Inversion of Incidence Mappings

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Abstract

Denote by $H(t,q)$, $t \leq q$, the incidence matrix (with respect to inclusion) of the t–sets versus the q–sets of the n–set $\{1,2,\dots,n\}$. This matrix is considered as a linear map of $\mathbb{Q}$–vector spaces

$$C_q(n) \longrightarrow C_t(n),$$

where $C_s(n)$ is the $\mathbb{Q}$–vector space having the s–sets as a basis ($s \leq n$).

As a basic tool, we introduce a connection of the vector spaces to a graded $\mathbb{Q}$–algebra (which is at the same time an Artinian local ring). We define mappings $\triangle$ and $\mathbb{X}$ of this algebra of degree -1 and 1, respectively. These two mappings correspond up to a scalar factor to the linear mappings $H(s-1,s)$ and $H(s-1,s)^T$, respectively.

Then, a relation between the algebra maps $\triangle$ and $\mathbb{X}$ is established. This relation allows to rewrite a term $\triangle^\beta \mathbb{X}^\alpha$ with α, β non–negative integers (subject to some restrictions) as a sum $\sum_{k=0}^{\beta} \binom{\beta}{k} \cdot \mathbb{X}^{\alpha-k} \cdot \triangle^{\beta-k}$ (up to some scalar factors). As a main result of this relation surjectivity of the map $\triangle^{q-t}$ (related to $H(t,q)$ up to a scalar factor) is proved under the assumption $\binom{n}{t} \leq \binom{n}{q}$. Moreover, a right inverse for the matrix $H(t,q)$ is given explicitely.

This result is exploited to give an inverse of the (square) incidence matrix $H(t,q)$ in the case $t = n-q$.

These results extend some work done by J.B. Graver and W.B. Jurkat.

1. For $n \in \mathbb{N}$ we put

$$\underline{\underline{n}} = \{1, 2, \dots, n\}.$$

For $0 \leq t \leq q \leq n$ we denote by $H\,(t,q)$ the incidence matrix (with respect to inclusion) of the t–sets versus the q–sets of the n–set $\underline{\underline{n}}$ and by $C_q(n)$ the $\mathbb{Q}$–vector space having the basis $\{[M]\}_{M \subseteq \underline{\underline{n}}, |M|=q}$. Then $H\,(t,q)$ defines a linear mapping ("incidence mapping")

$$C_q(n) \longrightarrow C_t(n),\ [M] \longrightarrow \sum_{N \subseteq M}^{|N|=t} [N],$$

which is denoted by the same symbol $H\,(t,q)$. –

The transpose $H\,(t,q)^T$ of $H\,(t,q)$ defines a linear mapping

$$C_t(n) \longrightarrow C_q(n),\ [N] \longrightarrow \sum_{N \subseteq M}^{|M|=q} [M],$$

which is denoted by the same symbol $H\,(t,q)^T$. Finally we define the "augmentation mapping"

$$H\,(-1,0) : C_0(n) \longrightarrow 0.$$

Assume that $\dim_{\mathbb{Q}} C_t(n) = \binom{n}{t} \leq \binom{n}{q} = \dim_{\mathbb{Q}} C_q(n)$. Then it is known that the mapping $H\,(t,q)$ is surjective ([3], 2.3, 2.4), therefore in case of equality of the two dimensions under consideration an isomorphism. In this case moreover we exhibit a method to compute explicitely $H\,(t,q)^{-1}$ by defining a structure of a graded commutative algebra on the graded $\mathbb{Q}$–vector space $C_*(n) := \bigoplus_{q=0}^{n} C_q(n)$ such that at the same time $C_*(n)$ becomes an Artinian local ring.

We provide an example for the computation of $H\,(n-q,q)^{-1}$:

- *Assume n odd and $q = \lfloor \frac{n}{2} \rfloor + 1$. Then one has*

$$H\,(q-1,q)^{-1} = \sum_{j=1}^{q} \frac{(-1)^{j+1}}{j} \cdot H\,(q-j,q)^T \circ H\,(q-j,q-1).$$

There is the following generalization:

•• *Assume* $q \le \frac{n}{2}$ *if* n *is even or* $q \le \lfloor \frac{n}{2} \rfloor + 1$ *if* n *is odd. Then the mapping*

$$K\,(q, q-1) : C_{q-1}(n) \longrightarrow C_q(n)$$

defined by

$$K\,(q, q-1) = \sum_{j=1}^{q} \frac{(-1)^{j+1}}{j} \binom{n-2q+j+1}{j}^{-1} \cdot H\,(q-j, q)^T \circ H\,(q-j, q-1)$$

is a right inverse of $H\,(q-1, q)$.

2. One defines the structure of a commutative $\mathbb{Q}$–algebra on $C_*(n) = \bigoplus\limits_{q=0}^{n} C_q(n)$ by setting for subsets M, N of $\underline{\underline{n}}$

$$[M] \cdot [N] = \begin{cases} [M \cup N], & \text{if } M \cap N = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

This $\mathbb{Q}$–algebra which we denote by $\mathfrak{C}_*(n)$ is isomorphic to

$$\mathcal{A}\,(n) := \mathbb{Q}\,[T_1, \dots, T_n]/(T_1^2, T_2^2, \dots, T_n^2) = \mathbb{Q}\,[X_1, \dots, X_n],$$

where $T_1, \dots, T_n$ are algebraically independent elements and $X_j = T_j \bmod (T_1^2, \dots, T_n^2)$. The isomorphism $\mathcal{C}_*(n) \longrightarrow \mathcal{A}\,(n)$ is induced by

$$\begin{aligned} [M] = \big[\{j_1, \dots, j_q\}\big] &\longrightarrow X_{j_1} X_{j_2} \cdot \ldots \cdot X_{j_q}, \text{ if } |M| = q \ge 1, \\ [\emptyset] &\longrightarrow 1 \in \mathbb{Q}. \end{aligned}$$

In the sequel we identify $\mathfrak{C}_*(n)$ and $\mathcal{A}\,(n)$.

