Séminaire Lotharingien de Combinatoire, B39e(1997), 26pp.

An Algorithm for the Decomposition of Ideals of the Group Ring of a Symmetric Group

B. Fiedler
Institut für Informatik, Universität Leipzig
Augustusplatz 10/11, D-04109 Leipzig
email: fiedler@rz.uni-leipzig.de

Abstract

We present an algorithm which produces a decomposition of left or right ideals of the group ring of a symmetric group into minimal left or right ideals and a corresponding set of primitive pairwise orthogonal idempotents by means of a computer. The algorithm can be used to determine generating idempotents of (left or right) ideals which are given as sums or intersections of (left or right) ideals.

We discuss several subjects such as minimal sets of test permutations and the application of fast Fourier transforms which contribute to a good efficiency of the algorithm. Further we show possibilities of use of the algorithm in the computer algebra of tensor expressions.

1 Introduction

Let $\mathbb{C}[\mathcal{S}_r]$ be the group ring of a symmetric group $\mathcal{S}_r$ over the field of complex numbers $\mathbb{C}$. In this paper we present a computer algorithm which can produce a decomposition of an arbitrary left ideal $I := \mathbb{C}[\mathcal{S}_r] \cdot a$ of $\mathbb{C}[\mathcal{S}_r]$ with given generating element $a \in \mathbb{C}[\mathcal{S}_r]$ into a direct sum of minimal left ideals

$$I \;=\; \bigoplus_{j=1}^{l} I_j \tag{1}$$

and which moreover yields a generating idempotent e of I and a system of orthogonal primitive idempotents e_j corresponding to this decomposition (1):

$$e \;=\; \sum_{j=1}^{l} e_j \quad , \quad e_j \in I_j \quad , \quad I = \mathbb{C}[\mathcal{S}_r] \cdot e \quad . \tag{2}$$

We have described already the basic ideas of this algorithm in [6]. We complete the version of the algorithm from [6] by some considerations about possibilities of the improvement of the efficiency of the algorithm. Such possibilities are the construction of minimal sets of test permutations which are used by our algorithm (section 4) and the application of fast Fourier transforms for symmetric groups (section 5) realized by the fast algorithms of Clausen and Baum [4, 3].

In section 3 we consider left ideals for which a generating element is not known, for instance intersections of left ideals and non-direct sums of left ideals. The algorithm can be used to construct a generating idempotent for such an ideal. Of course, a version of the algorithm for right ideals can be formed, too.

A special reason to develop our decomposition algorithm comes for us from the computer algebra of tensor expressions which aims to do conversions of tensor expressions according to the rules of the Ricci calculus by means of a computer algebra system. Symbolic calculations with tensor expressions require effective methods of the determination of normal forms of tensors and of the investigation of tensor symmetries.

The connection between tensors and the representation theory of symmetric groups is well-known [17, 10, 1, 2]. A very informative paper is [7] which demonstrates the application of important tools of the representation theory such as the Littlewood-Richardson rule and plethysms to the construction of normal form expressions for polynomial terms in the coordinates of the Riemannian curvature tensor and its covariant derivatives.

We have discussed several aspects of the use of our decomposition algorithm in the computer algebra of tensor expressions in [6]. Some essential points of these considerations are summarized in section 6.

We have realized a Mathematica package called PERMS [5] which contains among other things tools for the decomposition algorithm.

2 Decomposition of ideals with given generating elements into minimal ideals

In this section we give a short description of our decomposition algorithm. More details can be found in [6]. In particular, [6] presents a version of the algorithm which works in the group ring $\mathbb{C}[G]$ of a finite group G.

We consider the group ring $\mathbb{C}[\mathcal{S}_r]$ of a symmetric group[1] $\mathcal{S}_r$ over the field of complex numbers $\mathbb{C}$. Let $I := \mathbb{C}[\mathcal{S}_r] \cdot a$ be a left ideal of this group ring with known generating element $a \in \mathbb{C}[\mathcal{S}_r]$. We search for a decomposition of I in a direct sum of minimal left ideals I_j of $\mathbb{C}[\mathcal{S}_r]$

$$I \quad = \quad I_1 \oplus I_2 \oplus \ldots \oplus I_l \quad . \tag{3}$$

[1] We use the convention $(p \circ q) : i \mapsto (p \circ q)(i) := p(q(i))$ for the multiplication of permutations.

Furthermore, we want to determine a generating idempotent e of I and the decomposition of e in a sum of orthogonal primitive idempotents e_j which corresponds to the decomposition (3),

$$\begin{array}{l} e \;=\; e_1 + e_2 + \ldots + e_l \quad , \quad e_j \in I_j \quad , \quad I = \mathbb{C}[\mathcal{S}_r] \cdot e \\ e_j \cdot e_j = e_j \quad , \quad e_j \cdot e_k = 0 \quad (j \neq k) \quad . \end{array} \tag{4}$$

Our algorithm is based on the well-known fact that the group ring $\mathbb{C}[\mathcal{S}_r]$ can be decomposed into minimal left ideals by means of Young symmetrizers (see e.g. [1, Chapter IV/§4 and §6] or [9, vol. I/pp.73,74]).

Theorem 1 *Let $\mathcal{ST}_\lambda$ denote the set of all standard tableaux of a Young frame which is characterized by a partition $\lambda \vdash r$ of the natural number r as usual. Further, let y_t be the Young symmetrizer of a Young tableaux t belonging to r. Then the group ring $\mathbb{C}[\mathcal{S}_r]$ has the decomposition*

$$\mathbb{C}[\mathcal{S}_r] \;=\; \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \mathcal{ST}_\lambda} \mathbb{C}[\mathcal{S}_r] \cdot y_t \quad . \tag{5}$$

A Young symmetrizer y_t of a Young tableau t is defined by

$$y_t \;:=\; \sum_{q \in \mathcal{V}_t} \sum_{p \in \mathcal{H}_t} \chi(q)\, p \circ q \quad \in \mathbb{C}[\mathcal{S}_r] \quad . \tag{6}$$

Here, $\mathcal{H}_t$ and $\mathcal{V}_t$ are the groups of the horizontal and vertical permutations of the Young tableau t, respectively, and $\chi(q)$ denotes the signature of the permutation q.

Every Young symmetrizer y_t differs from a primitive idempotent e_t only by a factor $0 \neq \mu \in \mathbb{C}$, i.e. $e_t = \mu y_t$. A left ideal $\mathbb{C}[\mathcal{S}_r] \cdot y_t$, generated by a Young symmetrizer, is minimal[2]. However, the Young symmetrizers y_t of standard tableaux $t \in \mathcal{ST}_\lambda$ are not pairwise orthogonal in general ([10, p.76], [1, p.103]).

If we multiply equation (5) by the generating element a of the ideal I, we obtain

$$I \;=\; \mathbb{C}[\mathcal{S}_r] \cdot a \;=\; \sum_{\lambda \vdash r} \sum_{\substack{t \in \mathcal{ST}_\lambda \\ y_t \cdot a \neq 0}} \mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a \quad . \tag{7}$$

Unfortunately, the sums in (7) are no longer direct. Thus, the following problem arises. We have to delete left ideals $\mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a$ in (7) in such a way that we obtain a direct sum which results still in I. We will solve this problem.

First we notice (see [6])

Lemma 1 *If $y_t \cdot a \neq 0$, then the left ideals $I_t := \mathbb{C}[\mathcal{S}_r] \cdot y_t$ and $W_t := \mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a$ are equivalent that means there exists a vector space isomorphism $\phi : I_t \to W_t$ which fulfils $\phi(p \cdot f) = p \cdot \phi(f)$ for all $p \in \mathcal{S}_r, f \in I_t$. Since I_t is a minimal left ideal, W_t is minimal, too.*

---

[2]About Young symmetrizers and Young tableaux see for instance [2, 7, 8, 9, 10, 11, 12, 13, 16, 17].

Next we show a simple method to produce a generating idempotent of a left ideal $W_t = \mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a$.

Proposition 1 *Let $e \in \mathbb{C}[\mathcal{S}_r]$ be a primitive idempotent and let $a \in \mathbb{C}[\mathcal{S}_r]$ be a group ring element with $e \cdot a \neq 0$. Then there exists a permutation $p \in \mathcal{S}_r$ such that*

$$e \cdot a \cdot p \cdot e \quad \neq \quad 0 \,. \tag{8}$$

Moreover, the group ring element $b := p \cdot e \cdot a$ formed with this p is essentially idempotent and generates the left ideal $W = \mathbb{C}[\mathcal{S}_r] \cdot e \cdot a$.

Proof. The left ideal $W = \mathbb{C}[\mathcal{S}_r] \cdot e \cdot a$ possesses a generating idempotent f [1, p. 54] which can be written as $f = x \cdot e \cdot a$ with a certain $x \in \mathbb{C}[\mathcal{S}_r]$. Now, the relation

$$e \cdot a \cdot x \cdot e \quad \neq \quad 0 \tag{9}$$

follows from $0 \neq f = f \cdot f = x \cdot e \cdot a \cdot x \cdot e \cdot a$. But then a permutation $p \in \mathcal{S}_r$ has to exist which satisfies (9) with $x = p$ since otherwise the left-hand side of (9) would vanish for every $x \in \mathbb{C}[\mathcal{S}_r]$.

As e is a primitive idempotent, we get

$$e \cdot a \cdot p \cdot e = \mu\, e$$

with a complex number $\mu \in \mathbb{C}$ [1, p. 56] and $\mu \neq 0$ on account of (8). Consequently, $b := p \cdot e \cdot a$ is essentially idempotent, because

$$b \cdot b = p \cdot (e \cdot a \cdot p \cdot e) \cdot a = \mu\, b \,,$$

and b generates W since $\mathbb{C}[\mathcal{S}_r] \cdot p = \mathbb{C}[\mathcal{S}_r]$. $\square$

Remark. The assertion of proposition 1 remains true if we replace the permutation p by a group ring element h. If $h \in \mathbb{C}[\mathcal{S}_r]$ satisfies (8), then, obviously, $b := h \cdot e \cdot a$ is essentially idempotent. Furthermore, b generates the left ideal $W = \mathbb{C}[\mathcal{S}_r] \cdot e \cdot a$, too, since $W' := \mathbb{C}[\mathcal{S}_r] \cdot h \cdot e \cdot a$ is a non-vanishing left subideal of W which has to coincide with W because of the minimality of W. This version of an idempotent construction is used in section 5.

By proposition 1 it is possible to construct a generating idempotent for every minimal left ideal $\mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a$ in (7) with $y_t \cdot a \neq 0$.

