Eigenspace Decompositions with Respect to Symmetrized Incidence Mappings

Helmut Krämer

Mathematisches Seminar der Universität Hamburg
Bundesstraße 55
D–20146 Hamburg

Abstract

Let $\mathbb{K}$ denote one of the fields $\mathbb{Q}, \mathbb{F}_2$ and define $H\,(t,q)$, $t \leq q$, to be the $\mathbb{K}$–incidence matrix of the t–sets vs. the q–sets of the n–set $\{1, 2, \ldots, n\}$. This matrix is considered as a linear map of $\mathbb{K}$–vector spaces

$$_{\mathbb{K}}C_q(n) \longrightarrow {}_{\mathbb{K}}C_t(n),$$

where $_{\mathbb{K}}C_s(n)$ $(s \leq n)$ is the $\mathbb{K}$–vector space having the s–sets as a basis. The symmetrized $\mathbb{K}$–incidence matrix (of $H\,(t,q)$) is defined to be the symmetric matrix $\widetilde{H}\,(t,q) := H\,(t,q)^T \cdot H\,(t,q)$ which is also considered as an endomorphism of $_{\mathbb{K}}C_q(n)$. In case $\mathbb{K} = \mathbb{Q}$ we exhibit explicitely a decomposition of $_{\mathbb{Q}}C_q(n)$ into eigenspaces with respect to $\widetilde{H}\,(t,q)$. A closer examination of the proof of this result yields a canonical decomposition of $\ker H\,(t,q)$ (provided $\binom{n}{t} < \binom{n}{q}$) extending work done by J.B. Graver and W.B. Jurkat.

In case $\mathbb{K} = \mathbb{F}_2$ denote $\widetilde{H}\,(q \mid n) := \widetilde{H}\,(q-1,q)$. Then $\widetilde{H}\,(q \mid n)$ is a projection hence diagonalizable if n is odd (otherwise nilpotent). In both cases the rank of $\widetilde{H}\,(q \mid n)$ is determined; among other results an explicit decomposition of $_{\mathbb{F}_2}C_q(n)$ into the two eigenspaces with respect to $\widetilde{H}\,(q \mid n)$ is obtained provided n is odd.

As a basic tool we use the graded commutative $\mathbb{K}$–algebra

$$_{\mathbb{K}}\mathfrak{C}_*(n) = \mathbb{K}[T_1, \ldots, T_n]/(T_1^2, T_2^2, \ldots, T_n^2).$$

Here the $\mathbb{K}$–vector spaces of the elements of degree q of ${}_{\mathbb{K}}\mathfrak{C}_*(n)$ are isomorphic to ${}_{\mathbb{K}}C_q(n)$.

1. We denote for $n \in \mathbb{N}$

$$\underline{\underline{n}} = \{1, 2, \ldots, n\},$$

let in additon $\mathbb{K}$ be a field. Assume $0 \leq t \leq q \leq n$. Then $H(t,q)$ denotes the $\mathbb{K}$–incidence matrix $\Big(\iota(N,M)\Big)$. Here M runs through the q–sets of $\underline{\underline{n}}$, N runs through the t–sets of $\underline{\underline{n}}$ and we define

$$\iota(N,M) = \begin{cases} 1, & N \subseteq M, \\ 0, & \text{otherwise.} \end{cases} \quad (0, 1 \in \mathbb{K}).$$

Let ${}_{\mathbb{K}}C_q(n)$ be the $\mathbb{Q}$–vector space with basis $\{[M]\}_{M \in \binom{\underline{\underline{n}}}{q}}$, such that $H(t,q)$ defines a linear mapping

$${}_{\mathbb{K}}C_q(n) \longrightarrow {}_{\mathbb{K}}C_t(n),\ [M] \longrightarrow \sum_{\substack{N, \\ |N|=t}} \iota(N,M)[N]$$

which again is denoted by the same symbol $H(t,q)$. The transposed matrix $H(t,q)^T$ defines a linear mapping

$${}_{\mathbb{K}}C_t(n) \longrightarrow {}_{\mathbb{K}}C_q(n),\ [N] \longrightarrow \sum_{\substack{M, \\ |M|=q}} \iota(N,M)[M],$$

which again is denoted by the same symbol $H(t,q)^T$. Finally we define an "augmentation map"

$$H(-1,0) : {}_{\mathbb{K}}C_0(n) \longrightarrow 0.$$

We are dealing here with the "symmetrized incidence mapping" $\widetilde{H}(t,q)$. This is defined to be the mapping

$$\widetilde{H}(t,q) := H(t,q)^T \circ H(t,q) : {}_{\mathbb{K}}C_q(n) \longrightarrow {}_{\mathbb{K}}C_q(n)$$

which is already diagonalizable in case $\mathbb{K} = \mathbb{Q}$ as we soon will see.

The following proposition is well known.

Proposition 1. *We have*

$$\widetilde{H}\,(t,q)\Big([M]\Big) = \sum_{\substack{M',\\ |M'|=q}} 1 \cdot \binom{|M \cap M'|}{t} \cdot [M'].$$

Proof. We have

$$\widetilde{H}\,(t,q)\Big([M]\Big) = \sum_{\substack{N,\\ |N|=t}} \sum_{\substack{M',\\ |M'|=q}} \iota\,(N,M)\,\iota\,(N,M') \cdot [M'],$$

in addition

$$\begin{aligned} \sum_{|N|=t}^{N} \iota\,(N,M) \cdot \iota\,(N,M') &= 1 \cdot \#\Big\{ N \Big| |N| = t,\ N \subseteq M \cap M' \Big\} \\ &= 1 \cdot \tbinom{|M \cap M'|}{t}. \end{aligned}$$

□

Therefore all entries of the matrix $\widetilde{H}\,(t,q)$ are non–negative in case $\mathbb{K} = \mathbb{Q}$. If $\mathfrak{z}(M)$ denotes the row sum of the matrix $\widetilde{H}\,(t,q)$ indexed by the q–set M we have

$$\mathfrak{z}(M) = \sum_{\substack{M',\\ |M'|=q}} \binom{|M \cap M'|}{t}$$

and this sum is independent from M; so we denote the constant row sum by $\mathfrak{z}$.