Let $\mathfrak{C}_*(n)_p$ denote the $\mathbb{Q}$–module of the elements of degree p of the graded algebra $\mathfrak{C}_*(n)$. Then one has

$$\mathfrak{C}_*(n)_p = \begin{cases} C_p(n), & 0 \le p \le n, \\ 0, & p > n. \end{cases}$$

To the mappings $H\,(q-1, q)$, $0 \le q \le n$, corresponds the map $\triangle : \mathfrak{C}_*(n) \longrightarrow \mathfrak{C}_*(n)$ of degree -1 defined by

$$\begin{aligned} \triangle\Big|_{\mathbb{Q}} = 0,\ \triangle X_j &= 1,\ j \in \underline{\underline{n}}, \\ \triangle\,(X_{j_1} \cdot X_{j_2} \cdot \ldots \cdot X_{j_q}) &= \sum_{k=1}^{q} X_{j_1} \cdot \ldots \cdot \widehat{X}_{j_k} \cdot \ldots \cdot X_{j_q},\ 2 \le q \le n, \end{aligned}$$

$(1 \leq j_1 < j_2 < \ldots < j_q \leq n, \hat{}$ denotes the deleting operator).

Finally we define $\mathbb{X} = \sum_{j=1}^{q} X_j$.

Proposition 1. *Assume $0 \leq t \leq q \leq n$. Then with the agreement $\triangle^\circ = id$, $\mathbb{X}^\circ = 1$ the following identities hold*

$$\begin{aligned} (q-t)! \cdot H(t,q) &= \triangle^{q-t}\Big|_{C_q(n)}, \\ (q-t)!\, H(t,q)^T(w) &= \mathbb{X}^{q-t} \cdot w, \; w \in C_t(n). \end{aligned}$$

Proof. To prove the first statement we show by induction with respect to m, $0 \leq m \leq q$, that the identity

$$(1)\ldots \qquad \triangle^m\Big|_{C_q(n)} = m!\, H(q-m,q)$$

holds. Of course, this is true for $m = 0,1$. Assume this identity has been proved already in case $m-1 \geq 1$.

The following relation is well known in case $0 \leq s \leq t \leq q$

$$(1)\ldots \qquad H(s,t) \circ H(t,q) = \binom{q-s}{t-s} H(s,q)$$

(see for example [2], Chapt 15, Lemma 8.1).

Therefore one has

$$\begin{aligned} \triangle^m\Big|_{C_q(n)} &= \triangle\Big|_{C_{q-m+1}(n)} \circ \triangle^{m-1}\Big|_{C_q(n)} \\ &= H(q-m,q-m+1) \cdot (m-1)! H(q-m+1,q) \\ &= m \cdot (m-1)! \cdot H(q-m,q) = m!\, H(q-m,q). \quad - \end{aligned}$$

To prove the second statement we show by induction with respect to $m \geq 0$, $t+m \leq n$, that for $w \in C_t(n)$ the following relation holds

$$\mathbb{X}^m \cdot w = m!\, H(t,t+m)^T(w).$$

It is sufficient for our purpose to take w as an element $X_{j_1} \cdot \ldots \cdot X_{j_t}$ of the "canonical" basis of $C_t(n)$. The claim is evident in case $m = 0$; moreover one has

$$\mathbb{X} w = \sum_{k \neq j_1, j_2, \ldots, j_t}^{k} X_{j_1} X_{j_2} \cdot \ldots \cdot X_{j_t} \cdot X_k = H(t,t+1)^T(w).$$

Assume that the statement has already been proved in case $m-1 \geq 0$. By transposing one gets from Eq (1)

$$H\,(t, t+m)^T = m \cdot H\,(t+1, t+m)^T \circ H\,(t, t+1)^T,$$

Therefore according to the induction hypothesis (applied to $\mathbb{X}w \in C_{t+1}(n)$)

$$\begin{aligned} \mathbb{X}^m w = \mathbb{X}^{m-1} \cdot \mathbb{X}w \;&= (m-1)!\; H\,(t+1, t+m)^T \circ H\,(t, t+1)^T (w) \\ &= m!\; H\,(t, t+m)^T\,(w). \end{aligned}$$

$\square$

Suppose $w \in \mathfrak{C}_*(n)$. Then we define the "foundation" of w (in signs Fund (w)) to be the product of all X_j which appear in the basis decomposition of w with a coefficient unequal to zero.

For example one has with pairwise distinct $X_{j_1}, X_{j_2}, \ldots, X_{j_{2t}}$

$$\text{Fund}\,\Big((X_{j_1} - X_{j_2})(X_{j_3} - X_{j_4}) \cdot \ldots \cdot (X_{j_{2t-1}} - X_{j_{2t}})\Big) = \prod_{k=1}^{2t} X_{j_k}.$$

In a self–explaining manner we can treat the foundation also as a subset of $\underline{\underline{n}}$.