The determination of the permutation p for the forming of the essentially idempotent element b can be done by a computer program which tests the validity of condition (8) for the finitely many group elements $p \in \mathcal{S}_r$ one after another. The search stops if the first $p \in \mathcal{S}_r$ is found which fulfils (8). We have realized such an algorithm in PERMS. Though symmetric groups have a very large cardinality

$|\mathcal{S}_r| = r!$ in general, all examples treated with this algorithm claim a small number of search steps to reach a permutation $p \in \mathcal{S}_r$ which satisfies (8).

In section 4 we give minimal sets of permutations which contain certainly a permutation satisfying (8). Furthermore, in section 5 we will see that the use of fast Fourier transforms makes the search for such permutations unnecessary.

Now we consider an orthogonalization problem. Let be given the direct sum of two left ideals

$$(\mathbb{C}[\mathcal{S}_r] \cdot e_1) \oplus (\mathbb{C}[\mathcal{S}_r] \cdot e_2)$$

with generating idempotents e_1, e_2 which are non-orthogonal. We search for new generating idempotents f_1, f_2 of these ideals which fulfil

$$f_1 \cdot f_2 \;=\; f_2 \cdot f_1 \;=\; 0\,.$$

To solve this problem the following characterization of the set of all generating idempotents of a given left ideal is very helpful.

Proposition 2 *Let G be a finite group and $e \in \mathbb{C}[G]$ be a generating idempotent of a left ideal $I = \mathbb{C}[G] \cdot e$. Then a group ring element $f \in \mathbb{C}[G]$ is a generating idempotent[3] of I if and only if there exists a group ring element $x \in \mathbb{C}[G]$ such that*

$$f \;=\; e - x \cdot e + e \cdot x \cdot e \quad . \tag{10}$$

Proof. First we show that every group ring element (10) is a generating idempotent of I. Since e is an idempotent we obtain $f \cdot e = f$. Further, $e \cdot f = e$ follows immediately from $-e \cdot x \cdot e + e \cdot e \cdot x \cdot e = 0$, i.e. f is a generating element of I. Now the idempotent property of f is verified by

$$f \cdot f = (e - x \cdot e + e \cdot x \cdot e) \cdot f = e - x \cdot e + e \cdot x \cdot e = f \quad .$$

On the other hand, every generating idempotent f of I can be represented in the form (10). From $f \in I$ there follows $f - e \in I$. Therefore we can write $f - e = -y \cdot e$ with a certain $y \in \mathbb{C}[G]$. Then $e \cdot f = e$ yields $e \cdot y \cdot e = 0$ such that $f = e - y \cdot e + e \cdot y \cdot e$ is correct. □

Corollary 1 [4] *Let $e \in \mathbb{C}[G]$ be an idempotent. Then the following assertions hold true for all $x \in \mathbb{C}[G]$:*

1. *$n := x \cdot e - e \cdot x \cdot e$ is nilpotent, i.e. $n \cdot n = 0$.*

[3] The idea to produce a new idempotent f from a given idempotent e in this way was taken out of [14, p. 137]. However, in [14] the forming of new idempotents is carried out only by means of group elements $x = g \in G$ of the underlying group G.

[4] This remarkable property is mentioned in [14, p. 138], too. According to [14], first Zalesskii becomes aware of it.

2. *$u := id - n$ is an invertible element or a unit of $\mathbb{C}[G]$ with the inverse $u^{-1} = id + n$, where id denotes the identity element of G.*

3. *The idempotent $f = e - x\cdot e + e\cdot x\cdot e$ in accordance with proposition 2 fulfils $f = u\cdot e\cdot u^{-1}$.*

Proof. Corollary 1 can be proved by straightforward calculations (see [6]). □

Since formula (10) describes the complete set of generating idempotents of a left ideal I we can search the idempotents f_1, f_2 of the orthogonalisation problem in the form of (10). The next proposition is the basis of our orthogonalization procedure.

Proposition 3 *Let $I = \mathbb{C}[\mathcal{S}_r]\cdot e$ and $\tilde{I} = \mathbb{C}[\mathcal{S}_r]\cdot\tilde{e}$ be two left ideals of $\mathbb{C}[\mathcal{S}_r]$, generated by the idempotents $e, \tilde{e}$. We assume that e is primitive, i.e. I is minimal. Further we require $e\cdot\tilde{e}\neq e$ such that I and $\tilde{I}$ form a direct sum $I\oplus\tilde{I}$ since $I\not\subseteq\tilde{I}$. Then there holds true:*

1. *A permutation $p\in\mathcal{S}_r$ can be found such that*

$$e\cdot(id-\tilde{e})\cdot p\cdot e \;\neq\; 0 \quad . \tag{11}$$

   Moreover, a complex number $\lambda\in\mathbb{C}$ belonging to that p is available such that $f := e - x\cdot e + e\cdot x\cdot e$ with $x := \lambda\,(id-\tilde{e})\cdot p$ is a generating idempotent of I which satisfies $\tilde{e}\cdot f = 0$.

2. *For a given idempotent f according to 1 a permutation $\tilde{p}\in\mathcal{S}_r$ exists such that*

$$f\cdot(id-\tilde{e})\cdot\tilde{p}\cdot f \;\neq\; 0 \quad . \tag{12}$$

   Besides, a complex number $\tilde{\lambda}\in\mathbb{C}$ can be chosen such that $\tilde{f} := \tilde{e} - \tilde{x}\cdot\tilde{e}$ with $\tilde{x} := \tilde{\lambda}\,(id-\tilde{e})\cdot\tilde{p}\cdot f$ is a generating idempotent of $\tilde{I}$ which fulfils $f\cdot\tilde{f} = \tilde{f}\cdot f = 0$.

Proof. From $e\cdot\tilde{e}\neq e$ we obtain $e\cdot(id-\tilde{e})\neq 0$. Then, proposition 1 yields the existence of a $p\in\mathcal{S}_r$ such that $e\cdot(id-\tilde{e})\cdot p\cdot e\neq 0$. Thus (11) is proved. Since e is primitive, a relation

$$e\cdot(id-\tilde{e})\cdot p\cdot e \;=\; \mu\, e \tag{13}$$

is valid with a complex number $\mu\in\mathbb{C}$ [1, p. 56], and $\mu\neq 0$ on account of (11). Now, if f is an idempotent according to statement 1 of proposition 3 which generates I by proposition 2, we get

$$\begin{aligned}\tilde{e}\cdot f &= \tilde{e}\cdot e - \tilde{e}\cdot x\cdot e + \tilde{e}\cdot e\cdot x\cdot e\\ &= \tilde{e}\cdot e + \lambda\mu\,\tilde{e}\cdot e\end{aligned}$$

by considering $\tilde{e} \cdot x = 0$ and (13). Then $\lambda = -1/\mu$ leads to $\tilde{e} \cdot f = 0$.

As f likewise generates I, there follows $f \cdot \tilde{e} \neq f$. Otherwise, there would be $I \subseteq \tilde{I}$ in contradiction to $e \cdot \tilde{e} \neq e$. Now the existence of a $\tilde{p} \in \mathcal{S}_r$ which satisfies (12) arises from the application of statement 1 of proposition 3 to the idempotents $f, \tilde{e}$. We change to a new idempotent $\tilde{f} := \tilde{e} - \tilde{x} \cdot \tilde{e} + \tilde{e} \cdot \tilde{x} \cdot \tilde{e}$ of $\tilde{I}$ with $\tilde{x} := \tilde{\lambda}\,(id - \tilde{e}) \cdot \tilde{p} \cdot f$. Then $\tilde{e} \cdot \tilde{x} = 0$ yields $\tilde{e} \cdot \tilde{x} \cdot \tilde{e} = 0$. Being a generating idempotent of the minimal left ideal I, f is primitive and (12) results consequently in $f \cdot (id - \tilde{e}) \cdot \tilde{p} \cdot f = \tilde{\mu}\, f$ with $0 \neq \tilde{\mu} \in \mathbb{C}$. Thus, we get for $\tilde{f} = \tilde{e} - \tilde{x} \cdot \tilde{e}$

$$f \cdot \tilde{f} = f \cdot \tilde{e} - \tilde{\lambda}\, f \cdot (id - \tilde{e}) \cdot \tilde{p} \cdot f \cdot \tilde{e} = (1 - \tilde{\lambda}\tilde{\mu})\, f \cdot \tilde{e} \quad ,$$

and the choice $\tilde{\lambda} = 1/\tilde{\mu}$ gives $f \cdot \tilde{f} = 0$. The relation $\tilde{f} \cdot f = 0$ follows simply from $\tilde{e} \cdot f = 0$. □

Now, we can describe our

DECOMPOSITION ALGORITHM (L) FOR LEFT IDEALS:

Let be given a left ideal $I = \mathbb{C}[\mathcal{S}_r] \cdot a$ of $\mathbb{C}[\mathcal{S}_r]$ with known generating element $a \in \mathbb{C}[\mathcal{S}_r]$. We start with formula (7), i.e.

$$I \;=\; \mathbb{C}[\mathcal{S}_r] \cdot a \;=\; \sum_{\lambda \vdash r} \sum_{\substack{t \in \mathcal{ST}_\lambda \\ y_t \cdot a \neq 0}} \mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a \quad . \tag{14}$$

Then we can carry out the following steps:

1. The first left ideal in the sum (14) is minimal. We denote it by I_1 and we can determine a generating idempotent e_1 of I_1 by means of proposition 1.

2. We search for the first minimal left ideal in (14) which is not contained in I_1 that means for which there holds true

$$y_t \cdot a \cdot e_1 \quad \neq \quad y_t \cdot a \quad . \tag{15}$$

   We denote it by I_2 and we construct a generating idempotent e_2 of I_2 by means of proposition 1. Furthermore we determine new generating idempotents $\hat{f}_1, f_2$ of I_1, I_2 according to proposition 3 which are orthogonal. Then we form the leftideal $\tilde{I}_2 := I_1 \oplus I_2$ for which $\tilde{f}_2 := \hat{f}_1 + f_2$ is a generating idempotent.

3. Now we search for the next minimal left ideal in (14) which is not contained in $\tilde{I}_2$ that means for which there holds true

$$y_t \cdot a \cdot \tilde{f}_2 \quad \neq \quad y_t \cdot a \quad . \tag{16}$$

   We denote it by I_3. We construct a generating idempotent e_3 of I_3 and pass over to new orthogonal idempotents $\hat{f}_2, f_3$ instead of $\tilde{f}_2, e_3$. This leads us to the left ideal $\tilde{I}_3 := \tilde{I}_2 \oplus I_3$ which has the generating idempotent $\tilde{f}_3 := \hat{f}_2 + f_3$.