• ([3], **Lemma 5.1.1**) *Suppose A is a real $n \times n$–matrix with non–negative entries and constant row sum k. Then $(1,1,\dots,1)^T$ is an eigenvector of A with eigenvalue k. Moreover if μ is another (complex) eigenvalue of A then it holds that*

$$|\mu| \leq k.$$

Suppose now $n > 1$. Then k is an eigenvalue of geometric multiplicity 1 if and only if A is irreducible.

The last assertion follows from the so–called Perron–Frobenius–Theory.

We note another result which applies to the matrix $\widetilde{H}\,(t,q)$:

•• ([3], **Theorem 3.2.1**) *Suppose that A is a real or complex $n \times n$–matrix. Then A is irreducible if and only if the directed graph $D(A)$ associated to A is strongly connected.*

Now if $A = \widetilde{H}(t,q)$, $t < q$, $\mathbb{K} = \mathbb{Q}$, the graph $D(A)$ has $\binom{n}{q}$ as set of vertices V. If $L, M \in V$ then the directed arc (L, M) is in the set E of edges of $D(A)$ if and only if $\binom{|L \cap M|}{t} \neq 0$, that is $|L \cap M| \geq t$ ist. In this case (M, L) is an arc in $D(A)$, too.

We conclude therefore that $D(A)$ is strongly connected if and only if the corresponding undirected graph is connected. This is indeed the case as can be easily seen as follows: Fix $L, M \in V$. Then there exist q–sets $L = L_1, L_2, \ldots, L_r = M$ with the property

$$|L_i \cap L_{i+1}| = q - 1 \geq t, \; 1 \leq i \leq r - 1.$$

If one denotes the eigenspace of $\widetilde{H}(t,q)$ with eigenvalue $\lambda \in \mathbb{R}$ by

$$\mathrm{Eig}\left(\widetilde{H}(t,q), \lambda\right) \subset {}_{\mathbb{R}}C_q(n)$$

then the arguments stated above yield

$$\mathrm{Eig}\left(\widetilde{H}(t,q), \mathfrak{z}\right) = \mathbb{R} \cdot \left(\sum_{\substack{M, \\ |M|=q}} [M] \right).$$

In the following we make the convention $\binom{n}{-1} = 0$.

Theorem 1. *We assume $\mathbb{K} = \mathbb{Q}$ and $0 \leq t < q \leq n$. Then $\widetilde{H}(t,q)$ is diagonalizable (as a mapping of $\mathbb{Q}$–vector spaces). More exactly the following holds: In case $0 \leq s \leq \min\{q, n-q\}$ we define*

$$\mu(q,t;s) = \binom{q-s}{q-t} \cdot \binom{n-t-s}{q-t}.$$

1) *Assume $t \geq \min\{q, n-q\}$. Then we have*

   i) $\mu(q,t;0) > \mu(q,t;1) > \ldots > \mu(q,t;\min\{q,n-q\}) > 0$ *and*

   ii) $\mathrm{Eig}\ \left(\widetilde{H}(t,q),\, \mu(q,t;s)\right) = H(s,q)^T \left(\ker H(s-1,s)\right),$

iii) $\dim H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big) = \binom{n}{s} - \binom{n}{s-1}$,

such that

iv) ${}_{\mathbb{Q}}C_q(n) = \bigoplus_{s=0}^{\min\{q,n-q\}} H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)$

is a decomposition of ${}_{\mathbb{Q}}C_q(n)$ *into eigenspaces with respect to the endomorphism* $\widetilde{H}\,(t,q)$.

2) Assume $t < \min\{q, n-q\}$. *Then we have*

i) $\mu\,(q,t;0) > \mu\,(q,t;1) > \ldots > \mu\,(q,t;t) > 0$,

$$\mu\,(q,t;t+1) = \ldots = \mu\,\Big(q,t;\min\{q,n-q\}\Big) = 0.$$

In case $0 \leq s \leq t$ *we have*

ii) $\mathrm{Eig}\;\Big(\widetilde{H}\,(t,q),\,\mu\,(q,t;s)\Big) = H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)$,

iii) $\dim H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big) = \binom{n}{s} - \binom{n}{s-1}$.

Furthermore it holds that

iv) $\mathrm{Eig}\;\Big(\widetilde{H}\,(t,q), 0\Big) = \ker H\,(t,q)$,

v) $\dim\ker H\,(t,q) = \binom{n}{q} - \binom{n}{t}$,

such that

vi) ${}_{\mathbb{Q}}C_q(n) = \left(\bigoplus_{s=0}^{t} H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)\right) \oplus \ker H\,(t,q)$

is a decomposition of ${}_{\mathbb{Q}}C_q(n)$ *into eigenspaces with respect to the endomorphism* $\widetilde{H}\,(t,q)$.

Corollary 1. *We assume* $q+t=n$. *Then the following identity holds*

$$\big|\det H\,(t,q)\big| = \prod_{s=0}^{t-1} \binom{q-s}{q-t}^{\binom{n}{s}-\binom{n}{s-1}}.$$

This statement can also be derived from [7], Theorem 2.