Proposition 2. *i) Assume $v, w \in \mathfrak{C}_*(n)$ and Fund $(v) \cap$ Fund $(w) = \emptyset$. Then we have that*

$$\triangle\,(vw) = v\triangle\,(w) + w\triangle\,(v).$$

ii) Denote $\mathbb{Z} = \sum\limits_{k=1}^{p} X_{j_k}$, the X_{j_k} pairwise distinct. Assume $1 \leq m \leq p$ (and put $\mathbb{Z}^0 = 1$). Then we have that

$$\triangle\,(\mathbb{Z}^m) = m\,(p-m+1)\mathbb{Z}^{m-1}.$$

Proof. Ad i) Assume in the first instance $v = X_{i_1} \cdot \ldots \cdot X_{i_s}$, $w = X_{j_1} \cdot \ldots \cdot X_{j_t}$ are elements from a basis of $\mathfrak{C}_*(n)$. According to hypothesis one has

$$\big|\{i_1, \ldots, i_s, j_1, \ldots, j_t\}\big| = s + t.$$

Therefore

$$\begin{aligned}\triangle(vw) &= \sum_{k=1}^{s} X_{i_1}\cdot\ldots\cdot\widehat{X}_{i_k}\cdot\ldots\cdot X_{i_s}\cdot X_{j_1}\cdot\ldots\cdot X_{j_t}\\ &+\sum_{l=1}^{t} X_{i_1}\cdot\ldots\cdot X_{i_s}\cdot X_{j_1}\cdot\ldots\cdot\widehat{X}_{j_l}\cdot\ldots\cdot X_{j_t}\\ &= w\triangle(v)+v\triangle(w).\end{aligned}$$

The general case follows now from the law of distributivity.

Ad ii) Without loss of generality assume $\mathbb{Z}=\sum_{j=1}^{p} X_j$. Then we have

$$(2)\ldots\qquad \mathbb{Z}^k=\begin{cases} k!\displaystyle\sum_{1\le j_1<\ldots<j_k\le p} X_{j_1}\cdot\ldots\cdot X_{j_k}, & 1\le k\le p,\\ 0, & k>p.\end{cases}$$

In this sum a term $X_{i_1}\cdot\ldots\cdot X_{i_{m-1}}$, $(1\le i_1<\ldots<i_{m-1}\le p)$ occurs exactly in the terms $\triangle(X_{i_1}\cdot\ldots\cdot X_{i_{m-1}}\cdot X_t)$, $t\in\underline{\underline{p}}\backslash\{i_1,\ldots,i_{m-1}\}$ with factor 1; therefore it occurs in $\triangle(\mathbb{Z}^m)$ with factor $m!(p-m+1)=m\,(p-m+1)(m-1)!$. The conclusion follows now from equation (2). □

We remark that $\triangle$ is no derivation of $\mathfrak{C}_*(n)$.

Let $\mathfrak{m}:=(X_1,\ldots,X_n)$ denote the maximal ideal of $\mathfrak{C}_*(n)$. Then it holds that $\mathfrak{m}^{n+1}=0$. If $\mathfrak{p}$ is a prime ideal of $\mathfrak{C}_*(n)$, then from $\mathfrak{m}^{n+1}\subseteq\mathfrak{p}$ one concludes $\mathfrak{m}=\mathfrak{p}$. Therefore $\mathfrak{C}_*(n)$ is an Artinian local ring. Let $\widetilde{\triangle}$ denote a derivation of $\mathfrak{C}_*(n)$ (into itself). Then it must hold for all $j\in\underline{\underline{n}}$ that

$$0=\widetilde{\triangle}(X_j^2)=2X_j\widetilde{\triangle}(X_j),$$

therefore $\widetilde{\triangle}(X_j)\in\mathfrak{m}$. – So $\widetilde{\triangle}$ maps $\mathfrak{m}$ (and $\mathfrak{C}_*(n)$, too) into $\mathfrak{m}$ (compare with [4], §1, Exercise 4).

Now we extend Prop.2, ii):

Proposition 3. *Assume* $0\le s\le n-1$, $\alpha\in\mathbb{N}$, $\alpha+s\le n$ *and* $w\in C_s(n)$. *Then the following identity holds:*

$$\triangle(\mathbb{X}^\alpha\cdot w)=\mathbb{X}^\alpha\cdot\triangle(w)+\alpha\,(n-\alpha-2s+1)\mathbb{X}^{\alpha-1}\cdot w.$$

Proof. In case $s = 0$ the statement is true according to Prop. 2, ii). Assume now $s \geq 1$ and take

$$w = X_{j_1} \cdot \ldots \cdot X_{j_s}$$

as a basis element of $C_s(n)$. One defines

$$\mathbb{Y} = \sum_{k=1}^{s} X_{j_k},\ \mathbb{Z} = \mathbb{X} - \mathbb{Y}.$$

Since w is a word in *all* $X_{j_1}, \ldots, X_{j_s}$ one has $\mathbb{Y} \cdot w = 0$. Assume $t \geq 0$. Then the following holds

$$\mathbb{X}^t w = (\mathbb{Y} + \mathbb{Z})^t \cdot w = \sum_{r=0}^{t} \binom{t}{r} \mathbb{Z}^r \mathbb{Y}^{t-r} \cdot w = \mathbb{Z}^t w,$$

furthermore according to the definition of $\mathbb{Y}$ and $\mathbb{Z}$ Fund $(w) \cap$ Fund $(\mathbb{Z}^t) = \emptyset$; from Prop. 2 one concludes

$$(3)\ldots \qquad \left\{ \begin{aligned} \triangle(\mathbb{X}^\alpha \cdot w) &= \triangle(\mathbb{Z}^\alpha \cdot w) = \mathbb{Z}^\alpha \cdot \triangle(w) + w \cdot \triangle(\mathbb{Z}^\alpha) \\ &= \mathbb{Z}^\alpha \cdot \triangle(w) + \alpha(n - s - \alpha + 1)\,\mathbb{Z}^{\alpha-1} w \\ &= \mathbb{Z}^\alpha \cdot \triangle(w) + \alpha(n - s - \alpha + 1)\,\mathbb{X}^{\alpha-1} w. \end{aligned} \right.$$

Now we use

$$\mathbb{Y}^t \cdot \triangle(w) = \begin{cases} 0, & t > 1, \\ s \cdot w, & t = 1. \end{cases}$$