4. We continue this procedure until we have investigated all left ideals in (14). The result is the searched decomposition
$$I \;=\; I_1 \oplus I_2 \oplus \ldots \oplus I_m \tag{17}$$
of I and a generating idempotent $\tilde{f}_m$ of I.

According to statement 2 of proposition 3 every idempotent $\hat{f}_k$ of the left ideal $\tilde{I}_k$ of the $(k+1)$-th step can be written as $\hat{f}_k = (id - x_k) \cdot \tilde{f}_k$ with $x_k \in \mathbb{C}[\mathcal{S}_r]$ which we have already determined to carry out the orthogonalization $(e_{k+1}, \tilde{f}_k) \mapsto (f_{k+1}, \hat{f}_k)$. Thus we can write

$$\begin{aligned}
\tilde{f}_m &= \hat{f}_{m-1} + f_m \\
&= (id - x_{m-1}) \cdot \tilde{f}_{m-1} + f_m \\
&= (id - x_{m-1}) \cdot (\hat{f}_{m-2} + f_{m-1}) + f_m \\
&= (id - x_{m-1}) \cdot (id - x_{m-2}) \cdot \tilde{f}_{m-2} + (id - x_{m-1}) \cdot f_{m-1} + f_m \\
&\vdots \\
&= \sum_{k=1}^{m-1} (id - x_{m-1}) \cdot (id - x_{m-2}) \cdot \ldots \cdot (id - x_k) \cdot f_k + f_m \,.
\end{aligned} \tag{18}$$

Formula (18) presents a decomposition $\tilde{f}_m = \sum_{k=1}^{m} h_k$ of $\tilde{f}_m$ the summands of which fulfil

$$\begin{aligned}
h_k := (id - x_{m-1}) \cdot (id - x_{m-2}) \cdot \ldots \cdot (id - x_k) \cdot f_k &\in I_k \\
h_m := f_m &\in I_m \,.
\end{aligned} \tag{19}$$

Therefore, $\tilde{f}_m = \sum_{k=1}^{m} h_k$ is the decomposition of $\tilde{f}_m$ corresponding to the direct sum $I = \bigoplus_{k=1}^{m} I_k$ and the h_k are orthogonal generating idempotents of the minimal left ideals I_k [1, p. 55].

Theorem 2 *Let $I = \mathbb{C}[\mathcal{S}_r] \cdot a$ be a left ideal of $\mathbb{C}[\mathcal{S}_r]$, generated by a given group ring element $a \in \mathbb{C}[\mathcal{S}_r]$, $a \neq 0$. Then the algorithm* (L) *produces a generating idempotent $e \in \mathbb{C}[\mathcal{S}_r]$ of I and a decomposition $e = h_1 + h_2 + \ldots + h_m$ of e into primitive pairwise orthogonal idempotents h_k which defines a decomposition $I = \bigoplus_{k=1}^{m} I_k$ of I into minimal left ideals $I_k = \mathbb{C}[\mathcal{S}_r] \cdot h_k$.*

Essentially, the correctness of theorem 2 follows from the description of algorithm (L). A detailed proof is given in [6, Theorem 4.1].

It is clear that there are still some possibilities to increase the efficiency of the algorithm (L). Since minimal left ideals $\mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a$ and $\mathbb{C}[\mathcal{S}_r] \cdot y_{t'} \cdot a$ automatically form a direct sum if the standard tableaux $t \in \mathcal{ST}_\lambda$ and $t \in \mathcal{ST}_{\lambda'}$ belong to different partitions $\lambda, \lambda' \vdash r$, we have to apply the algorithm only to sets of left ideals $\mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a$ which lie in the same two-sided ideal
$$I_\lambda \;:=\; \bigoplus_{t \in \mathcal{ST}_\lambda} \mathbb{C}[\mathcal{S}_r] \cdot y_t \quad , \tag{20}$$

characterized by a partition $\lambda \vdash r$ that means which belong to the same class of equivalent minimal left ideals.

Furthermore, the knowledge of multiplicities of equivalent minimal left ideals in the searched decomposition of I is very helpful. If for a fixed $\lambda \vdash r$ the algorithm has found a direct sum of minimal left ideals $\mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a$ with $t \in \mathcal{ST}_\lambda$, the number of which is equal to the known multiplicity, then the investigation of the remaining left ideals $\mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a$ of λ can be canceled. The calculation of such multiplicities is possible if the given left ideal $I = \mathbb{C}[\mathcal{S}_r] \cdot a$ can be identified with the representation space of a certain linear representation α of the symmetric group $\mathcal{S}_r$, the character of which is known. In particular, if α turns out to be an induced representation of an outer tensor product of representations or of a certain representation of a wreath product of certain symmetric groups, then the Littlewood-Richardson rule (see [10, pp. 94], [9, vol. I p. 84], [11, pp. 68] and [7]) or plethysms (see [9], [15, pp. 8]) are important tools to determine such multiplicities.

A further efficiency problem of the algorithm arises from the fact that the naive calculation of the product $a \cdot b$ of two group ring elements $a, b \in \mathbb{C}[\mathcal{S}_r]$ entails a high need of calculation time and computer memory, if we have a large r. The use of fast Fourier transforms shows a way out of this problem (section 5).

We finish this section with a remark on decompositions of rigth ideals.

Proposition 4 *Let $J := a \cdot \mathbb{C}[\mathcal{S}_r]$ be a right ideal of $\mathbb{C}[\mathcal{S}_r]$, generated by a given group ring element $a \in \mathbb{C}[\mathcal{S}_r]$, $a \neq 0$. Then a version* (R) *of the algorithm* (L) *can be stated which produces a generating idempotent $e \in \mathbb{C}[\mathcal{S}_r]$ of J and a decomposition $e = h_1 + h_2 + \ldots + h_m$ of e into primitive pairwise orthogonal idempotents h_k defining a decomposition $J = \bigoplus_{k=1}^m J_k$ of J into minimal right ideals $J_k = h_k \cdot \mathbb{C}[\mathcal{S}_r]$.*

Proof. The version (R) of the decomposition algorithm starts with the decomposition

$$\mathbb{C}[\mathcal{S}_r] \;=\; \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \mathcal{ST}_\lambda} y_t \cdot \mathbb{C}[\mathcal{S}_r] \tag{21}$$

of $\mathbb{C}[\mathcal{S}_r]$ into minimal right ideals $y_t \cdot \mathbb{C}[\mathcal{S}_r]$ which can be proved in the same way as (5) (see [9, vol. I/pp.73,74] and [1, footnote on p.105]). Then we obtaine a description of the version (R) from the description of the version (L) if we write all products of group ring elements, which occur in the description (L), in reverse order. Furthermore, we have to transform the given versions of lemma 1 and of the propositions 1, 2 and 3 for left ideals into statements on right ideals by inversion of the factor sequence in all appearing products. Then, the proofs of propositions 1, 2, 3 can be rewritten easily into proofs for right ideals. □

The involution

$$* : \mathbb{C}[\mathcal{S}_r] \to \mathbb{C}[\mathcal{S}_r] \quad , \quad a = \sum_{p\in\mathcal{S}_r} a(p)\, p \;\mapsto\; a^* := \sum_{p\in\mathcal{S}_r} a(p)\, p^{-1} \tag{22}$$

of $\mathbb{C}[\mathcal{S}_r]$ presents an other possibility to decompose right ideals by means of the algorithm (L). The involution $*$ maps (primitive) itempotents to (primitive) idempotents, direct sums of left (rigth) ideals to direct sums of rigth (left) ideals and preserves the minimality of ideals. Thus we can construct a decomposition of a given right ideal $J = a\cdot\mathbb{C}[\mathcal{S}_r]$ by passing over to the left ideal $I = J^* = \mathbb{C}[\mathcal{S}_r]\cdot a^*$, determining a generating Idempotent e and a decomposition

$$I \;=\; \mathbb{C}[\mathcal{S}_r]\cdot e \;=\; \bigoplus_{k=1}^{m} \mathbb{C}[\mathcal{S}_r]\cdot e_k \quad , \quad e \;=\; \sum_{k=1}^{m} e_k$$

of I by means of (L) and returning to the right ideal J,

$$J \;=\; I^* \;=\; e^*\cdot\mathbb{C}[\mathcal{S}_r] \;=\; \bigoplus_{k=1}^{m} e_k^*\cdot\mathbb{C}[\mathcal{S}_r] \quad , \quad e^* \;=\; \sum_{k=1}^{m} e_k^* \quad .$$

However, the transformation $*$ causes extra costs in time and memory. In particular, the use of two algorithms (L) and (R) is more advantageous in the case of applying fast Fourier transforms.

3 Determination of generating idempotents by means of the decomposition algorithm

In this section we discuss some cases of left (or right) ideals for which no generating elements are known, but for which the determination of generating idempotents can be done by the help of the decomposition algorithm.

At first, we give some lemmas on annihilator ideals of a group ring $\mathbb{C}[G]$ of a finite group G. We denote by $Z_r(I)$ the right annihilator ideal and by $Z_l(I)$ the left annihilator ideal of an ideal I of $\mathbb{C}[G]$,

$$\begin{aligned} Z_r(I) &:= \{\, h \in \mathbb{C}[G] \mid \forall\, f \in I : f\cdot h = 0 \,\} \\ Z_l(I) &:= \{\, h \in \mathbb{C}[G] \mid \forall\, f \in I : h\cdot f = 0 \,\} \quad . \end{aligned}$$

Lemma 2 *Let I and J be a left ideal and a right ideal of $\mathbb{C}[G]$, respectively. Then there holds true*

$$J \;=\; Z_r(I) \quad \Longleftrightarrow \quad I \;=\; Z_l(J) \quad . \tag{23}$$

Proof. Let $J = Z_r(I)$. The ideal I has a generating idempotent e. Then $Z_r(I) = J$ is generated[5] by the idempotent $id - e$ and further $Z_l(J)$ is generated by $id - (id - e) = e$, i.e. $Z_l(J) = I$. The reverse case follows in the same way. □

Lemma 3 *Let I be a left ideal of $\mathbb{C}[G]$ and $a \in \mathbb{C}[G]$ be a group ring element with $a \neq 0$. Then there holds true:*

1. *There is $I = \mathbb{C}[G] \cdot a$ if and only if $Z_r(I) = \{h \in \mathbb{C}[G] \mid a \cdot h = 0\}$.*
2. *There is $I = \{f \in \mathbb{C}[G] \mid f \cdot a = 0\}$ if and only if $Z_r(I) = a \cdot \mathbb{C}[G]$.*

Proof. We prove statement 1. Let I satisfy $I = \mathbb{C}[G] \cdot a$. We use the notation $K := \{h \in \mathbb{C}[G] \mid a \cdot h = 0\}$. Obviously, there is $Z_r(I) \subseteq K$ since $a \cdot Z_r(I) = 0$. If e is a generating idempotent of I, then $id - e$ generates $Z_r(I)$. Now we assume that there exists a group ring element $h \in e \cdot \mathbb{C}[G]$, $h \neq 0$, with $a \cdot h = 0$. But since e can be written as $e = x \cdot a$ with a certain $x \in \mathbb{C}[G]$, we would obtain the contradiction $x \cdot a \cdot h = h \neq 0$. Consequently, there is $K = Z_r(I)$.