2. For the proof of the theorem we make use of a graded $\mathbb{K}$–algebra which was essentially introduced in the previous paper [6]. We denote this algebra by ${}_{\mathbb{K}}\mathfrak{C}_*(n)$. It is defined by

$$ {}_{\mathbb{K}}\mathfrak{C}_*(n) = \mathbb{K}[T_1, \dots, T_n]/(T_1^2, \dots, T_n^2) = \mathbb{K}[X_1, \dots, X_n], $$

here $T_1, \dots, T_n$ are algebraically independent elements and X_j denotes the residue–class $T_j \mathrm{mod}(T_1^2, \dots, T_n^2), j \in \underline{\underline{n}}$. This algebra will be used in the sequel in the cases $\mathbb{K} = \mathbb{Q}$ and $\mathbb{K} = \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$. Let ${}_{\mathbb{K}}\mathfrak{C}_*(n)_p$ denote the $\mathbb{K}$–vector space of the elements of degree p in this algebra; then we have that

$$ {}_{\mathbb{K}}\mathfrak{C}_*(n)_p = 0, \ p > n, $$

and

$$ {}_{\mathbb{K}}\mathfrak{C}_*(n)_p \cong {}_{\mathbb{K}}C_p(n), \ 0 \leq p \leq n. $$

The isomorphisms under consideration are induced by the mappings

$$ \begin{aligned} \mathbb{K} \ni 1 \quad &\longrightarrow \quad [\emptyset], \\ X_{j_1} \cdot X_{j_2} \cdot \ldots \cdot X_{j_p} \quad &\longrightarrow \quad \Big[\{j_1, j_2, \dots, j_p\}\Big], \\ (1 \leq j_1 < j_2 &< \ldots < j_p \leq n). \end{aligned} $$

In case $0 \leq q \leq n$ we will identify the spaces ${}_{\mathbb{K}}\mathfrak{C}_*(n)_q$ and ${}_{\mathbb{K}}C_q(n)$. – To the incidence mappings $H\,(q-1, q)$, $0 \leq q \leq n$, corresponds the $\mathbb{K}$–linear map Δ of ${}_{\mathbb{K}}\mathfrak{C}_*(n)$ with degree –1 induced by

$$ \Delta\Big|_{\mathbb{Q}} = 0, \ \Delta X_j = 1, \ j \in \underline{\underline{n}}, $$

$$ \Delta\left(X_{j_1} \cdot X_{j_2} \cdot \ldots \cdot X_{j_q}\right) = \sum_{k=1}^{q} X_{j_1} \cdot \ldots \cdot \widehat{X}_{j_k} \cdot \ldots \cdot X_{j_q}, \ 2 \leq q \leq n, $$

where we assume that the X_{j_k} are pairwise distinct and $\widehat{\ }$ denotes the deletion operator.

Finally we define $\mathbb{X} := \sum_{j=1}^{n} X_j$.

Proposition 2. *We agree upon* $\Delta^\circ = \mathrm{id}, \mathbb{X}^\circ = 1$. *Then we have*

i) in case $\mathbb{K} = \mathbb{Q}$

$$\begin{aligned}(q-t)!\, H\,(t,q) &= \Delta^{q-t}\Big|_{C_q(n)},\\ (q-t)!\, H\,(t,q)^T(w) &= \mathbb{X}^{q-t}\cdot w,\ w\in C_t(n),\end{aligned}$$

ii) in case $\mathbb{K} = \mathbb{F}_2$

$$\Delta^2 = 0,\ \mathbb{X}^2 = 0,$$

$$H\,(t,t+1)^T(w) = \mathbb{X}\cdot w,\ w\in C_t(n),\ 0\le t\le n-1.$$

Proof. For the proof of i) we refer to [6], Proposition 1. –

In the second statement it is obvious that Δ^2 vanishes on the vectorspace ${}_{\mathbb{F}_2}C_1(n)$. In case $2\le q\le n$ we rewrite

$$\Delta\,(X_{j_1}\cdot\ldots\cdot X_{j_q}) = \sum_{k=1}^{q}(-1)^k X_{j_1}\cdot\ldots\cdot\widehat{X}_{j_k}\cdot\ldots\cdot X_{j_q}$$

and apply a standard argument from simplicial homology. The remaining assertions are obvious. □

We remark that $\big({}_{\mathbb{F}_2}\mathfrak{C}_*(n),\,\Delta\big)$ is isomorphic to a Koszul–complex. We will return to this topic in the last section of this paper.

Let us write $w\in{}_{\mathbb{K}}\mathfrak{C}_*(n)$ as a sum of monomials (with respect to $X_1,\ldots,X_n$) with coefficients from $\mathbb{K}$. Then we have defined in [6] the foundation of w (in signs Fund (w)), to be the product of all X_j which appear in this decomposition with non–vanishing coeffients. Sometimes we will identify Fund (w) with a subset of $\underline{\underline{n}}$. This convention is used in the next proposition.

Proposition 3. *Assume* $v,w\in{}_{\mathbb{K}}\mathfrak{C}_*(n)$ *and Fund* $(v)\cap$ *Fund* $(w)=\emptyset$. *Then it holds that*

$$\Delta\,(v\cdot w) = w\Delta\,(v) + v\Delta\,(w).$$

For the proof we refer to [6], Prop. 2. $\square$

Finally we define the "falling factorial"

$$[r]_k = r\,(r-1)(r-2)\cdot \ldots \cdot (r-k+1),\ [r]_0 = 1.$$

Proposition 4. *i) In case $\mathbb{K} = \mathbb{Q}$ let α, β be non–negative integers. We assume $0 \le s \le n-1,\ 1 \le \alpha,\ \alpha + s \le n,\ 0 \le \beta \le \alpha$, and $w \in {}_{\mathbb{Q}}C_s(n)$. Then the following identity holds*

$$\Delta^{\beta}(\mathbb{X}^{\alpha} \cdot w) = \sum_{k=0}^{\beta} \binom{\beta}{k} [\alpha]_k [n - \alpha - 2s + \beta]_k \cdot \mathbb{X}^{\alpha - k} \cdot \Delta^{\beta - k}(w).$$

ii) In case $\mathbb{K} = \mathbb{F}_2$ we assume $0 \le s \le n-1$ and $w \in {}_{\mathbb{F}_2}C_s(n)$. Then the following identity holds

$$\Delta\,(\mathbb{X}w) = \mathbb{X} \cdot \Delta\,(w) + (n \cdot 1) \cdot w.$$

Proof. For the first statement we refer to [6], Prop. 4.