Therefore

$$\begin{aligned} \mathbb{X}^\alpha \triangle(w) &= \left(\sum_{r=0}^{\alpha} \tbinom{\alpha}{r} \mathbb{Y}^r \mathbb{Z}^{\alpha-r} \right) \cdot \triangle(w) = (\alpha \mathbb{Y} \cdot \mathbb{Z}^{\alpha-1} + \mathbb{Z}^\alpha) \cdot \triangle(w) \\ &= \alpha s \mathbb{Z}^{\alpha-1} w + \mathbb{Z}^\alpha \cdot \triangle(w) = \alpha \cdot s \cdot \mathbb{X}^{\alpha-1} w + \mathbb{Z}^\alpha \cdot \triangle(w). \end{aligned}$$

Replacing the term $\mathbb{Z}^\alpha \triangle(w)$ in Eq. (3) yields

$$\triangle(\mathbb{X}^\alpha w) = \mathbb{X}^\alpha \cdot \triangle(w) + \alpha(n - 2s - \alpha + 1)\,\mathbb{X}^{\alpha-1} \cdot w,$$

as claimed. – $\square$

Prop. 3 is a special case of

Proposition 4. *The same assumptions about s, α are in force as in Prop. 3. Assume $r \in \mathbb{R}$ and $k \in \mathbb{N}$ and define*

$$[r]_k = r\,(r-1)\cdot(r-2)\cdot\ldots\cdot(r-k+1),\ [r]_0 = 1.$$

Let β be a non–negative integer and assume $0 \le \beta \le \alpha$. Then the following identity holds

$$\triangle^\beta(\mathbb{X}^\alpha\cdot w) = \sum_{k=0}^{\beta}\binom{\beta}{k}[\alpha]_k[n-\alpha-2s+\beta]_k\cdot\mathbb{X}^{\alpha-k}\cdot\triangle^{\beta-k}(w).$$

Proof. There is nothing to prove in case $\beta = 0$; in case $\beta = 1$ the claim boils down to Prop. 3. If $0 \le j \le \beta$ we put

$$[\beta, j] = \mathbb{X}^{\alpha-j}\cdot\triangle^{\beta-j}(w)$$

and prove first of all by induction with respect to $\beta \ge 1$, that the identity

$$(4)\ldots\qquad \triangle^\beta(\mathbb{X}^\alpha w) = \sum_{j=0}^{\beta} c\,(\beta,j)\cdot[\beta,j],\quad c\,(\beta,j)\in\mathbb{Q},$$

holds; furthermore this will yield recursion formulas for the coefficients $c\,(\beta, j)$. In case $\beta = 1$ one has according to Prop. 3

$$c\,(1,0) = 1,\ c\,(1,1) = \alpha\,(n-\alpha-2s+1) =: \lambda\,(\alpha,s).$$

Since $\triangle^{k-j}(w) \in C_{s-k+j}(n)$, Prop. 3 now yields

$$\begin{aligned}
&\triangle\left([k,j]\right) = \triangle\big(\mathbb{X}^{\alpha-j}\cdot\triangle^{k-j}(w)\big) = \\
&= \mathbb{X}^{\alpha-j}\cdot\triangle^{k-j+1}(w) + (\alpha-j)\cdot\big(n-(\alpha-j)-2(s-k+j)+1\big)\cdot\mathbb{X}^{\alpha-j-1}\cdot\triangle^{k-j}(w) = \\
&= \mathbb{X}^{\alpha-j}\cdot\triangle^{k-j+1}(w) + \lambda\,(\alpha-j, s-k+j)\cdot\mathbb{X}^{\alpha-j-1}\cdot\triangle^{k-j}(w),
\end{aligned}$$

that is

$$\triangle\left([k,j]\right) = [k+1,j] + \lambda(\alpha-j, s-k+j)[k+1,j+1].$$

According to the induction hypothesis we obtain

$$\begin{aligned}
\triangle^{\beta+1}(\mathbb{X}^\alpha w) &= \triangle\left(\triangle^\beta(\mathbb{X}^\alpha\cdot w)\right) = \\
c\,(\beta,0)[\beta+1,0] &+ \sum_{j=1}^{\beta}\big\{c\,(\beta,j) + c\,(\beta,j-1)\cdot\lambda\,(\alpha-j+1, s-\beta+j-1)\big\}\cdot[\beta+1,j] \\
&+ \lambda\,(\alpha-\beta,s)\cdot c\,(\beta,\beta)[\beta+1,\beta+1],
\end{aligned}$$

which proves Eq. (4); at the same time we have proved the following recursion formulas

(5.1) ... $$c(\beta+1,0)=c(\beta,0),\ c(\beta+1,\beta+1)=\lambda(\alpha-\beta,s)\,c(\beta,\beta),$$

(5.2) ... $$c(\beta+1,j)=c(\beta,j)+\lambda(\alpha-j+1,s-\beta+j-1)\cdot c(\beta,j-1),(1\leq j\leq\beta).$$

Eq. (5.1) yields immediately

(6)... $$\begin{cases} c(\beta,0)=1 \\ \\ c(\beta,\beta)=[\alpha]_\beta[n-\alpha-2s+\beta]_\beta \end{cases} \qquad (0\leq\beta\leq\alpha).$$

Now we claim that the following holds in case $\beta\geq j$

(7)... $$c(\beta,j)=\binom{\beta}{j}[\alpha]_j[n-\alpha-2s+\beta]_j$$

which we prove by induction with respect to the pairs (β,j), $\beta\geq j$. The claim is true for pairs $(\beta,0)$ according to Eq. (6). Assume that it has already been proved that in case $j\geq 1$ the claim is true for all pairs $(\gamma,j-1)$, $\gamma\geq j-1$. We rewrite a term in the recursion formula (5.2)

$$\lambda(\alpha-j+1,s-\beta+j-1)=(\alpha-j+1)(n-\alpha-2s+2\beta-j+2).$$

Now we proceed by induction with respect to $\beta,\beta\geq j$. In case $\beta=j$ the statement is true according to Eq. (6). Assume that the statement has already be proved for some $\beta\geq j$. Then we have according to Eq. (5.2)

$$\begin{aligned} c(\beta+1,j)= & \tbinom{\beta}{j}[\alpha]_j[n-\alpha-2s+j]_j+ \\ & +(n-\alpha-2s+2\beta-j+2)\cdot\tbinom{\beta}{j-1}[\alpha]_j[n-\alpha-2s+\beta]_{j-1}. \end{aligned}$$