Conversely, if $Z_r(I)$ fulfils $Z_r(I) = \{h \in \mathbb{C}[G] \mid a \cdot h = 0\}$, then there follows from the just drawn conclusions that $Z_r(\mathbb{C}[G] \cdot a) = \{h \in \mathbb{C}[G] \mid a \cdot h = 0\} = Z_r(I)$. Thus lemma 2 yields $I = \mathbb{C}[G] \cdot a$.

Statement 2 can be proved in the same way as statement 1. □

Lemma 4 *Let $I_1, I_2, \ldots, I_m \subseteq \mathbb{C}[G]$ be left ideals of $\mathbb{C}[G]$. Then there holds true*

$$Z_r\Big(\bigcap_{k=1}^{m} I_k\Big) \;=\; \sum_{k=1}^{m} Z_r(I_k) \quad . \tag{24}$$

Proof. Instead of (24), we show the equivalent relation

$$\bigcap_{k=1}^{m} I_k \;=\; Z_l\Big(\sum_{k=1}^{m} Z_r(I_k)\Big) \quad . \tag{25}$$

The inclusion $\bigcap_{k=1}^{m} I_k \supseteq Z_l(\sum_{k=1}^{m} Z_r(I_k))$ follows from the fact that every $f \in Z_l(\sum_{k=1}^{m} Z_r(I_k))$ fulfils $f \cdot h_k = 0$ for all $h_k \in Z_r(I_k)$ and all $k = 1, \ldots, m$, i.e. $f \in Z_l(Z_r(I_k)) = I_k$ for all $k = 1, \ldots, m$.

Conversely, every $f \in \bigcap_{k=1}^{m} I_k$ satisfies $f \cdot h = f \cdot h_1 + \ldots + f \cdot h_m = 0$ for all $h = h_1 + \ldots + h_m \in \sum_{k=1}^{m} Z_r(I_k)$, $h_k \in Z_r(I_k)$. This leads to $\bigcap_{k=1}^{m} I_k \subseteq Z_l(\sum_{k=1}^{m} Z_r(I_k))$. □

Proposition 5 *Let $a \in \mathbb{C}[\mathcal{S}_r]$, $a \neq 0$, be a group ring element of $\mathbb{C}[\mathcal{S}_r]$ which annihilates a given left ideal $I \subseteq \mathbb{C}[\mathcal{S}_r]$, i.e. $I = \{f \in \mathbb{C}[\mathcal{S}_r] \mid f \cdot a = 0\}$. Then we can construct a generating idempotent of I by means of the decomposition algorithm* (R).

[5] On the one hand, every $h \in \mathbb{C}[G]$ can be written as $h = e \cdot h + (id - e) \cdot h$ such that there follows $h = (id - e) \cdot h$ for $h \in Z_r(I)$ since $e \cdot h = 0$. On the other hand, every group ring element $(id - e) \cdot y$ lies in $Z_r(I)$, because $x \cdot e \cdot (id - e) \cdot y = 0$ for every $x \cdot e \in I$.

Proof. From lemma 3 there follows that $Z_r(I) = a \cdot \mathbb{C}[\mathcal{S}_r]$. Thus we can use algorithm (R) to determine a generating idempotent e of $Z_r(I)$ from which we obtain the generating idempotent $id - e$ of I. $\square$

Proposition 6 *Let $I_1, I_2, \ldots, I_m \subseteq \mathbb{C}[\mathcal{S}_r]$ be a finite set of left ideals of the group ring $\mathbb{C}[\mathcal{S}_r]$. We assume that every left ideal I_k is defined either by a generating element $a_k \in \mathbb{C}[\mathcal{S}_r]$, i.e. $I_k = \mathbb{C}[\mathcal{S}_r] \cdot a_k$, or by an annihilating element $n_k \in \mathbb{C}[\mathcal{S}_r]$, i.e. $I_k = \{f \in \mathbb{C}[\mathcal{S}_r] \mid f \cdot n_k = 0\}$. Then we can construct generating idempotents of the left ideals $\sum_{k=1}^{m} I_k$ and $\bigcap_{k=1}^{m} I_k$ by means of the algorithms* (L) *or* (R).

Proof. At first we consider the (non-direct) sum $J = \sum_{k=1}^{m} I_k$. We can assume that we know a generating element a_k for every left ideal I_k because we can determine generating idempotents e_l according to proposition 5 for all such left ideals I_l which are characterized only by annihilating elements n_l.

By using equation (5) we can write

$$J \;=\; \sum_{\lambda \vdash r} \sum_{k=1}^{m} \sum_{\substack{t \in \mathcal{ST}_\lambda \\ y_t \cdot a_k \neq 0}} \mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a_k \quad . \tag{26}$$

Now we apply our decomposition algorithm (L) to the set $\mathcal{I}$ of all minimal left ideals $\mathbb{C}[\mathcal{S}_r] \cdot y_t \cdot a_k$ in (26), to select such a subset $\tilde{\mathcal{I}}$ of $\mathcal{I}$ that J is the direct sum of the ideals in $\tilde{\mathcal{I}}$. The occurrence of more than one generating element a_k does not disturb the effectiveness of the algorithm. Obviously, algorithm (L) yields a generating idempotent e of J in this case, too.

To process the intersection $K = \bigcap_{k=1}^{m} I_k$, we determine generating idempotents e_l by means of algorithm (L) for all such left ideals I_l which are described by a generating element a_l. Then $id - e_l$ is an annihilating element of $I_l = \mathbb{C}[\mathcal{S}_r] \cdot e_l$. Thus we can assume that all left ideals I_k are characterized by annihilating elements n_k. Taking into account lemma 3 and 4 we obtain

$$Z_r(K) \;=\; \sum_{k=1}^{m} Z_r(I_k) \;=\; \sum_{k=1}^{m} n_k \cdot \mathbb{C}[\mathcal{S}_r] \quad .$$

Now the application of algorithm (R) to $\sum_{k=1}^{m} n_k \cdot \mathbb{C}[\mathcal{S}_r]$ produces a generating idempotent e of $Z_r(K)$ wich leads to the generating idempotent $id - e$ of K. $\square$

4 Minimal sets of test permutations

In this section we will determine minimal sets of permutations in which we can find certainly such a permutation that a condition of type (8), (11) or (12) is fulfilled.

Proposition 7 *Let us consider the algorithm* (L) *which is based on the decomposition* (5) *of the group ring* $\mathbb{C}[\mathcal{S}_r]$. *Then every of the conditions* (8), (11) *and* (12) *can be reduced to a condition*

$$y_t \cdot w \cdot p \cdot y_t \quad \neq \quad 0 \tag{27}$$

with a certain Young symmetrizer y_t of a standard tableau t and a certain group ring element $w \in \mathbb{C}[\mathcal{S}_r]$, satisfying $y_t \cdot w \neq 0$, in the course of every step of the algorithm.

Proof. Let us consider the k-th step of the algorithm (L). The first time we have to use condition (8), if we have found a group ring element $y_{t_k} \cdot a \neq 0$ such that $y_{t_k} \cdot a \cdot \tilde{f}_{k-1} \neq y_{t_k} \cdot a$ with the resulting idempotent $\tilde{f}_{k-1}$ of the (k-1)-th step. Then we have to construct an idempotent from $y_{t_k} \cdot a$ by means of proposition 1. But in this case condition (8) takes a form (27), i.e.

$$y_{t_k} \cdot a \cdot p \cdot y_{t_k} \quad \neq \quad 0 \quad . \tag{28}$$

The idempotent produced from $y_{t_k} \cdot a$ according to proposition 1 is $e_k = \alpha_k \, p_k \cdot y_{t_k} \cdot a$ with a constant $\alpha_k \in \mathbb{C}$ and a permutation $p_k \in \mathcal{S}_r$ fulfilling (28).

Next we have to determine an idempotent f_k from e_k according to statement 1 of proposition 3 such that $\tilde{f}_{k-1} \cdot f_k = 0$. Let us consider the condition

$$y_{t_k} \cdot a \cdot (id - \tilde{f}_{k-1}) \cdot s \cdot y_{t_k} \quad \neq \quad 0 \tag{29}$$

of the type (27). The permutation $s \in \mathcal{S}_r$ is searched. From $y_{t_k} \cdot a \cdot \tilde{f}_{k-1} \neq y_{t_k} \cdot a$ there follows $y_{t_k} \cdot a \cdot (id - \tilde{f}_{k-1}) \neq 0$. Thus proposition 1 yields the existence of a permutation $s_k \in \mathcal{S}_r$ which satisfies (29). Since y_{t_k} differs from a primitive idempotent only by a constant factor the relation

$$y_{t_k} \cdot a \cdot (id - \tilde{f}_{k-1}) \cdot s_k \cdot y_{t_k} \quad = \quad \kappa \, y_{t_k} \quad , \quad \kappa = \text{const.} \neq 0 \tag{30}$$

arises from (29). Multiplying (30) by p_k and a from the left and the right, respectively, we obtain (11),

$$e_k \cdot (id - \tilde{f}_{k-1}) \cdot s_k \cdot p_k^{-1} \cdot e_k \;=\; \alpha_k \, \kappa \, e_k \;\neq\; 0 \quad ,$$

fulfilled by the permutation $s_k \cdot p_k^{-1}$. Now we can form the idempotent f_k. Because of corollary 1 there holds true $f_k = u_k \cdot e_k \cdot u_k^{-1}$ with a certain unit $u_k \in \mathbb{C}[\mathcal{S}_r]$ which we can calculate according to proposition 2 and corollary 1.