The second statement is obvious in case $s = 0$. Assume now $s \ge 1$. Let $\widetilde{w} \in {}_{\mathbb{Q}}C_s(n)$ be a sum of monomials (with respect to $X_1, \ldots, X_n$) with integer coefficients. Then as we have seen in the first part of the proof it holds that

$$\Delta\,(\mathbb{X}\widetilde{w}) = \mathbb{X} \cdot \Delta\,(\widetilde{w}) + (n - 2s) \cdot \widetilde{w}.$$

Reducing this equation modulo 2 now yields the claim. $\square$

Proposition 5. ([4], Chapt. 15, Corollary 8.5).

We assume $\mathbb{K} = \mathbb{Q}$ and $s \le \min\{q, n-q\}$. Then the mapping

$$H\,(s,q)^T :\ {}_{\mathbb{Q}}C_s(n) \longrightarrow\ {}_{\mathbb{Q}}C_q(n)$$

is injective.

Proof. We use the relation derived in Prop. 4, i) and assume $\alpha = \beta = q - s \geq 0$. Let us rewrite this relation in terms of matrices. The left hand side of the relation is $\left((q-s)!\right)^2 H\,(s,q) \circ H\,(s,q)^T$; the right hand side is sum of the positive semi–definite matrices

$$H\,(2s-q+k,s)^T \circ H\,(2s-q+k,s),\ q-2s \leq k \leq q-s,$$

with non–negative integer coefficients. Also the unit matrix occurs here (take $k = q-s$) with the coefficient

$$[q-s]_{q-s} \cdot [n-2s]_{q-s}$$

which doesn't vanish since $s \leq n-q$. We conclude that in case $s \leq n-q$

$$H\,(s,q) \circ H\,(s,q)^T$$

is an isomorphism, hence the mapping $H\,(s,q)^T$ is injective. $\square$

3. In this section we first come to the proof of Theorem 1.

Ad 1) So assume $t \geq \min\{q, n-q\}$. Suppose $\binom{q-s}{q-t} = 0$. This yields

$$t < s \leq \min\{q, n-q\},$$

a contradiction. In the same straightforward manner we conclude that the second factor occuring in $\mu\,(q,t;s)$ doesn't vanish. Now it is easily seen that the $\mu\,(q,t;s),\ k = 0, 1, \ldots, \min\{q, n-q\}$ are strictly decreasing. This establishes statement i). –
Assume now $w \in \ker \Delta = \ker H\,(s-1,s) \subset C_s(n)$. According to Prop. 4 we have that

$$\Delta^{q-t}(\mathbb{X}^{q-s}w) = [q-s]_{q-t} \cdot [n-s-t]_{q-t} \cdot \mathbb{X}^{t-s} \cdot w.$$

We multiply this equation with $\mathbb{X}^{q-t}$ and obtain

$$(\mathbb{X}^{q-t} \cdot \Delta^{q-t}) \cdot (\mathbb{X}^{q-s}w) = [q-s]_{q-t} \cdot [n-s-t]_{q-t} \cdot \mathbb{X}^{q-s} \cdot w.$$

Now we use Prop. 2. This yields

$$\mathbb{X}^{q-s} \cdot w = (q-s)!\, H\,(s,q)^T(w),$$

$$\mathbb{X}^{q-t} \cdot \Delta^{q-t}(w) = \Big((q-t)!\Big)^2 \widetilde{H}\,(t,q)(w).$$

Therefore we have now

$$\widetilde{H}\,(t,q)\Big(H\,(s,q)^T\cdot w\Big) = \mu\,(q,t;s)\cdot\Big(H\,(s,q)^T\cdot w\Big),$$

and in turn

$$(1)\ldots\qquad H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)\subseteq\ \mathrm{Eig}\ \Big(\widetilde{H}\,(t,q),\,\mu\,(q,t;s)\Big).$$

Since $s\le\min\{q,n-q\}$ the inequality $s\le\lfloor\frac{n}{2}\rfloor$ holds.

Now we use the following

Lemma. *Assume $h,k\in\{0,1,\ldots,n\}$ and $\binom{n}{h}\le\binom{n}{k}$. Then the mapping $H\,(h,k): C_k(n)\longrightarrow C_h(n)$ is surjective.*

For a proof of the Lemma we refer to [5], 2.3., 2.4.
For another independent proof see [6], Theorem 1. □

According to the Lemma we have

$$\dim\ker H\,(s-1,s)=\binom{n}{s}-\binom{n}{s-1}.$$

We now invoke Prop. 5 and obtain

$$\dim H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)=\binom{n}{s}-\binom{n}{s-1}.$$

Since eigenspaces to different eigenvalues are independent, we conclude

$$\sum_{s=0}^{\min\{q,n-q\}} H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)=\bigoplus_{s=0}^{\min\{q,n-q\}} H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big),$$

and this subspace of ${}_{\mathbb{Q}}C_q(n)$ has the dimension

$$\sum_{s=0}^{\min\{q,n-q\}}\left(\binom{n}{s}-\binom{n}{s-1}\right)=\binom{n}{q}=\dim\ {}_{\mathbb{Q}}C_q(n).$$