Using the identities $\binom{\beta}{j}=\binom{\beta}{j-1}\cdot\dfrac{\beta-j+1}{j}$ and $\binom{\beta}{j}+\binom{\beta}{j-1}=\binom{\beta+1}{j}$ we conclude

$$\begin{aligned} c(\beta+1,j) &= [\alpha]_j[n-\alpha+2s+\beta]_{j-1}\cdot\Big\{\tbinom{\beta+1}{j}(n-\alpha-2s)+\tbinom{\beta}{j}(\beta-j+1)+ \\ &\quad +\tbinom{\beta}{j-1}(2\beta-j+2)\Big\} \\ &= [\alpha]_j[n-\alpha-2s+\beta]_{j-1}\cdot\tbinom{\beta+1}{j}(n-\alpha-2s+\beta+1) \\ &= \tbinom{\beta+1}{j}[\alpha]_j[n-\alpha-2s+(\beta+1)]_j \end{aligned}$$

This proves Eq. (7). – $\square$

3. Assume now $0 \leq t \leq q \leq n$ and $\binom{n}{t} \leq \binom{n}{q}$. We denote $\nabla := \triangle^{q-t}$: $C_q(n) \longrightarrow C_t(n)$ and fix some $w \in C_t(n)$. Our aim is to construct explicitely a primage $u \in C_q(n)$ of w with respect to ∇. Therefore we make the following ansatz (we will see later that this method will work): If $0 \leq j \leq t$ we denote

$$U_j := \mathbb{X}^{q-t+j} \cdot \triangle^j(w)$$

and put

$$u = \sum_{j=0}^{t} x_j U_j$$

where the $x_j \in \mathbb{Q}$ have to be determined. To compute ∇U_j apply Prop. 4 (replace here w by $\triangle^j(w) \in C_{t-j}(n)$) and obtain with the convention

$$(8)\ldots \quad c(k,j) := \binom{q-t}{k}[q-t+j]_k \cdot [n-2t+j]_k, \quad (0 \leq j \leq t,\ 0 \leq k \leq q-t)$$

$$\nabla U_j = \sum_{k=0}^{q-t} c(k,j) \cdot \mathbb{X}^{q-t+j-k} \cdot \triangle^{q-t+j-k}(w) \in C_t(n),$$

therefore

$$(9)\ldots \qquad \nabla u = \sum_{k=0}^{q-t}\sum_{j=0}^{t} x_j c(k,j) \cdot \mathbb{X}^{q-t+j-k} \cdot \triangle^{q-t+j-k}(w).$$

We order the right hand side of this equation with respect to the terms

$$V_m := \mathbb{X}^m \cdot \triangle^m(w),\ 0 \leq m \leq t,$$

by defining

$$V_{k,j} = \mathbb{X}^{q-t+j-k} \cdot \triangle^{q-t+j-k}(w).$$

Then it holds that $V_{k,j} = 0$ if $q-t+j-k > t$ and

$$V_{k,j} = V_m$$

exactly if $k = (q-t)-l$, $j = m-l$, $0 \leq l \leq \min\{m, q-t\}$. Therefore Eq. (9) yields now

$$\nabla u = \sum_{m=0}^{t} d_m V_m$$

with

$$(10)\ldots \qquad d_m = \sum_{l=0}^{\min\{m,q-t\}} c\left((q-t)-l,\, m-l\right)\cdot x_{m-l}.$$

We observe that $V_0 = w$. – So u is in fact a preimage, if the system of linear equations in the unknowns $x_0, \ldots, x_t$

$$d_0 = 1,$$

$$d_m = 0, 1 \le m \le t,$$

is solvable. This will be the case if all $c(q-t, m)$ don't vanish. In fact we have in the trivial case $q - t = 0$

$$c(0, m) = 1,\ 0 \le m \le t,$$

(and the system has the solution $x_0 = 1,\ x_1 = x_2 = \ldots = x_t = 0$). In case $q - t > 0$ it is sufficient (see Eq. (8)) to show that the

$$[n - 2t + m]_{q-t}$$

don't vanish. Now the smallest factor f_m in the above mentioned falling factorial is

$$f_m = n - 2t + m - (q-t) + 1 = n - (q+t) + m + 1.$$

Assume first $\binom{n}{t} = \binom{n}{q}$. Then it holds that $t + q = n$, therefore $f_m = m + 1$. In the second case the condition $\binom{n}{k} < \binom{n}{q}$ is equivalent to $q + t + 1 \le n$, therefore $f_m \ge m + 2$.

So our ansatz has worked and we have proved

Theorem 1. *Assume $0 \le t \le q \le n$ and $\binom{n}{t} \le \binom{n}{q}$. Then the mapping*

$$\nabla := \triangle^{q-t} : C_q(n) \longrightarrow C_t(n)$$

is surjektiv. There exist $x_0, x_1, \ldots, x_t \in \mathbb{Q}$ such that the mapping

$$\nabla^{[-1]} := \sum_{j=0}^{t} x_j \cdot \mathbb{X}^{q-t+j} \cdot \triangle^j \Big|_{C_t(n)}$$

is a right inverse with respect to ∇ (in case $\binom{n}{t} = \binom{n}{q}$ $\nabla^{[-1]}$ is the inverse of ∇).

