begin{aligned}\tfrac{t^n}{t-1}\,J_\mu[\tfrac{X_n}{1-1/t};q,1/t\,]-\tfrac{1}{t-1}\,J_\mu[\tfrac{X_n+(1-t)(1-q)/tz}{1-1/t};q,1/t\,]\,\Omega[-tzX_n]\,\big|_{z^0}&\\ =\Big(\sum_{i=1}^n t^{i-1}q^{\mu_i}\Big)\,J_\mu[\tfrac{X_n}{1-1/t};q,1/t\,]\ .&\end{aligned}$$

Multiplying by $1-t$ and making the replacement $tz\to z$, before taking the coefficient of z^0, from I.7 we get that

$$\begin{aligned}-t^n\tilde H_\mu[X_n;q,t\,]\ \ +\ \ \tilde H_\mu[X_n+\tfrac{M}{z};q,t\,]\Omega[-zX_n]\,\big|_{z^0}&\\ =\Big((1-t)\sum_{i=1}^n t^{i-1}q^{\mu_i}\Big)\,\tilde H_\mu[X_n;q,t\,]\ .&\end{aligned}\tag{4.25}$$

Now simple manipulations give the identity

$$(1-t)\sum_{i=1}^n t^{i-1}q^{\mu_i}\ \ =\ \ -t^n-D_\mu(q,t)\ .$$

Substituting this in 4.25 finally yields

$$\tilde H_\mu[X_n+\tfrac{M}{z};q,t\,]\Omega[-zX_n]\,\big|_{z^0}\ \ =\ \ -D_\mu(q,t)\,\tilde H_\mu[X_n;q,t\,]\ ,$$

which is 1.11 a). We have seen (Remark 1.1) that 1.11 a) implies 1.1 b). Thus the only thing that remains is to verify the normalization in 1.12. However, this follows from the identity

$$K_{(n),\mu}(q,t)\ \ =\ \ t^{n(\mu)}$$

which is proved in ex. 2, p. 362 of [15].

Proof of 1.15

It is shown in [15] ((5.13) (iv) p. 324) that

$$P_\mu[X;q,t\,]\ \ =\ \ P_\mu[X;1/q,1/t\,]\ .$$

This given, from 4.22 and 4.23 we get that

$$\begin{aligned}J_\mu[X,q,t]\ &=\ h_\mu(q,t)\,P_\mu[X;1/q,1/t\,]\\ &=\ t^{n(\mu)}q^{n(\mu')}(-t)^{|\mu|}h_\mu(1/q,1/t)\,P_\mu[X;1/q,1/t\,]\\ &=\ t^{n(\mu)}q^{n(\mu')}(-t)^{|\mu|}J_\mu[X,1/q,1/t]\ .\end{aligned}$$

Thus making the replacements $t \to 1/t$, $X \to X/(1-1/t)$ and using I.7 we obtain

$$
\begin{aligned}
\tilde{H}_\mu[X_n,q,t] &= q^{n(\mu')}(-t)^{-|\mu|} J_\mu[\tfrac{X}{1-1/t}, 1/q, t\,] \\
&= q^{n(\mu')}(-1)^{|\mu|} J_\mu[\tfrac{-X}{1-t}, 1/q, t\,] \\
&= q^{n(\mu')} t^{n(\mu)} (-1)^{|\mu|} \tilde{H}_\mu[-X, 1/q, 1/t\,] \ ,
\end{aligned}
$$

and I.2 gives 1.15.

We terminate with the

Proof of formula 1.18 c)

The starting point is the Macdonald "Cauchy" formula ((4.13) p. 324 of [15])

$$
\Omega[XY\tfrac{1-t}{1-q}] \;=\; \sum_\mu P_\mu[X;q,t] Q_\mu[Y;q,t] \ .
$$

Using 4.22 we can rewrite this as

$$
\Omega[XY\tfrac{1-t}{1-q}] \;=\; \sum_\mu \frac{J_\mu[X;q,t] J_\mu[Y;q,t]}{h_\mu(q,t) h'_\mu(q,t)} \ .
$$

Making the replacements $t \to 1/t$ and then $X \to X/(1-1/t)$, $Y \to Y/(1-1/t)$ we get (recalling I.23)

$$
\begin{aligned}
\Omega[\tfrac{-t\,XY}{(1-t)(1-q)}] &= \sum_\mu \frac{J_\mu[\frac{X}{1-1/t}; q, 1/t] J_\mu[\frac{Y}{1-1/t}; q, 1/t]}{\tilde{h}_\mu(q,t) \tilde{h}'_\mu(q,t)} (-t)^{|\mu|} t^{2n(\mu)} \\
&= \sum_\mu \frac{\tilde{H}_\mu[X;q,t\,] \tilde{H}_\mu[Y;q,t\,]}{\tilde{h}_\mu(q,t) \tilde{h}'_\mu(q,t)} (-t)^{|\mu|} \\
&= \sum_\mu \frac{\tilde{H}_\mu[{}^{-}tX;q,t\,] \tilde{H}_\mu[Y;q,t\,]}{\tilde{h}_\mu(q,t) \tilde{h}'_\mu(q,t)} \ ,
\end{aligned}
$$

and 1.18 c) follows by making the replacement ${}^{-}tX \to X$ and using 1.17.

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