Therefore strict equality must hold in Eq (1). At the same time all other statements are proved.

ad 2): The proof of assertion i) is straigthforward. Also, along the same lines as in the corresponding statement in case 1) we conclude

$$(2)\ldots\quad H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)\subseteq \mathrm{Eig}\Big(\widetilde{H}\,(t,q),\,\mu\,(q,t;s)\Big),$$
$$0\le s\le \min\{1,n-q\},$$

and

$$\dim H\,(t,q)^T\Big(\ker H\,(s-1,s)\Big)=\binom{n}{s}-\binom{n}{s-1}$$

provided $0\le s\le \min\{q,n-q\}$.

Now we assume $t<\min\{q,n-q\}$ and obtain $t+q+1\le n$. This inequality is equivalent to the condition $\binom{n}{t}<\binom{n}{q}$. According to the Lemma in the first part of the proof $H\,(t,q)$ is surjective, in turn

$$\dim\ker H\,(t,q)=\binom{n}{q}-\binom{n}{t}.$$

Obviously it holds that

$$(3)\ldots\qquad \ker H\,(t,q)\subseteq \mathrm{Eig}\Big(\widetilde{H}\,(t,q),0\Big).-$$

Now we apply the first half of assertion i) and obtain

$$\textstyle\sum_{s=0}^{t} H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)+\ker H\,(t,q)=$$
$$=\ \textstyle\bigoplus_{s=0}^{t} H\,(s,q)^T\Big(\ker H\,(s-1,s)\Big)\oplus\ker H\,(t,q).$$

This subspace of ${}_{\mathbb{Q}}C_q(n)$ therefore has the dimension

$$\sum_{s=0}^{t}\left(\binom{n}{s}-\binom{n}{s-1}\right)+\binom{n}{q}-\binom{n}{t}=\binom{n}{q}=\dim\ {}_{\mathbb{Q}}C_q(n).$$

We conclude that strict equality must hold in Eq (2), (3). At the same time, all other statements have been proved. $\square$

Remarks:

a) We note the particular result

$$\mathrm{Eig}\Big(\widetilde{H}\,(t,q),\ \mu\,(q,t;0)\Big) = \mathbb{Q}\cdot\left(\sum_{\substack{M,\\ |M|=q}}[M]\right).$$

This allows us to compute the constant row–sums $\mathfrak{z}$ of $\widetilde{H}\,(t,q)$. We obtain

$$\mathfrak{z} = \mu\,(q,t;0) = \binom{q}{t}\cdot\binom{n-t}{q-t}.$$

b) From the second part of the proof we derive

$$\ker H\,(t,q) = \mathrm{Eig}\Big(\widetilde{H}\,(t,q),0\Big) = \ker\Big(H\,(t,q)^T\circ H\,(t,q)\Big).$$

Of course this is also a consequence of the following well–known equality

$$\mathrm{rank} H\,(t,q) = \mathrm{rank}\big(H\,(t,q)^T\circ H\,(t,q)\big)\Big(= \mathrm{rank}\big(H\,(t,q)\circ H\,(t,q)^T\big)\Big),$$

(see for instance [1], Chapt. II, 2.5 Lemma).

Now we turn to the proof of the corollary.
Assume first $t=q$. The claim is trivially true since $\widetilde{H}\,(t,q)$ is the unit matrix.
Now assume $t<q$. Of course

$$|\det H\,(t,q)| = \sqrt{\det\widetilde{H}\,(t,q)},$$

and $\det\ \widetilde{H}\,(t,q)$ is the product of the eigenvalues counted with the corresponding multiplicities.
Since $q=n-t$, case 1) of the Theorem applies and yields

$$\mu\,(q,t;s) = \binom{q-s}{q-t}^2,\ 0\le s\le t=\min\{q,n-q\}.$$

□

Let us once again return to the proof of the Theorem, case 2). We consider the sum

$$U = \sum_{s=0}^{\min\{q,n-q\}} H\,(s,q)^T\big(\ker H\,(s-1,s)\big).$$

Our arguments have shown that all subspaces occuring in this sum are subspaces of eigenspaces with respect to $\widetilde{H}(t,q)$ but the eigenspaces under consideration do not necessarily have *distinct* eigenvalues. In fact the last $\min\{q, n-q\} - t$ eigenvalues are zero according to assertion i). So in general we cannot conclude by standard arguments that U is a direct sum. However, this is true as can be seen from our next result which was announced in the previous paper ([6], Theorem 3).

Theorem 2. *Assume $\binom{n}{t} < \binom{n}{q}$. Then it holds that*

$$\ker H(t,q) = \bigoplus_{s=t+1}^{\min\{q,n-q\}} H(s,q)^T \Big(\ker H(s-1,s)\Big).$$

Proof. Assume $t+1 \le s \le \min\{q, n-q\}$ and define

$$V_s := H(s,q)^T \Big(\ker H(s-1,s)\Big).$$

We have already remarked that the condition imposed in Theorem 2 is equivalent to $t+q+1 \le n$. Now we use the following

Lemma. *Assume $0 \le s \le \min\{q, n-q\}$ and $w_s \in \ker H(s-1,s)$. Then we have*

$$\Delta^{q-r}(\mathbb{X}^{q-s} \cdot w_s) = \begin{cases} \alpha(q,s) \cdot w_s, \ \alpha(q,s) \neq 0, & \text{if } r = s, \\ 0\,, & \text{if } r < s. \end{cases}$$