If one denotes in case $0 \leq k \leq q-t$, $0 \leq j \leq t$,

$$c\,(k,j) := \binom{q-t}{k} \cdot [q-t+j]_k \cdot [n-2t+j]_k,$$

then one can choose the $x_0, \ldots, x_t$ *as the (existing) solution of the system of linear equations*

$$c\,(q-t,0) \cdot x_0 = 1,$$
$$\sum_{l=0}^{\min\{m,q-t\}} c\,\big((q-t)-l, m-l\big) \cdot x_{m-l} = 0,\ 1 \leq m \leq t.$$

Corollary 1. *If one defines*

$$y_j = (q-t)!j!(q-t+j)! \cdot x_j,\ 0 \leq j \leq t,$$

then the mapping $K\,(q,t) : C_t(n) \longrightarrow C_q(n)$ *defined by*

$$K\,(q,t) := \sum_{j=0}^{t} y_j \cdot H\,(t-j,q)^T \circ H\,(t-j,t)$$

is a right inverse of $H\,(t,q)$.

Proof (of the corollary). According to Prop. 1 one has

$$\nabla = (q-t)! \cdot H\,(t,q)$$

and

$$\mathbb{X}^{q-t+j} \cdot \triangle^j \Big|_{C_t(n)} = (q-t+j)!j! \cdot H\,(t-j,q)^T \circ H\,(t-j,t). \qquad \square$$

Now we exploit the the theorem and the corollary. The following result is properly spoken another corollary; however, we state it as

Theorem 2. *Assume* $0 \leq t < q \leq n$ *and* $t+q=n$. *Then the following identity holds:*

$$H\,(t,q)^{-1} = \sum_{j=0}^{t} (-1)^j \frac{q-t}{q-t+j}\, H\,(t-j,q)^T \circ H\,(t-j,t).$$

Proof. We introduce a new parameter $p = q - t$. Then we obtain $n - 2t = p$ and

$$c(k, j) = \binom{p}{k} \cdot \left([p + j]_k\right)^2.$$

We switch now to new indeterminates α_j in the system of linear equations of the Theorem by defining

$$x_j = (-1)^j \frac{\alpha_j}{((p + j)!)^2}, \ \alpha_j \in \mathbb{Q}.$$

An elementary computation which we omit yields

$$\alpha_0 = 1$$

and the following recursion formula

$$\alpha_j = \sum_{l=1}^{\min\{p,j\}} (-1)^{l+1} \binom{p}{l} \alpha_{j-l}, \ 1 \leq j \leq t.$$

We compute the α_j by elementary difference calculus; let us define therefore functions

$$f_p : \mathbb{N}_0 \longrightarrow \mathbb{Q}$$

by the following conditions

$$(11)\ldots \qquad \begin{cases} f_p(0) = 1, \\ f_p(j) = \sum\limits_{l=1}^{\min\{p,j\}} (-1)^{l+1} \binom{p}{l} f_p(j - l), \ 1 \leq j. \end{cases}$$

Then we claim

$$(12)\ldots \qquad f_p(j) = \binom{j + p - 1}{p - 1}.$$

In order to prove Eq. (12) we need the following

Lemma. *Assume $p \geq 1$ and $0 \leq l \leq p$. Then the following identity holds*

$$\binom{p}{l} - \binom{p}{l-1} + \binom{p}{l-2} \mp \ldots + (-1)^l \binom{p}{0} = \binom{p-1}{l}.$$

Proof (of the lemma): Denote the left hand side of this identity by $S_l(p)$. Then we have

$$S_l(p) = \binom{p}{l} - S_{l-1}(p).$$

Now we proceed by induction with respect to l. □

Denote by $\triangle f_p$ the first difference series of f_p, that is

$$(\triangle f_p)(j) = f_p(j+1) - f_p(j),\ j \in \mathbb{N}_0.$$

Then we have by placing $f_p(k) = \alpha_k$ and $h = \min\{p, j\}$, in case $j \geq 1$

$$\begin{aligned}\alpha_{j+1} - \alpha_j \quad &= \left[\tbinom{p}{1} - \tbinom{p}{0}\right] \cdot (\alpha_j - \alpha_{j-1}) - \left[\tbinom{p}{2} - \tbinom{p}{1} + \tbinom{p}{0}\right](\alpha_{j-1} - \alpha_{j-2}) \pm \dots \\ &+ (-1)^{h+1}\left[\tbinom{p}{h} - \tbinom{p}{h-1} + \tbinom{p}{h-2} \mp \dots + (-1)^h \tbinom{p}{0}\right] \cdot (\alpha_{j-h+1} - \alpha_{j-h}).\end{aligned}$$

According to the lemma this rewrites to

$$(\triangle f_p)(j) = \sum_{l=1}^{\min\{j,p\}} (-1)^{l+1} \binom{p-1}{l} (\triangle f_p)(j-l),\ j \geq 1.$$

Furthermore we have

$$(\triangle f_p)(0) = p - 1 = f_{p-1}(1),\ p \geq 2.$$

This yields in case $p \geq 2$

(13)... $$(\triangle f_p)(j) = f_{p-1}(j+1),\ j \in \mathbb{N}_0,$$

since $\triangle f_p$ satisfies the same recursion formula (see Eq. (12)) as f_{p-1} does.

Now Eq. (12) is certainly true if $p = 1$. Assume our claim is true if $p - 1 \geq 1$. Then according to Eq. (13) we have

$$(\triangle f_p)(j) = \binom{j+p-1}{p-2},\ j \in \mathbb{N}_0.$$

Therefore the function $F : \mathbb{N}_0 \to \mathbb{Q}$ defined by

$$F(j) = \binom{j+p-1}{p-1}$$

is a discrete indefinite integral of $\triangle f_p$; that is $\triangle F = \triangle f_p$. It follows that

$$F = f_p + \text{ const},$$

but $F(0) = f_p(0)$, so we have $F = f_p$. Therefore the claim Eq. (12) is proved.