At last, we have to construct an idempotent $\hat{f}_{k-1}$ from $\tilde{f}_{k-1}$ according to statement 2 of proposition 3 such that $f_k \cdot \hat{f}_{k-1} = \hat{f}_{k-1} \cdot f_k = 0$. To this end, we search for a permutation $\hat{s}_k \in \mathcal{S}_r$ which satisfies the condition

$$y_{t_k} \cdot a \cdot u_k^{-1} \cdot (id - \tilde{f}_{k-1}) \cdot s \cdot y_{t_k} \quad \neq \quad 0 \tag{31}$$

of the type (27). From $f_k \cdot \tilde{f}_{k-1} \neq f_k$ there follows $y_{t_k} \cdot a \cdot u_k^{-1} \cdot (id - \tilde{f}_{k-1}) \neq 0$ such that proposition 1 guarantees the existence of such a permutation $\hat{s}_k$. Then relation (31) leads to

$$y_{t_k} \cdot a \cdot u_k^{-1} \cdot (id - \tilde{f}_{k-1}) \cdot \hat{s}_k \cdot y_{t_k} \;=\; \hat{\kappa}\, y_{t_k} \quad , \quad \hat{\kappa} = \text{const.} \neq 0 \quad . \tag{32}$$

Multiplying (32) by $u_k \cdot p_k$ and $a \cdot u_k^{-1}$ from the left and the right, respectively, we obtain (12),

$$f_k \cdot (id - \tilde{f}_{k-1}) \cdot h_k \cdot f_k \;=\; \alpha_k\, \hat{\kappa}\, f_k \;\neq\; 0 \quad ,$$

satisfied by the group ring element $h_k := \hat{s}_k \cdot p_k^{-1} \cdot u_k^{-1}$. Now we can form an idempotent $\hat{f}_{k-1} := \tilde{f}_{k-1} - \hat{x} \cdot \tilde{f}_{k-1}$ with $\hat{x} := \hat{\lambda}\,(id - \tilde{f}_{k-1}) \cdot h_k \cdot f_k$. The permutation $\tilde{p}$, used in proposition 3, is replaced by h_k. But the proof of statement 2 of proposition 3 works in this case, too. □

Now we present a set of permutations for a given relation (27) which is essentially smaller than $\mathcal{S}_r$ but contains definitely a permutation p satisfying (27).

Proposition 8 *Let $t_0 \in \mathcal{ST}_\lambda$ be a fixed standard tableau of a given partition $\lambda \vdash r$ of a natural number $r \in \mathbb{N}$ and let y_{t_0} be the Young symmetrizer of t_0. We denote by $\mathcal{P}_{t_0}$ the set*

$$\mathcal{P}_{t_0} \;\;:=\;\; \{s \in \mathcal{S}_r \mid s \circ t_0 \in \mathcal{ST}_\lambda\} \tag{33}$$

of all such permutations $s \in \mathcal{S}_r$ which transform t_0 to the rest of the standard tableaux of λ. Then, for every $w \in \mathbb{C}[\mathcal{S}_r]$ with $y_{t_0} \cdot w \neq 0$ we can find a permutation $s_0 \in \mathcal{P}_{t_0}$ such that

$$y_{t_0} \cdot w \cdot s_0 \cdot y_{t_0} \;\;\neq\;\; 0 \quad . \tag{34}$$

Proof. Since $y_{t_0} \cdot w \neq 0$, the left ideal $W_{t_0} := \mathbb{C}[\mathcal{S}_r] \cdot y_{t_0} \cdot w$ is equivalent to the minimal left ideal $I_{t_0} := \mathbb{C}[\mathcal{S}_r] \cdot y_{t_0}$ (lemma 1). Therefore, W_{t_0} is a subideal of the two-sided ideal

$$I_\lambda \;\;:=\;\; \bigoplus_{t \in \mathcal{ST}_\lambda} \mathbb{C}[\mathcal{S}_r] \cdot y_t \tag{35}$$

which contains all ideals of the equivalence class of minimal left ideals of $\mathbb{C}[\mathcal{S}_r]$, characterized by λ. Thus $y_{t_0} \cdot w \in I_\lambda$ can be written as

$$y_{t_0} \cdot w \;\;=\;\; \sum_{t \in \mathcal{ST}_\lambda} x_t \cdot y_t \quad , \quad x_t \in \mathbb{C}[\mathcal{S}_r] \quad , \tag{36}$$

with suitable group ring elements x_t. At least one of the summands of (36) does not vanish.

We use the usual dictionary order for Young tableaux of the same partition $\lambda \vdash r$ [9, vol. I, p. 73]. A tableau t_2 of λ is regarded as greater than a tableau t_1 of λ, if the simultaneous run through the rows of both tableaux from left to right and from top to bottom reaches earlier in t_2 a number which is greater than the number on the corresponding place in t_1.

Now, let $t_1 \in \mathcal{ST}_\lambda$ be the smallest standard tableau of λ such that the summand of t_1 in (36) fulfils $x_{t_1} \cdot y_{t_1} \neq 0$. Since all Young symmetrizers y_t of standard tableaux $t \in \mathcal{ST}_\lambda$ with $t > t_1$ satisfy $y_t \cdot y_{t_1} = 0$ and, on the other hand, the symmetrizer y_{t_1} is essentially idempotent, i.e. $y_{t_1} \cdot y_{t_1} = \mu\, y_{t_1}$, $\mu = \text{const.} \neq 0$, the multiplication of (36) by y_{t_1} from the right yields

$$y_{t_0} \cdot w \cdot y_{t_1} \;\;=\;\; \mu\, x_{t_1} \cdot y_{t_1} \;\neq\; 0 \quad . \tag{37}$$

Let $s_0 \in \mathcal{P}_{t_0}$ be that permutation which transforms t_0 into t_1, i.e. $t_1 = s_0 \circ t_0$. Then there holds true $y_{t_1} = s_0 \cdot y_{t_0} \cdot s_0^{-1}$ and we obtain from (37) the relation

$$y_{t_0} \cdot w \cdot s_0 \cdot y_{t_0} \cdot s_0^{-1} \;\neq\; 0$$

which leads to (34). □

The cardinality of $\mathcal{P}_{t_0}$ amounts to $|\mathcal{P}_{t_0}| = |\mathcal{ST}_\lambda| \ll |\mathcal{S}_r| = r!$. Now we will show that this value represents already the minimum.

Proposition 9 *Let y_{t_0} be the Young symmetrizer of an arbitrary Young tableau t_0 of a given partition $\lambda \vdash r$. Further, let $\mathcal{P} \subseteq \mathcal{S}_r$ be such a subset of permutations from $\mathcal{S}_r$ that for every group ring element $w \in \mathbb{C}[\mathcal{S}_r]$ with $y_{t_0} \cdot w \neq 0$ there exists a permutation $s \in \mathcal{P}$ which satisfies*

$$y_{t_0} \cdot w \cdot s \cdot y_{t_0} \;\;\neq\;\; 0 \quad . \tag{38}$$

Then there holds true

$$|\mathcal{P}| \;\;\geq\;\; |\mathcal{ST}_\lambda| \quad . \tag{39}$$

Proof. We assume $|\mathcal{P}| < |\mathcal{ST}_\lambda|$. Let us denote by $\mathcal{Y}_\mathcal{P}$ the set

$$\mathcal{Y}_\mathcal{P} \;:=\; \{y_t \mid y_t \text{ Young symmetrizer of } \; t = s \circ t_0 \;,\; s \in \mathcal{P}\}$$

of the Young symmetrizers of all those Young tableaux t which are generated from t_0 by all permutations of $\mathcal{P}$. The property of $\mathcal{P}$, required in proposition 9, has the consequence that for every $w \in \mathbb{C}[\mathcal{S}_r]$ with $y_{t_0} \cdot w \neq 0$ there exists a Young symmetrizer $y_t \in \mathcal{Y}_\mathcal{P}$ such that

$$y_{t_0} \cdot w \cdot y_t \;\;\neq\;\; 0 \quad . \tag{40}$$

If the permutation $s_0 \in \mathcal{P}$ satisfies (38) for a given w then the Young symmetrizer $y_{t_1} \in \mathcal{Y}_\mathcal{P}$ of the Young tableau $t_1 := s_0 \circ t_0$ fulfils $y_{t_1} = s_0 \cdot y_{t_0} \cdot s_0^{-1}$ and, consequently, there follows $y_{t_0} \cdot w \cdot y_{t_1} \neq 0$ from (38).

Let us consider the right ideal

$$J_{\mathcal{P}} \;:=\; \sum_{y_t \in \mathcal{Y}_{\mathcal{P}}} y_t \cdot \mathbb{C}[\mathcal{S}_r] \tag{41}$$

to which we can apply the algorithm (R) according to proposition 6. Since the Young symmetrizers y_t are proportional to primitive idempotents, the right ideals $y_t \cdot \mathbb{C}[\mathcal{S}_r]$ are minimal and algorithm (R) produces a subset $\mathcal{Y}' \subseteq \mathcal{Y}_{\mathcal{P}}$ of $\mathcal{Y}_{\mathcal{P}}$ such that

$$J_{\mathcal{P}} \;:=\; \bigoplus_{y_t \in \mathcal{Y}'} y_t \cdot \mathbb{C}[\mathcal{S}_r] \quad . \tag{42}$$

The right ideal $J_{\mathcal{P}}$ possesses a generating idempotent e which decomposes into primitive pairwise orthogonal idempotents e_{y_t} according to (42),

$$e \;=\; \sum_{y_t \in \mathcal{Y}'} e_{y_t} \qquad , \qquad e_{y_t} \in y_t \cdot \mathbb{C}[\mathcal{S}_r] \quad .$$

Now we form the left ideal $I_{\mathcal{P}}$ generated by e

$$I_{\mathcal{P}} \;:=\; \mathbb{C}[\mathcal{S}_r] \cdot e \;=\; \bigoplus_{y_t \in \mathcal{Y}'} \mathbb{C}[\mathcal{S}_r] \cdot e_{y_t} \quad . \tag{43}$$

Because all Young tableaux t in (41) belong to the same partition $\lambda \vdash r$, all right ideals $y_t \cdot \mathbb{C}[\mathcal{S}_r]$ in (41) are equivalent to the right ideal $y_{t_0} \cdot \mathbb{C}[\mathcal{S}_r]$. Therefore, for every $y_t \in \mathcal{Y}'$ there exists a non-vanishing group ring element $e_{y_t} \cdot x \cdot y_{t_0} \neq 0$, $x \in \mathbb{C}[\mathcal{S}_r]$, which describes the equivalence mapping $y_{t_0} \cdot \mathbb{C}[\mathcal{S}_r] \rightarrow e_{y_t} \cdot \mathbb{C}[\mathcal{S}_r]$ [1, p.56]. On the other hand, the existence of the elements $e_{y_t} \cdot x \cdot y_{t_0} \neq 0$ guarantees that every left ideal $\mathbb{C}[\mathcal{S}_r] \cdot e_{y_t}$ in (43) is equivalent to the left ideal $I_{t_0} := \mathbb{C}[\mathcal{S}_r] \cdot y_{t_0}$. Thus $I_{\mathcal{P}}$ is a subideal of the two-sided ideal I_λ (35) belonging to the given partition $\lambda \vdash r$. The dimension of $I_{\mathcal{P}}$ is smaller than the dimension of I_λ since

$$\dim I_{\mathcal{P}} \;=\; |\mathcal{Y}'| \dim I_{t_0} \;\leq\; |\mathcal{P}| \dim I_{t_0} \;<\; |\mathcal{ST}_\lambda| \dim I_{t_0} \;=\; (\dim I_{t_0})^2 \;=\; \dim I_\lambda \quad .$$

Now we consider the left ideal K which is annihilated by e, i.e.

$$K \;:=\; \{ f \in \mathbb{C}[\mathcal{S}_r] \mid f \cdot e = 0 \} \quad .$$

Because there holds true $\mathbb{C}[\mathcal{S}_r] = I_{\mathcal{P}} \oplus K$ and $\dim I_{\mathcal{P}} < \dim I_\lambda$, the left ideal K has to obtain a minimal left ideal $\check{I}$ which belongs to the equivalence class of minimal left ideals of λ.