Proof (of the lemma): From Prop. 4 we derive

$$\Delta^{q-s}(\mathbb{X}^{q-s} \cdot w_s) = \alpha(q,s) \cdot w_s$$
$$\alpha(q,s) = [q-s]_{q-s} \cdot [n-2s]_{q-s} \neq 0$$

provided $s \le \min\{q, n-q\}$.
Now assume $r < s$. Then we obtain

$$\Delta^{q-r}(\mathbb{X}^{q-s} \cdot w_s) = \Delta^{s-r}\Big(\Delta^{q-s}(\mathbb{X}^{q-s} \cdot w_s)\Big) = \Delta^{s-r}\Big(\alpha(q,s) \cdot w_s\Big) = 0.$$

$\square$

Now take $r = t$ in the lemma and apply Prop. 2. Then we have proved anew that V_s are contained in $\ker H\,(t, q)$. – Let us show now that the sum

$$V := \sum_{s=t+1}^{\min\{q,n-q\}} V_s$$

is direct.

We take $v \in V$ and write

$$(4)\ldots \qquad v = \sum_{s=t+1}^{\min\{q,n-q\}} \mathbb{X}^{q-s} w_s = 0,\ w_s \in \ker H\,(s-1, s).$$

In case $t + 1 = \min\{q, n - q\}$, nothing is to be proved. Otherwise apply $\Delta^{q-(t+1)}$ in Eq (4). According to the Lemma we obtain

$$\Delta^{q-(t+1)}(v) = \alpha\,(q, t+1)\, w_{t+1} = 0,$$

which in turn shows $w_{t+1} = 0$. Suppose now that it has already be shown that in Eq (4) the following equalities hold

$$w_{t+1} = w_{t+2} = \ldots = w_p = 0,\ p < \min\{q, n-q\}.$$

Then we obtain again according to the Lemma

$$\Delta^{q-(p+1)}(v) = \alpha\,(q, p+1) \cdot w_{p+1} = 0,$$

and therefore $w_{p+1} = 0$. –

We recall from the proof of Theorem 1

$$\dim V_s = \binom{n}{s} - \binom{n}{s-1}.$$

This in turn implies now

$$\dim V = \sum_{s=t+1}^{\min\{q,n-q\}} \left(\binom{n}{s} - \binom{n}{s-1} \right) = \binom{n}{q} - \binom{n}{t} = \dim \ker H\,(t, q)$$

which finishes the proof of the Theorem. $\square$

We recall that the two conditions imposed in Theorem 1, viz "$t \geq \min\{q, n-q\}$" and "$t < \min\{q, n-q\}$", respectively, are equivalent to the conditions "$\binom{n}{t} \geq \binom{n}{q}$" and "$\binom{n}{t} < \binom{n}{q}$". In the first case $\widetilde{H}\,(t,q)$ is an isomorphism according to Theorem 1. If we use only [6] in the proof of that theorem, which is possible, then we have proved anew independently from [5], 2.3, 2.4 that $H\,(t,q)$ is an isomorphism provided $\binom{n}{t} \geq \binom{n}{q}$.
(Of course this proof is (much) more complicated.) In particular we conclude that $\ker H(q-1,q) \neq 0$ if and only if $q \leq \lfloor \frac{n}{2} \rfloor$. Now we quote

• • • ([5], 4.2, [6], **4.**). *Assume $q \leq \lfloor \frac{n}{2} \rfloor$. Then* $\ker H\,(q-1,q)$ *is generated by elements of the type*

$$(X_{j_1} - X_{j_2})(X_{j_3} - X_{j_4}) \cdot \ldots \cdot (X_{j_{2q-1}} - X_{j_{2q}}).$$

If we combine this result with Theorem 2 we obtain systems of generators of $\ker H\,(t,q)$; however, these systems are in general different from those exhibited in [5]. – In the same way we have explicite systems of generators of the eigenspaces with respect to $\widetilde{H}\,(t,q)$.

Finally we make a remark concerning the eigenspaces of $H\,(t,q) \circ H\,(t,q)^T$. We restrict ourselves to quote the following result:

• • • • ([4], Chapt. 10, LEMMA 3.2) *For any matrix A the non–zero eigenvalues of AA^T and A^TA are the same, and have the same multiplicities.*

4. In this last section we take $\mathbb{K} = \mathbb{F}_2$. We investigate now the mappings

$$\widetilde{H}_q(q|n) =: \widetilde{H}\,(q-1,q) : \ {}_{\mathbb{F}_2}C_q(n) \longrightarrow \ {}_{\mathbb{F}_2}C_q(n),$$

using the algebra ${}_{\mathbb{F}_2}\mathfrak{C}_*(n)$. According to Prop. 1 we have

$$\widetilde{H}\,(q|n)\Big([M]\Big) = (q \cdot 1) \cdot [M] + \sum_{|M \cap M'| = q-1}^{|M'|=q} 1 \cdot [M'].$$

The reader might have wondered why we admit a field of positive characteristic. In fact, as we soon will see, $\widetilde{H}\,(q|n)$ is a projection (hence diagonalizable) if n is odd (otherwise nilpotent). We have already observed in **3.** that $({}_{\mathbb{F}_2}\mathfrak{C}_*(n), \Delta)$ is a complex in the sense of homological algebra. Let us rewrite this complex $\mathfrak{K}_n$ in the following way

$$0 \longrightarrow C_n(n) \xrightarrow{\Delta_n} C_{n-1}(n) \xrightarrow{\Delta_{n-1}} \ldots \xrightarrow{\Delta_2} C_1(n) \xrightarrow{\Delta_1} C_0(n) \longrightarrow 0,$$

where of course we use the notation $\Delta_q = \Delta\Big|_{C_q(n)}$.