Now we use the notations introduced in the corollary of Theorem 1 and obtain

$$y_j = (-1)^j \frac{p!\, j!\, (p+j)!}{((p+j)!)^2} \binom{j+p-1}{p-1} = (-1)^j \frac{p}{p+j} = (-1)^j \frac{q-t}{q-t+j}.$$

The statement of Theorem 2 now follows from the corollary to Theorem 1. $\square$

To exploit Theorem 1 and the corollary in the case $\binom{n}{t} \leq \binom{n}{q}$, we restrict ourselves to the condition $q - t = 1$. We then have $q \leq \frac{n}{2}$ if n is even and $q \leq \lfloor \frac{n}{2} \rfloor + 1$ if n is odd.

Here we have under the assumption $0 \leq j \leq t = q - 1$

$$\begin{aligned} c(0, j) &= 1, \\ c(1, j) &= (j+1)(n - 2q + j + 2). \end{aligned}$$

We obtain the following system of linear equations in the unknowns $x_0, \ldots, x_t$

$$\begin{aligned} c(1, 0)\, x_0 &= 1, \\ c(1, j)\, x_j + x_{j-1} &= 0,\ 1 \leq j \leq t, \end{aligned}$$

which has the solution

$$x_j = \frac{(-1)^j}{(j+1)![n - 2q + j + 2]_{j+1}},\ 0 \leq j \leq t.$$

Again with the notations of the corollary to Theorem 1 we obtain

$$y_j = j!\,(j+1)!\, x_j = \frac{(-1)^j}{j+1} \cdot \binom{n - 2q + j + 2}{j+1}^{-1}.$$

Now we change the indices from $j+1$ to j and obtain the statement of example $\bullet\bullet$.

Finally let us evaluate Theorem 2 in case $n = 5$, $q = 3$, $t = 2$. With the notation $\{j_1, j_2, j_3, j_4, j_5\} = \underline{\underline{5}}$ we obtain

$$H\,(2,3)^{-1}(X_{j_1}\cdot X_{j_2}) =$$

$$\frac{1}{6}\Big[2X_{j_1}X_{j_2}X_{j_3} \quad -X_{j_2}X_{j_3}X_{j_4} + 2X_{j_3}X_{j_4}X_{j_5} - X_{j_1}X_{j_4}X_{j_5} + 2X_{j_1}X_{j_2}X_{j_4} - X_{j_1}X_{j_3}X_{j_5} + \\ +2X_{j_1}X_{j_2}X_{j_5} - (X_{j_1}X_{j_3}X_{j_4} + X_{j_2}X_{j_3}X_{j_5} + X_{j_2}X_{j_4}X_{j_5})\Big],$$

and conclude that the matrix $H\,(2,3)^{-1}$ is "non–sparse".

4. Finally we demonstrate the usefulness (as we hope) of the algebra $\mathfrak{C}_*(n)$ by giving new proofs or extending, respectively, some results in [2].

• • • (l.c., 3.3) *Let* $v \in \mathrm{Ker}\ \triangle$, $w \in C_t(n)$, *such that* Fund $(v) \cap$ Fund $(w) = \emptyset$. *Then it holds that* $\triangle^{t+1}(v\cdot w) = 0$.

This is an immidediate consequence of the property of $\triangle$ to be a "quasi–derivation" (see Prop. 2).

Secondly, we exhibit a system of generators of Ker $H\,(t,q)$ provided $\binom{n}{t} < \binom{n}{q}$.

We denote $K_q(n) = \mathrm{Ker}\ \triangle\Big|_{C_q(n)}$. Then obviosly one has $K_q(n-1) \subset K_q(n)$. To determine remaining candidates u of $K_q(n)$ we make the following ansatz

$$(14)\ldots \qquad u = vX_n + w,\ v \in C_{q-1}(n-1),\ w \in C_q(n-1).$$

Since Fund $(v) \cap$ Fund $(X_n) = \emptyset$, Prop. 2, i) yields

$$0 = \triangle u = (\triangle v)\cdot X_n + (v + \triangle w),$$

therefore

$$(15)\ldots \qquad \triangle v = 0,\ v + \triangle w = 0.$$

Assume now that is has been proved by induction with respect to s that $K_t(s)$, $t \le \lfloor \frac{s}{2} \rfloor$, $s \le n-1$, is generated by elements of the shape

$$(X_{j_1} - X_{j_2})(X_{j_3} - X_{j_4})\cdot\ldots\cdot(X_{j_{2t-1}} - X_{j_{2t}})$$

with pairwise distinct X_{j_k}. Therefore we may assume (in Eq. (14), (15)) that

$$v = (X_{j_1} - X_{j_2}) \cdot \ldots \cdot \left(X_{j_{2(q-1)-1}} - X_{j_{2(q-1)}}\right).$$

There are two cases to be considered. In the first case assume $2\,(q-1) = n-1$; then we have n odd, $q = \lfloor \frac{n}{2} \rfloor + 1$. According to $\bullet \quad K_q(n) = 0$ and then there is no more to prove.

In the second case assume $2\,(q-1) < n-1$. Then there exists $X_j, j \leq n-1$ such that Fund $(v) \cap \{X_j\} = \emptyset$. We conclude that

$$w := -v \cdot X_j$$

solves the second equation in Eq. (15); furthermore one has $u = v\,(X_n - X_j)$.