Let $\check{e}$ be a generating idempotent of $\check{I}$. Every Young symmetrizer $y_t \in \mathcal{Y}_{\mathcal{P}}$ lies in $J_{\mathcal{P}}$ and can be written, consequently, as $y_t = e \cdot y_t$. Since furthermore there is $\check{e} \cdot e = 0$, we obtain

$$\forall\, y_t \in \mathcal{Y}_{\mathcal{P}} : \quad \check{e} \cdot y_t = 0 \quad .$$

On the other hand, the left ideals $\check{I} = \mathbb{C}[\mathcal{S}_r] \cdot \check{e}$ and $I_{t_0} = \mathbb{C}[\mathcal{S}_r] \cdot y_{t_0}$ are equivalent and the equivalence mapping $I_{t_0} \to \check{I}$ is described by a non-vanishing group ring element $y_{t_0} \cdot x \cdot \check{e} \neq 0$, $x \in \mathbb{C}[\mathcal{S}_r]$. Now, if we choose $w := x \cdot \check{e}$, then there holds true $y_{t_0} \cdot w \neq 0$, but

$$\forall\, y_t \in \mathcal{Y}_{\mathcal{P}} \ : \quad y_{t_0} \cdot w \cdot y_t = 0$$

in contradiction to (40). $\square$

In the next section we will see that the search process for permutations is canceled if we use fast Fourier transforms. But, in the case of small symmetric groups $\mathcal{S}_r$ with approximately $r \leq 7$, the algorithms (L) and (R) can be carried out with reasonable costs immediate in the group ring $\mathbb{C}[\mathcal{S}_r]$. Then the results of section 4 are important.

5 Application of fast Fourier transforms

As mentioned in section 2, the naive calculation of the product $a \cdot b$ of two group ring elements $a, b \in \mathbb{C}[\mathcal{S}_r]$ entails high costs in calculation time and computer memory, if we have a large r. This reduces the efficiency of the algorithms (L) and (R). Now we discuss some ideas of the use of fast Fourier transforms which show a way out of this problem.

The basis of the concept of discrete Fourier transforms for arbitrary finite groups is *Wedderburn's Theorem* which we give here only in the special form for symmetric groups $\mathcal{S}_r$ (see e.g. [1, p.61] or [3, p.38]).

Theorem 3 *For every partition $\lambda \vdash r$ the two-sided ideal $I_\lambda = \bigoplus_{t \in \mathcal{ST}_\lambda} \mathbb{C}[\mathcal{S}_r] \cdot y_t$ of the group ring $\mathbb{C}[\mathcal{S}_r]$ is isomorphic to a full algebra of complex $(d_\lambda \times d_\lambda)$-matrices:*

$$I_\lambda \ \simeq \ \mathbb{C}^{d_\lambda \times d_\lambda} \quad , \tag{44}$$

where d_λ denotes the dimension[6] of any minimal left ideal from the equivalence class of λ. Furthermore, the group ring $\mathbb{C}[\mathcal{S}_r]$ is isomorphic to an algebra of block diagonal matrices:

$$\mathbb{C}[\mathcal{S}_r] \ = \ \bigoplus_{\lambda \vdash r} I_\lambda \ \simeq \ \bigoplus_{\lambda \vdash r} \mathbb{C}^{d_\lambda \times d_\lambda} \quad . \tag{45}$$

Definition 1 Every isomorphism $D : \mathbb{C}[\mathcal{S}_r] \to \bigoplus_{\lambda \vdash r} \mathbb{C}^{d_\lambda \times d_\lambda}$ of $\mathbb{C}$-algebras is called a *discrete Fourier transform* for $\mathbb{C}[\mathcal{S}_r]$ (or simply, for $\mathcal{S}_r$).

[6] d_λ can be calculated from λ by means of the hook length formula (see e.g. [1, p.101], [9, p.81] or [7]).

We denote by D^λ the natural projections $D^\lambda : \mathbb{C}[\mathcal{S}_r] \rightarrow \mathbb{C}^{d_\lambda \times d_\lambda}$ of D which form a complete list of pairwise inequivalent irreducible representations of $\mathbb{C}[\mathcal{S}_r]$. Then D can be written as

$$D : a \;\mapsto\; A \;=\; \mathrm{diag}\Big(A^{(\lambda)}\,;\;\lambda \vdash r\Big) \;=\; \begin{pmatrix} A^{(\lambda_1)} & & \\ & \ddots & \\ & & A^{(\lambda_m)} \end{pmatrix}$$
$$a \in \mathbb{C}[\mathcal{S}_r]\;,\; A^{(\lambda)} = D^\lambda(a) \in \mathbb{C}^{d_\lambda \times d_\lambda} \quad .$$

Clausen and Baum have shown that the isomporphism $\sigma : \mathbb{C}[\mathcal{S}_r] \rightarrow \bigoplus_{\lambda\vdash r} \mathbb{C}^{d_\lambda \times d_\lambda}$, called *Young's seminormal form*, represents a discrete Fourier transform of $\mathcal{S}_r$ which can be handled by fast algorithms. In [3, chapter 9] they present efficient algorithms for the evaluation of σ and σ^{-1} (see also [4]). Upper bounds of the number of arithmetic operations for these evaluations are

$$\begin{array}{lcl} \sigma & : & \Big(\dfrac{5}{12}\,r^3 \;+\; \dfrac{1}{2}\,r^2 \;-\; \dfrac{11}{12}\,r\Big)\; r! \\ \sigma^{-1} & : & \Big(\dfrac{5}{12}\,r^3 \;+\; \dfrac{4}{3}\,\sqrt{2}\,r^{3/2} \;+\; \dfrac{7}{12}\,r\Big)\; r! \quad . \end{array}$$

Descriptions of Young's seminormal form can be found e.g in [2], [9, pp.75,76], [8], [3], [4].

Now it is clear that the use of the Fourier transform σ can essentially improve the efficiency of our algorithms (L) and (R) in the case of large symmetric groups $\mathcal{S}_r$. Instead of a group ring product $a \cdot b$ we can calculate $A \cdot B$ with $A = \sigma(a)$ and $B = \sigma(b)$ which can be done by isolated calculations of the block products $A^{(\lambda)} \cdot B^{(\lambda)}$ for all $\lambda \vdash r$. If these blocks are large matrices we could employ algorithms for fast matrix multiplication.

Since σ is an isomorphism of $\mathbb{C}$-algebras, the complete decomposition algorithms (L) and (R) can be carried out in the space $\sigma(\mathbb{C}[\mathcal{S}_r])$ of the Fourier transforms. We have to determine Fourier transforms only for the given input data, i.e. generating or annihilating elements of given ideals, and to calculate inverse Fourier transforms only for the results of the algorithms. Again, all calculations within the scope of these algorithms can be done blockwise.

There are two points on which the decomposition algorithms for $\sigma(\mathbb{C}[\mathcal{S}_r])$ should be modified.

(i) It is not necessary, to use Fourier transforms $\sigma(y_t)$ of Young symmetrizers y_t. We can replace the primitive idempotents $\sigma(y_t)$ by other natural primitive idempotents $Z_{(k)}$ of $\mathbb{C}^{d_\lambda \times d_\lambda}$.

Proposition 10 *Let $Z_{(k)} \in \mathbb{C}^{d_\lambda \times d_\lambda}$, $1 \leq k \leq d_\lambda$, be the matrix the elements $z_{(k),ij}$ of which fulfil*

$$z_{(k),ij} \;=\; \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{else} \end{cases} \quad .$$

Then the matrices $Z_{(1)}, \ldots, Z_{(d_\lambda)}$ form a set of primitive pairwise orthogonal idempotents, generating minimal left or right ideals, and $\mathbb{C}^{d_\lambda \times d_\lambda}$ decomposes into

$$\mathbb{C}^{d_\lambda \times d_\lambda} \;=\; \bigoplus_{k=1}^{d_\lambda} \mathbb{C}^{d_\lambda \times d_\lambda} \cdot Z_{(k)} \;=\; \bigoplus_{k=1}^{d_\lambda} Z_{(k)} \cdot \mathbb{C}^{d_\lambda \times d_\lambda} \quad . \tag{46}$$

Proof. The proof, that the $Z_{(k)}$ form a set of pairwise orthogonal idempotents, is trivial. For every matrix $A \in \mathbb{C}^{d_\lambda \times d_\lambda}$ the product $A \cdot Z_{(k)}$ (or $Z_{(k)} \cdot A$) is a matrix, the k-th column (row) of which is equal to the k-th column (row) of A, whereas all other elements vanish. Consequently, the left ideal $\mathcal{I}_k := \mathbb{C}^{d_\lambda \times d_\lambda} \cdot Z_{(k)}$ and the right ideal $\mathcal{J}_k := Z_{(k)} \cdot \mathbb{C}^{d_\lambda \times d_\lambda}$ have both the dimension d_λ such that they have to be minimal since $\sigma^{-1}(\mathcal{I}_k)$ and $\sigma^{-1}(\mathcal{J}_k)$ turn out to be minimal in $\mathbb{C}[\mathcal{S}_r]$ because of their dimensions. Relation (46) follows simply from the column structure or the row structure of the matrices $A \cdot Z_{(k)}$ or $Z_{(k)} \cdot A$, respectively. □

If we use the idempotents $Z_{(k)}$ instead of $\sigma(y_t)$ in every matrix algebra $\mathbb{C}^{d_\lambda \times d_\lambda}$, then the transfers of the propositions 1 and 3 and of the algorithms (L) and (R) to $\sigma(\mathbb{C}[\mathcal{S}_r])$ keep their validity.

(ii) The search process for permutations p which satisfy inequalities as (8), (11) and (12) can be replaced by a simple search of non-vanishing elements in matrices.