Proposition 6. *The complex $\mathfrak{K}_n$ is exact.*

This can be seen in different ways: First, the homology of the ball vanishes over any field. Or secondly, $\mathfrak{K}_n$ is isomorphic to a Koszulkomplex. The claim now follows from standard arguments about the vanishing of the homology modules of this complex. Third, the claim follows also from the much more general considerations in [2].

However, it will be useful for our purposes to prove the exactness of $\mathfrak{K}_n$ as follows: Since the statement is trivial if $n = 1$, we assume in the sequel always $n \geq 2$.

Proposition 7.

i) $\operatorname{Im} \Delta_q$ *is already generated by the images of the elements $X_n \cdot u$, where $u \in C_{q-1}(n-1)$.*

ii) $\operatorname{rank} \Delta_q = \binom{n-1}{q-1}$.

Proof. The elements different from zero recorded in assertion i) are exactly those $w \in C_q(n)$ with the property Fund $(w) \cap \{X_n\} \neq \emptyset$. If $q = n$, the claim is obvious. So assume now $1 \leq q \leq n-1$ and take $w \in C_q(n)$, Fund $(w) \cap \{X_n\} = \emptyset$. It follows that $X_n \cdot w \in C_{q+1}(n)$ and according to Prop. 3

$$\Delta_{q+1}(X_n \cdot w) = w + X_n \cdot \Delta_q(w).$$

But $\Delta_q \circ \Delta_{q+1} = 0$, so we obtain

$$\Delta_q(w) = \Delta_q\Big(X_n \cdot \Delta_q(w)\Big)$$

which proves the first claim.

To prove ii) it is sufficient to show that Δ_q restricted to the subspace $X_n \cdot C_{q-1}(n-1)$ is injective. So assume $X_n \cdot u$ is contained in that subspace. Then again according to Prop. 3 we obtain

$$0 = \Delta_q(X_n \cdot u) = u + X_n \cdot \Delta_{q-1}(u)$$

and hence $u = 0$, since X_n is no factor of u. Combined with assertion i) we obtain now

$$\operatorname{rank} \Delta_q = \dim C_{q-1}(n-1) = \binom{n-1}{q-1}.$$

$\square$

As announced we prove again the exactness of $\mathfrak{K}_n$ as follows: Assume $1 \leq q \leq n-1$. Then it holds that

$$\dim \ker \Delta_q = \binom{n}{q} - \binom{n-1}{q-1} = \binom{n-1}{q} = \dim \operatorname{Im} \Delta_{q+1}.$$

The exactness of $\mathfrak{K}_n$ at the positions $0, n$ is obvious.

The rank–formula in Prop. 7 is also a consequence of the more general considerations in [7]. Here the rank of the integer valued incidence matrix $H\,(t,q)$ reduced mod $p\mathbb{Z}$, p any prime, was determined. The rank–formula obtained there (loc. cit., Theorem 1) applied to our case yields

$$\operatorname{rank} \Delta_q = \binom{n}{q-1} - \binom{n}{q-2} + \binom{n}{q-3} \mp \ldots + (-1)^{q+1} \binom{n}{0}.$$

For a proof that both expressions obtained for the rank of Δ_q, coincide we refer to [6], Theorem 2, Lemma.

Theorem 3. *i)* $\widetilde{H}\,(q|n)^2 = \begin{cases} \widetilde{H}\,(q|n), & \textit{if } n \textit{ is odd,} \\ 0, & \textit{if } n \textit{ is even.} \end{cases}$

ii) If n is odd then

$$\operatorname{rank} \widetilde{H}\,(q|n) = \binom{n-1}{q-1}.$$

Assume $1 \leq q \leq n-1$. Then

$$\ker \widetilde{H}\,(q|n) = \operatorname{Im} H\,(q, q+1).$$

iii) If n is even then

$$\operatorname{rank} \widetilde{H}\,(q|n) = \binom{n-2}{q-1}.$$

Assume $1 \leq q \leq n-1$. Then

$$\ker \widetilde{H}\,(q|n) = X_n \cdot \operatorname{Im} \widetilde{H}\,(q-1|n-1) \oplus \operatorname{Im} H\,(q, q+1).$$

Proof. ad i). Suppose $u \in C_{q-1}(n)$. Then according to Prop. 4, ii) we have that

$$\Delta(\mathbb{X}u) = \mathbb{X} \cdot \Delta(u) + (n \cdot 1) \cdot u.$$

But $\mathbb{X}^2 = 0$, so we obtain

$$\mathbb{X}\Delta\mathbb{X}(u) = (n \cdot 1)\mathbb{X}u.$$

Now we take $u = \Delta w,\ w \in C_q(n)$. This yields

$$\mathbb{X}\Delta\mathbb{X}\Delta(w) = (n \cdot 1) \cdot \mathbb{X}\Delta(w).$$

We observe $\mathbb{X}\Delta(w) = \widetilde{H}(q|n)(w)$. The claim now follows.

To prove the remaining assertions let us make some preliminaries: Denote by $\widetilde{H}(q|n)_r$ the restriction of $\widetilde{H}(q|n)$ to the subspace $X_n \cdot C_{q-1}(n-1)$ of $C_q(n)$. Then according to Prop. 7, i)

$$\operatorname{Im}\widetilde{H}(q|n) = \operatorname{Im}\widetilde{H}(q|n)_r.$$

Take now $u \in C_{q-1}(n-1)$ and denote $\mathbb{X}_{(n-1)} := \sum_{j=1}^{n-1} X_j$. Then according to Prop. 3

$$\begin{aligned}\mathbb{X}\Delta(X_n \cdot u) &= (\mathbb{X}_{(n-1)} + X_n) \cdot \Big(u + X_n \cdot \Delta_{q-1}(u)\Big) = \\ &= X_n \cdot \Big(u + \mathbb{X}_{(n-1)}\Delta_{q-1}(u)\Big) + \mathbb{X}_{(n-1)} \cdot u.\end{aligned}$$

We rewrite this equation as follows

$$(5)\ldots \qquad \widetilde{H}(q|n)(X_n \cdot u) = X_n \cdot \Big(u + \widetilde{H}(q-1|n-1)(u)\Big) + \mathbb{X}_{(n-1)} \cdot u.$$

(Observe that $\widetilde{H}(0|n-1)$ is the zero–mapping.)