This proves

$\bullet\bullet\bullet\bullet$ (contained in l.c., 4.2) *Assume* $q \leq \lfloor \frac{n}{2} \rfloor$. *Then it holds that* Ker $H\,(q-1,q) \neq 0$ *and this kernel is generated by elements of the shape*

$$(X_{j_1} - X_{j_2})(X_{j_3} - X_{j_4}) \cdot \ldots \cdot (X_{j_{2q-1}} - X_{j_{2q}}).$$

In as subsequent paper (in which we use the algebra $\mathfrak{C}_*(n)$ to exhibit explicitely eigenspace decompositions of the matrix $H\,(t,q)^T \circ H\,(t,q)$) we will show

Theorem 3. *Assume* $\binom{n}{t} < \binom{n}{q}$. *Then it holds that*

$$\operatorname{Ker} H\,(t,q) = \bigoplus_{s=t+1}^{\min\{q,n-q\}} H\,(s,t)^T \left(\operatorname{Ker} H\,(s-1,s)\right).$$

This result provides together with $\bullet\bullet\bullet\bullet$ systems of generators of Ker $H\,(t,q)$ which are in general distinct from those which are exhibited in [2].

With the aid of Theorem 3 (and Prop. 4) one easily proves

$\bullet\bullet\bullet\bullet\bullet$ (l.c., 4.3) *Assume* $0 \leq t < q < n-t$. *Then* $H\,(t+1,q)$ *maps the kernel of* $H\,(t,q)$ *onto the kernel of* $H\,(t,t+1)$.

Finally we exhibit in a special case a system of generators of Ker $H\,(q-1,q)$ which is different from the "canonical" one constructed above as follows:

Assume $n = 7$, $q = 3$. Any enumeration σ of the points of the projective plane $\mathbb{P}$ consisting of 7 points and 7 lines yields the family $\mathfrak{G}_\sigma \subset \binom{\underline{7}}{3}$ consisting of the lines of $\mathbb{P}$. We claim that the elements

$$u_\sigma := 4 \cdot \sum_{M \in \mathfrak{G}_\sigma} [M] - \sum_{M \notin \mathfrak{G}_\sigma}^{|M|=3} [M]$$

also generate Ker $H\,(2,3)$.

First, we sketch a proof that the u_σ indeed are contained in the kernel of $H\,(2,3)$ as follows: Define for $q \in \underline{\underline{n}}$

$$H_q := H\,(q-1,q)^T \circ H\,(q-1,q) : C_q\,(n) \longrightarrow C_q\,(n).$$

Then it holds, if $[M]$ is an element of the canonical basis of $C_q(n)$,

$$(16)\ldots \qquad H_q\big([M]\big) = q \cdot [M] + \sum_{|M \cap M'|=q-1}^{|M'|=q} [M'].$$

Now it can be seen easily that

$$w_q := \sum_{|M|=q}^{M} [M]$$

is an eigenvector of H_q with eigenvalue $q\,(n-q+1)$. In addition we need the well known result (see for instance [1], Chapt. II, 2.5 Lemma), that

$$\text{rang } H\,(t,q) = \text{rang } H\,(t,q)^T \circ H\,(t,q) \left(= \text{rang } H\,(t,q) \circ H\,(t,q)^T\right),$$

which yields

$$\text{Ker } H\,(t,q) = \text{Ker } \left(H\,(t,q)^T \circ H\,(t,q)\right).$$

After this digression suppose now again $n = 7$, $q = 3$, $t = 2$. Since two different projective lines intersect in one point we have $|M_1 \cap M_2| = 1$ provided $M_1, M_2 \in \mathfrak{G}_\sigma$ and $M_1 \neq M_2$. This in turn yields according to Eq. (16):

$$H_3\left(\sum_{M \in \mathfrak{G}_\sigma} [M]\right) = 3 \cdot \sum_{M \in \mathfrak{G}_\sigma} [M] + \sum_{M' \notin \mathfrak{G}_\sigma}^{|M'|=3} \lambda_{M'} [M'].$$

The coefficients $\lambda_{M'} \in \mathbb{Q}$ are determined as follows: Any $M' \notin \mathfrak{G}_\sigma$ consists of three non–collinear points; therefore for given M' there are exactly three $M \in \mathfrak{G}_\sigma$ such that $|M' \cap M| = 2$. We conclude $\lambda_{M'} = 3$, in turn

$$H_3\left(\sum_{M \in \mathfrak{G}_\sigma} [M]\right) = 3 \cdot w_3 = H_3\left(\frac{1}{5} w_3\right),$$

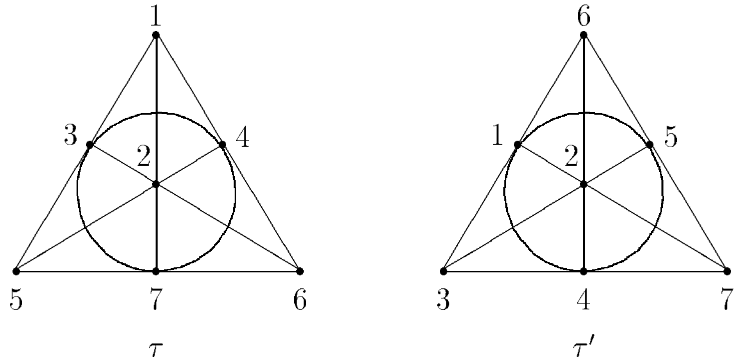
that is

$$\sum_{M \in \mathfrak{G}_\sigma} [M] - \frac{1}{5} w_3 \in \operatorname{Ker}\,\left(H\,(2,3)^T \circ H\,(2,3)\right) = \operatorname{Ker}\, H\,(2,3).-$$

Secondly, assume that v is an element of the "canonical" system of generators of Ker $H\,(2,3)$, say

$$v = (X_1 - X_2)(X_3 - X_4)(X_5 - X_6).$$

Suppose the enumerations τ, τ' of the points of $\mathbb{P}$ are given by



Then a lengthy but elementary computation which we omit yields

$$v = \frac{1}{5}\,(u_\tau - u_{\tau'}).$$

We conclude that the u_σ generate Ker $H\,(2,3)$, too.

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