Proposition 11 *Let $E \in \mathbb{C}^{d \times d}$ be a primitive idempotent of $\mathbb{C}^{d \times d}$, $d \geq 2$, and let $W \in \mathbb{C}^{d \times d}$ be a matrix with $E \cdot W \neq 0$. Search for a non-vanishing element*[7] *in each of the matrices E and $E \cdot W$. If such non-vanishing elements have been found in the j_0-th column of $E \cdot W$ and the k_0-th row of E, then the matrix $P = (p_{jk})$ with $p_{j_0 k_0} = 1$ and $p_{jk} = 0$ else fulfils*

$$E \cdot W \cdot P \cdot E \;\neq\; 0 \quad .$$

Thus we can determine matrices P which satisfy the transfers of the conditions (8), (11) and (12) to $\sigma(\mathbb{C}[\mathcal{S}_r])$ by means of proposition 11. We remark that, in general, $\sigma^{-1}(P)$ is not a permutation but some group ring element of $\mathbb{C}[\mathcal{S}_r]$. However, the idempotent constructions of the propositions 1 and 3 remain correct if we use group ring elements instead of the original permutations p satisfying (8), (11) or (12). (See also the remark after the proof of proposition 1.)

Our considerations have the remarkable consequence that the algorithms (L) and (R) can be regarded as algorithms for the decomposition of left or right ideals of arbitrary matrix algebras $\mathbb{C}^{d \times d}$, independent of their connection with group rings. This statement is supported by the fact that for every natural number $d \geq 2$ the hook length formula yields a dimension $d_\lambda = d$ for the partition $\lambda = (d\,1) \vdash d+1$, i.e. $\mathbb{C}^{d \times d}$ is isomorphic to the two-sided ideal $I_\lambda \subset \mathbb{C}[\mathcal{S}_{d+1}]$ belonging to that partition λ.

[7]Such non-vanishing elements exist because of $E \cdot W \neq 0$.

Proposition 12 *For every matrix algebra $\mathbb{C}^{d\times d}$, $d \geq 2$, the transfers of the algorithms* (L) *and* (R) *to $\mathbb{C}^{d\times d}$ can be used to decompose left or right ideals of $\mathbb{C}^{d\times d}$, corresponding to all types of ideals from sections 2 and 3, into minimal (left or right) ideals and to determine generating idempotents of these ideals, decomposed in primitive pairwise orthogonal idempotents according to the ideal decomposition.*

Example.[8] We determine a decomposition of the left ideal $I = \mathbb{C}^{3\times 3} \cdot A$ given by the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} .$$

Let $Z_{(1)}, Z_{(2)}, Z_{(3)}$ be the idempotents of $\mathbb{C}^{3\times 3}$ described in proposition 10. Since $Z_{(1)} \cdot A \neq 0$ we obtain $I_1 := \mathbb{C}^{3\times 3} \cdot Z_{(1)} \cdot A$ as first minimal subideal of I. Further there is $Z_{(1)} \cdot A \cdot Z_{(1)} \cdot Z_{(1)} \neq 0$ such that proposition 1 yields as generating idempotent of I_1

$$E_1 = Z_{(1)} \cdot Z_{(1)} \cdot A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Next we check $Z_{(2)} \cdot A \neq 0$ and $Z_{(2)} \cdot A \cdot E_1 \neq Z_{(2)} \cdot A$ such that the next minimal subideal of I is $I_2 := \mathbb{C}^{3\times 3} \cdot Z_{(2)} \cdot A$. I_1 and I_2 form a direct sum $\tilde{I}_2 = I_1 \oplus I_2 \subseteq I$. There is $Z_{(2)} \cdot A \cdot Z_{(2)} \cdot Z_{(2)} \neq 0$ such that the construction of proposition 1 gives

$$E_2 = \frac{1}{5} Z_{(2)} \cdot Z_{(2)} \cdot A = \begin{pmatrix} 0 & 0 & 0 \\ 4/5 & 1 & 6/5 \\ 0 & 0 & 0 \end{pmatrix}$$

as a generating idempotent of I_2.

Now we have to orthogonalize E_1 and E_2. Taking into account

$$Id - E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -4/5 & 0 & -6/5 \\ 0 & 0 & 1 \end{pmatrix} ,$$

we can check $E_1 \cdot (Id - E_2) \cdot Z_{(1)} \cdot E_1 \neq 0$. Thus we can make an ansatz $F_1 = E_1 - X \cdot E_1 + E_1 \cdot X \cdot E_1$ with $X = \lambda\,(Id - E_2) \cdot Z_{(1)}$ for a new idempotent F_1 of I_1 according to proposition 3. The condition $E_2 \cdot F_1 = 0$ leads to $\lambda = 5/3$ and

$$F_1 = \begin{pmatrix} -5/3 & -10/3 & -5 \\ 4/3 & 8/3 & 4 \\ 0 & 0 & 0 \end{pmatrix} .$$

Next we obtain

$$F_1 \cdot (Id - E_2) = \begin{pmatrix} 1 & 0 & -1 \\ -4/5 & 0 & 4/5 \\ 0 & 0 & 0 \end{pmatrix}$$

and $F_1 \cdot (Id - E_2) \cdot Z_{(1)} \cdot F_1 \neq 0$ such that we can use the ansatz $F_2 = E_2 - X \cdot E_2$ with $X = \lambda\,(Id - E_2) \cdot Z_{(1)} \cdot F_1$. Now the condition $F_1 \cdot F_2 = 0$ yields $\lambda = 1$ and

$$F_2 = \begin{pmatrix} 8/3 & 10/3 & 4 \\ -4/3 & -5/3 & -2 \\ 0 & 0 & 0 \end{pmatrix} .$$

[8] All calculations of this example have been carried out by means of Mathematica 2.2 [18].

Then $\tilde{I}_2$ has the generating idempotent

$$\tilde{F}_2 \;=\; F_1 + F_2 \;=\; \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad .$$

Since $Z_{(3)} \cdot A \neq 0$ but $Z_{(3)} \cdot A \cdot \tilde{F}_2 = Z_{(3)} \cdot A$ we finish with $I = \tilde{I}_2 = \mathbb{C}^{3\times 3} \cdot \tilde{F}_2$.

6 Application of the decomposition algorithms to investigations of tensors

In this section we summarize shortly ideas from [6] which refer to applications of our algorithms to investigations of symmetry properties of tensors and to problems of reduction of tensor expressions arising in symbolic computer calculations.

We make use of the following connection between tensors and elements of the group ring of a symmetric group. Let $\mathcal{T}_r V$ be the space of all complex-valued covariant tensors of order r on a vector space V over the field of complex numbers $\mathbb{C}$. The tensors $T \in \mathcal{T}_r V$ are multilinear mappings of the r-fold cartesian product of V onto $\mathbb{C}$,

$$T : \underbrace{V \times V \times ... \times V}_{\text{r factors}} \to \mathbb{C} \quad , \quad (v_1, \ldots, v_r) \mapsto T(v_1, \ldots, v_r) \;\; .$$

Let $T \in \mathcal{T}_r V$ be a covariant tensor of order r and let $b := \{v_1, \ldots, v_r\} \subset V$ be an arbitrary subset of r vectors from V. Then T and b induce a complex-valued function T_b on the symmetric group $\mathcal{S}_r$

$$T_b : \mathcal{S}_r \to \mathbb{C} \quad , \quad T_b : p \mapsto T_b(p) := T(v_{p(1)}, \ldots, v_{p(r)})$$

which we will identify with the group ring element $\sum_{p\in\mathcal{S}_r} T_b(p)\, p$ denoted by T_b, too. The action of a group ring element $a = \sum_{p\in\mathcal{S}_r} a(p)p \in \mathbb{C}[\mathcal{S}_r]$ on a tensor T is defined by

$$a : T \mapsto aT \quad , \quad (aT)_{i_1\ldots i_r} := \sum_{p\in\mathcal{S}_r} a(p)\, T_{i_{p(1)}\ldots i_{p(r)}} \quad .$$

Using the involution $*$ from (22) one can show by a straightforward calculation (see [6])

$$(aT)_b \;=\; T_b \cdot a^* \quad . \tag{47}$$

Now we consider tensors which possess a symmetry relating to permutations of their indices.

Definition 2 We call a pair (C, ϵ) a *tensor symmetry*, if $C \subseteq \mathcal{S}_r$ is a subgroup of the symmetric group $\mathcal{S}_r$ and $\epsilon : C \to S^1$ is a homomorphism of C onto a finite subgroup of the group of the unimodular numbers $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. We say that a tensor $T \in \mathcal{T}_r V$ possesses the symmetry (C, ϵ), if

$$\forall c \in C \quad : \quad cT = \epsilon(c)\, T \quad . \tag{48}$$

The T_b of a tensor with a tensor symmetry lie in a special left ideal.

Proposition 13 *Let (C, ϵ), $C \subseteq \mathcal{S}_r$, be a tensor symmetry. Then the group ring element*

$$\epsilon \;:=\; \sum_{c \in C} \epsilon(c)\, c \quad \in \mathbb{C}[\mathcal{S}_r] \;\; , \tag{49}$$

is essentially idempotent. Furthermore, every T_b of a covariant tensor $T \in \mathcal{T}_r V$ with the symmetry (C, ϵ) is contained in the left ideal $I := \mathbb{C}[\mathcal{S}_r] \cdot \epsilon$ of $\mathbb{C}[\mathcal{S}_r]$.

Proof. There holds true $\epsilon(c)\, \epsilon \cdot c = \epsilon$ for all $c \in C$ since

$$\epsilon(c)\, \epsilon \cdot c = \sum_{c' \in C} \epsilon(c)\epsilon(c')\, c' \cdot c = \sum_{c'' \in C} \epsilon(c)\epsilon(c'' \cdot c^{-1})\, c'' = \sum_{c'' \in C} \epsilon(c'')c'' = \epsilon.$$

Thus we obtain $\epsilon \cdot \epsilon = \sum_{c \in C} \epsilon(c)\, \epsilon \cdot c = |C|\, \epsilon$, i.e. ϵ is essentially idempotent.

Because of (47) equation (48) turns into $T_b \cdot c^{-1} = \epsilon(c)\, T_b$ or $T_b = \epsilon(c)\, T_b \cdot c$ for every vector set $b = \{v_1, \ldots, v_r\} \subset V$. Then the sum of these relations over all $c \in C$ yields $|C|\, T_b = T_b \cdot \epsilon$, i.e. T_b lies in the left ideal $\mathbb{C}[\mathcal{S}_r] \cdot \epsilon$. □

Now, let us consider tensors T which satisfy a certain system of linear identities.