Now assume in Eq (5) that $X_n \cdot u$ is contained in the kernel of $\widetilde{H}(q|n)$. Since $\mathbb{X}_{(n-1)} \cdot u$ does not contain X_n as a factor both terms on the right–hand side in Eq (5) must be zero, in particular

$$(6)\ldots \qquad u + \widetilde{H}(q-1|n-1)(u) = 0.$$

ad ii). We derive from the first part of the proof that now $\widetilde{H}\,(q-1|n-1)$ is nilpotent. Therefore Eq (6) possesses only the trivial solution $u=0$, so $\widetilde{H}\,(q|n)_r$ is injective. In turn

$$\operatorname{rank}\widetilde{H}\,(q|n)=\operatorname{rank}\widetilde{H}\,(q|n)_r=\dim C_{q-1}(n-1)=\binom{n-1}{q-1}.$$

Now take $1\le q\le n-1$. Since $\mathfrak{K}_n$ is exact

$$\operatorname{Im}\Delta_{q+1}\subseteq\ker\widetilde{H}\,(q|n).$$

According to Prop. 7, ii) we have

$$\dim\operatorname{Im}\widetilde{H}\,(q|n)+\dim\operatorname{Im}\Delta_{q+1}=\binom{n-1}{q-1}+\binom{n-1}{q}=\binom{n}{q}=\dim C_q(n).$$

This proves the remaining assertions.

ad iii). Assume first $q\ge 2$. Let $X_n\cdot u$ be in the kernel of $\widetilde{H}\,(q|n)_r$. Then as it was stated above u must solve Eq (6). Now according to i) $\widetilde{H}\,(q-1|n-1)$ is a projection. Therefore Eq (6) has exactly all $u\in\operatorname{Im}\widetilde{H}\,(q-1|n-1)$ as solutions. Now we apply i) and obtain

$$\begin{aligned}\dim\ker\widetilde{H}\,(q|n)_r &= \dim\big(X_n\cdot\operatorname{Im}\widetilde{H}\,(q-1|n-1)\big)=\\ &= \dim\operatorname{Im}\widetilde{H}\,(q-1|n-1)=\binom{n-2}{q-2},\end{aligned}$$

in turn

$$\operatorname{rank}\widetilde{H}\,(q|n)=\operatorname{rank}\widetilde{H}\,(q|n)_r=\binom{n-1}{q-1}-\binom{n-2}{q-2}=\binom{n-2}{q-1}.$$

(Observe that $\widetilde{H}\,(n|n)=0$.) These arguments carry easily over to the case $q=1$; we leave the details to the reader whom we remind of our convention $\binom{m}{-1}=0$.

Assume now $1\le q\le n-1$. Then we claim that the subspaces $X_n\cdot\operatorname{Im}\widetilde{H}$ $(q-1|n-1)$ and $\operatorname{Im}\Delta_{q+1}$ of $\ker\widetilde{H}\,(q|n)$ are disjoint. In fact according to Prop. 7, i) $\operatorname{Im}\Delta_{q+1}$ is already generated by the $\Delta_{q+1}(X_n\cdot u)$, $u\in C_q(n-1)$. But

$$\Delta_{q+1}(X_n\cdot u)=u+X_n\cdot\Delta_q(u).$$

Therefore we conclude

$$\begin{aligned}
&\dim \mathrm{Im}\widetilde{H}\,(q|n) + \dim\Big(X_n\cdot \mathrm{Im}\ \widetilde{H}\,(n-1|q-1) + \mathrm{Im}\Delta_{q+1}\Big)\\
&\qquad = \binom{n-2}{q-1} + \binom{n-2}{q-2} + \binom{n-1}{q} = \binom{n-1}{q-1} + \binom{n-1}{q}\\
&\qquad = \binom{n}{q} = \dim C_q(n).
\end{aligned}$$

This finishes the proof. □

Acknowledgement.

The author is indebted to the referee for helpful comments.

References

[1] Th. Beth, D. Jungnickel, H. Lenz: Design Theory. Mannheim/Wien/Zürich 1985 (B.I. Wissenschaftsverlag)

[2] Th. Bier: *Eine homologische Interpretation gewisser Inzidenzmatrizen mod p.* Math. Ann. 297 (1993) 289–302

[3] R. A. Brualdi, H. Ryser: Combinatorial Matrix Theory. Cambridge 1991 (Cambridge University Press)

[4] C.D. Godsil: Algebraic Combinatorics. New York/London 1993 (Chapman & Hall).

[5] J.B. Graver, W.B. Jurkat: *The Module Structure of Integral Designs.* Journal of Comb. Theory A **15** (1973), 75–90

[6] H. Krämer: *Inversion of Incidence Mappings.* Séminaire Lotharingien de Combinatoire, B39f (1997), 20pp. `http://cartan.u-strasbg.fr:80/~slc/`

[7] R.M. Wilson: *A Diagonal Form for the Incidence Matrices of t–Subsets vs. k–Subsets.* Europ. J. Combinatorics **11** (1990) 609–615.