Proposition 14 *Let $T \in \mathcal{T}_r V$ be a covariant tensor which fulfils a system of m linear identities*

$$u_j T \;=\; 0 \quad , \quad j = 1, \ldots, m \quad , \tag{50}$$

given by m group ring elements $u_1, \ldots, u_m \in \mathbb{C}[\mathcal{S}_r]$. Then every T_b of T lies in the intersection $J = \bigcap_{k=1}^{m} J_k$ of those left ideals J_k of $\mathbb{C}[\mathcal{S}_r]$ which are annihilated by the group ring elements u_k^*, i.e. $J_k := \{f \in \mathbb{C}[\mathcal{S}_r] \mid f \cdot u_k^* = 0\}$.

Proof. On account of (47) the relations (50) are equivalent to

$$\forall b = \{v_1, v_2, \ldots, v_r\} \subset V : \qquad T_b \cdot u_j^* = 0 \quad , \quad j = 1, 2, ..., m \quad . \; \Box$$

The summary of propositions 13 and 14 reads: If $T \in \mathcal{T}_r V$ is a tensor which possesses a tensor symmetry and/or satisfies linear identities, then all T_b of T lie in the intersection

$$I \;=\; \bigcap_{j=1}^{m} I_j$$

of certain left ideals $I_j \subseteq \mathbb{C}[\mathcal{S}_r]$ which are given by generating or annihilating elements. Using proposition 6 we can construct a generating idempotent $e = e_1 + \ldots + e_n$ of the intersection I and a decomposition of e into primitive pairwise orthogonal idempotents e_k. Then this idempotent e decomposes every T_b of T since we can write

$$\forall\, b = \{v_1, \ldots, v_r\} \subset V : \quad T_b \;=\; T_b \cdot e \;=\; T_b \cdot e_1 + \ldots + T_b \cdot e_n \quad .$$

This relation is equivalent to

$$T \;=\; e_1^* T + \ldots + e_n^* T \quad . \tag{51}$$

Formula (51) is a decomposition of the tensor T into parts belonging to so-called symmetry classes[9] which are generated by the idempotents e_k^* (see [1, p.127]).

The knowledge of a decomposition (51) of a tensor T can be very helpful in solving reduction problems which occur with symbolic tensor calculations by a computer. Let us consider a tensor expression τ which is a complex linear combination of certain isomers of a tensor $T \in \mathcal{T}_r V$,

$$\tau_{i_1 \ldots i_r} \;=\; \sum_{p \in P} \beta_p\, T_{i_{p(1)} \ldots i_{p(r)}} \quad , \quad \beta_p \in \mathbb{C} \;\; , \quad P \subseteq \mathcal{S}_r \quad , \tag{52}$$

where the sum runs over a subset P of the symmetric group $\mathcal{S}_r$. We assume that all T_b, belonging to T, lie in a left ideal $I := \mathbb{C}[\mathcal{S}_r] \cdot a$ with known generating element a. Then there arises the problem to determine linear dependences between the terms $T_{i_{p(1)} \ldots i_{p(r)}}$ and to reduce (52) to a linear combination of linearly independent terms.

Lemma 5 *A relation* (52) *exists between* $\tau, T \in \mathcal{T}_r V$ *if and only if there holds true*

$$\forall\, b = \{v_1, \ldots, v_r\} \subset V : \quad \tau_b(id) \;=\; \sum_{p \in P} \beta_p\, T_b(p) \quad . \tag{53}$$

Proof. (52) is equivalent to

$$\forall\, b = \{v_1, \ldots, v_r\} \subset V : \quad \tau_{i_1 \ldots i_r}\, v_1^{i_1} \ldots v_r^{i_r} \;=\; \sum_{p \in P} \beta_p\, T_{i_{p(1)} \ldots i_{p(r)}}\, v_1^{i_1} \ldots v_r^{i_r}$$

which can be written as (53). □

A set of complex numbers $\{x_p \mid p \in \mathcal{S}_r\}$ determines a linear identity for all elements of the left ideal I if

$$\forall\, f \in I : \quad \sum_{p \in \mathcal{S}_r} x_p f(p) \;=\; 0 \; . \tag{54}$$

[9] Let K be a right ideal of $\mathbb{C}[\mathcal{S}_r]$. Then the symmetry class of tensors from $\mathcal{T}_r V$ characterized by K is the subspace $\mathcal{S}_K := \{fT \mid f \in K\,,\, T \in \mathcal{T}_r V\}$ of $\mathcal{T}_r V$. Every generating idempotent e of K generates the symmetry class $\mathcal{S}_K$, i.e. $\mathcal{S}_K = \{eT \mid T \in \mathcal{T}_r V\}$.

If we know a non-trivial identity (54) with $x_p = 0$ for all $p \in \mathcal{S}_r \setminus P$, we can use it to eliminate a term $T_b(p)$ in (53). This leads to reduced variants of (53), (52)

$$\tau_b(id) = \sum_{p\in\tilde{P}} \tilde{\beta}_p \, T_b(p) \quad , \quad \tau_{i_1\ldots i_r} = \sum_{p\in\tilde{P}} \tilde{\beta}_p \, T_{i_{p(1)}\ldots i_{p(r)}} \quad , \quad \tilde{P} \subset P \, .$$

Since every $f \in I = \mathbb{C}[\mathcal{S}_r] \cdot a$ can be written as $f = g \cdot a = \sum_{p,p'\in\mathcal{S}_r} g(p)a(p')p \cdot p'$ with a $g \in \mathbb{C}[\mathcal{S}_r]$, we obtain from (54)

$$\forall g \in \mathbb{C}[\mathcal{S}_r] : \quad \sum_{p\in\mathcal{S}_r} \Big(\sum_{p'\in\mathcal{S}_r} a(p^{-1} \cdot p')x_{p'} \Big) g(p) = 0 \, .$$

From this relation there follows the homogeneous linear equation system

$$\sum_{p'\in\mathcal{S}_r} a(p^{-1} \cdot p')x_{p'} \;=\; 0 \quad , \quad p \in \mathcal{S}_r \tag{55}$$

for the numbers x_p that describe the linear identities of I. The coefficient matrix $A := [a(p^{-1} \cdot p')]_{p,p'\in\mathcal{S}_r}$ of (55) is a $\mathcal{S}_r$-circulant of $a \in \mathbb{C}[\mathcal{S}_r]$ (see [3, p.141]).

The set $\{p \cdot a \mid p \in \mathcal{S}_r\}$ generates the left ideal $I = \mathbb{C}[\mathcal{S}_r] \cdot a$ and can be reduced to a basis of I. Because

$$p \cdot a \;=\; \sum_{p'\in\mathcal{S}_r} a(p')p \cdot p' \;=\; \sum_{p''\in\mathcal{S}_r} a(p^{-1} \cdot p'')p'' \, ,$$

we see that

$$\operatorname{rank} A \;=\; \dim I \, . \tag{56}$$

Now, if a decomposition (51) of T is known we can write (53) as

$$\tau_b(id) \;=\; \sum_{k=1}^{n} \sum_{p\in P} \beta_p \, (e_k^* T)_b(p) \tag{57}$$

and reduce the subsums $\sum_{p\in P} \beta_p \, (e_k^* T)_b(p)$ separately. The $(e_k^* T)_b$ lie in the minimal left ideals $I_k := \mathbb{C}[\mathcal{S}_r] \cdot e_k$ since $(e_k^* T)_b = T_b \cdot e_k$. Then the equation system of type (55) for the reduction identities belonging to an ideal I_k possesses the circulant E_k of e_k as coefficient matrix. But $\operatorname{rank} E_k = \dim I_k$ is much smaller than $\operatorname{rank} A = \dim I$ such that we have better conditions to solve (55).

Further details of the handling of a system (55) can be found in [6]. In particular, a method is described in [6] which allows to determine a set of linearly independent rows of a system (55) without using the Gaussian algorithm.

References

[1] H. Boerner. *Darstellungen von Gruppen*, volume 74 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955.

[2] H. Boerner. *Representations of Groups.* North-Holland Publishing Company, Amsterdam, 2. revised edition, 1970.

[3] M. Clausen and U. Baum. *Fast Fourier Transforms.* BI Wissenschaftsverlag, Mannheim, Leipzig, Wien, Zürich, 1993.

[4] M. Clausen and U. Baum. Fast Fourier transforms for symmetric groups. In L. Finkelstein and W.M. Kantor, editors, *Groups and Computation*, volume 11 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 27 – 39, Providence, Rhode Island, 1993. DIMACS, American Mathematical Society. The DIMACS workshop on Groups and Computation was held at DIMACS, Rutgers University, Oct. 7 – 10, 1991.

[5] B. Fiedler. *PERMS 2.1.* Mathematisches Institut, Universität Leipzig, Leipzig, 1997. Will be sent in to MathSource, Wolfram Research Inc.

[6] B. Fiedler. A use of ideal decomposition in the computer algebra of tensor expressions. *Z. f. Analysis u. ihre Anw.*, 16(1):145 – 164, 1997.

[7] S.A. Fulling, R.C. King, B.G. Wybourne, and C.J. Cummins. Normal forms for tensor polynomials: I. The Riemann tensor. *Class. Quantum Grav.*, 9:1151 – 1197, 1992.

[8] G.D. James and A. Kerber. *The Representation Theory of the Symmetric Group*, volume 16 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley Publishing Company, Reading, Mass., London, Amsterdam, Don Mills, Ont., Sidney, Tokyo, 1981.

[9] A. Kerber. *Representations of Permutation Groups*, volume 240, 495 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1971, 1975.

[10] D.E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups.* Clarendon Press, Oxford, 2. edition, 1950.

[11] I.G. Macdonald. *Symmetric Functions and Hall Polynomials.* Clarendon Press, Oxford, 1979.

[12] W. Müller. *Darstellungstheorie von endlichen Gruppen.* Teubner Studienbücher Mathematik. B. G. Teubner, Stuttgart, 1980.

[13] M.A. Naimark and A.I. Štern. *Theory of Group Representations*, volume 246 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[14] D.S. Passman. *The Algebraic Structure of Group Rings.* Pure and Applied Mathematics. John Wiley & Sons, New York, London, Sydney, Toronto, 1977.

[15] F. Sänger. Plethysmen von irreduziblen Darstellungen symmetrischer Gruppen. Dissertation, Rheinisch-Westfälische Technische Hochschule, Mathematisch-Naturwissenschaftliche Fakultät, Aachen, 1980.

[16] B.L. van der Waerden. *Algebra*, volume I, II. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 9., 6. edition, 1993.

[17] H. Weyl. *The Classical Groups, their Invariants and Representations.* Princeton University Press, Princeton, New Jersey, 1939.

[18] S. Wolfram. *Mathematica.* Addison-Wesley, Bonn, München, Paris, Reading, Mass., 2. edition, 1992. In German.