Cacti, Braids and Complex Polynomials

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Abstract

The study of the topological classification of complex polynomials began in the XIX-th century by Luroth (1871), Clebsch (1873) and Hurwitz (1891). In the works of Zdravkovska [23] and Khovanskii and Zdravkovska [17] the problem is reduced to a purely combinatorial one, that of the study of a certain action of the braid groups on a class of tree-like figures that we, following [14], call "cacti".

Using explicit computation of the braid group orbits, enumerative results of [14], and also establishing some combinatorial invariants of the action, we provide the topological classification of polynomials of degree up to 9 (previous results were known up to degree 6).

Résumé

L'étude de la classification topologique des polynômes complexes a commencé au XIX-ème siecle par Luroth (1871), Clebsch (1873) et Hurwitz (1891). Dans les travaux de Zdravkovska [23] et Khovanskii et Zdravkovska [17] le problème est réduit à une étude purement combinatoire d'une certaine action du groupe des tresses sur un genre de figures arborescentes que nous appelons, d'après [14], les "cactus".

En utilisant le calcul explicite des orbites du groupe des tresses, les résultats énumératifs de [14], et aussi en mettant en évidence certains invariants combinatoires de l'action, nous achevons la classification topologique des polynômes de degré jusqu'à 9 (une telle classification a été connue jusqu'au degré 6).

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1 Introduction

Theory of polynomials is a source of numerous interesting combinatorial problems. For example, the classification of real polynomials, their critical values being real and different, leads to the study of alternating permutations, and thus to the Euler numbers, Bernoulli numbers, Genocchi numbers, Euler–Bernoulli triangle etc. [4]. In this case the number of critical values attains its maximum ($n-1$ critical values for a polynomial of degree n). On the other extreme, the study of complex polynomials with the minimal number of critical values (at most two: such polynomials are called *Shabat polynomials* or *generalized Chebyshev polynomials*) leads to the combinatorics of plane trees, and is also related to Galois theory and group theory [21].

In the present paper we study the intermediate case of polynomials of degree n with a certain number k of critical values, which will usually (but not always) be greater than 2 and smaller than $n-1$. Namely, we will study the classification of complex polynomials with respect to the *topological equivalence*. Let $P_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and $P_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be two polynomials of degree n, where $\overline{\mathbb{C}} \simeq S^2$ is the Riemann complex sphere. We say that P_1 and P_2 are *topologically equivalent* if there exist two orientation preserving homeomorphisms $h_1 : S^2 \to S^2$ and $h_2 : S^2 \to S^2$ such that the following diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{h_1} & S^2 \\ \Big\downarrow P_1 & & \Big\downarrow P_2 \\ S^2 & \xrightarrow{h_2} & S^2 \end{array}$$

is commutative.

The study of polynomials and, more generally, branched covers of S^2, up to topological equivalence began in the XIX-th century by Luroth [18] and Clebsch [8]. It was continued in the XX-th century by Thom [22], Arnold [3] and many others. We would specifically like to attract the reader's attention to a recent paper [17], which was for us the main source of information and references on the problem.

A complex polynomial of degree n with k critical values may be graphically represented by what is called "picture" in [23], and "cactus" in [14]. A *cactus* is a tree-like figure glued of n polygons with k sides each (see details in Section 2). Zdravkovska [23] proved that the classes of topologically equivalent polynomials are in one to one correspondence with the orbits of a certain action of the braid group B_k on the set of cacti. In fact, the action itself was introduced by Hurwitz as early as in 1891 [15]. Of course, Hurwitz did not use the term "braid group", because this group was invented only 34 years later [5].

To a certain extent the problem becomes purely technical: compute the orbits of the above action. But an orbit, once computed, is practically useless if we are unable to say whether it does or does not contain all the cacti of the needed type. Hence the enumerative formula of [14] becomes very important: together with the computer program which computes the orbits, these two tools permit us to finish with all the cases where there is only one orbit.

In the case of two or more orbits we look for combinatorial invariants which might explain the reasons for splitting. These are given in Section 5. We were able to produce the topological classification of polynomials of degree ≤ 9 (such a classification was previously known for degree ≤ 6, see [17]). Our results may be summarized as follows:

- The typical case is that of a single orbit.
- For all the examples with more than one orbit we give explicit combinatorial reasons for this. (Only for "a half of a case" the explanation remains somewhat insufficient: see Example 5.12.)
- The main reason why polynomials form a separate orbit is the possibility to represent them as compositions of polynomials of smaller degrees.

2 Cacti

A building block for constructing a cactus is a polygon with k sides whose vertices are "colored" with colors $1, 2, \ldots, k$ in the counterclockwise direction. (The number k will later correspond to the number of critical values of a polynomial.) Another graphical image for a building block is a star-tree, whose leaves are colored in the same way by $1, 2, \ldots, k$: see Figure 1. The center of the star is a point inside the polygon, and its rays go to the polygon vertices.

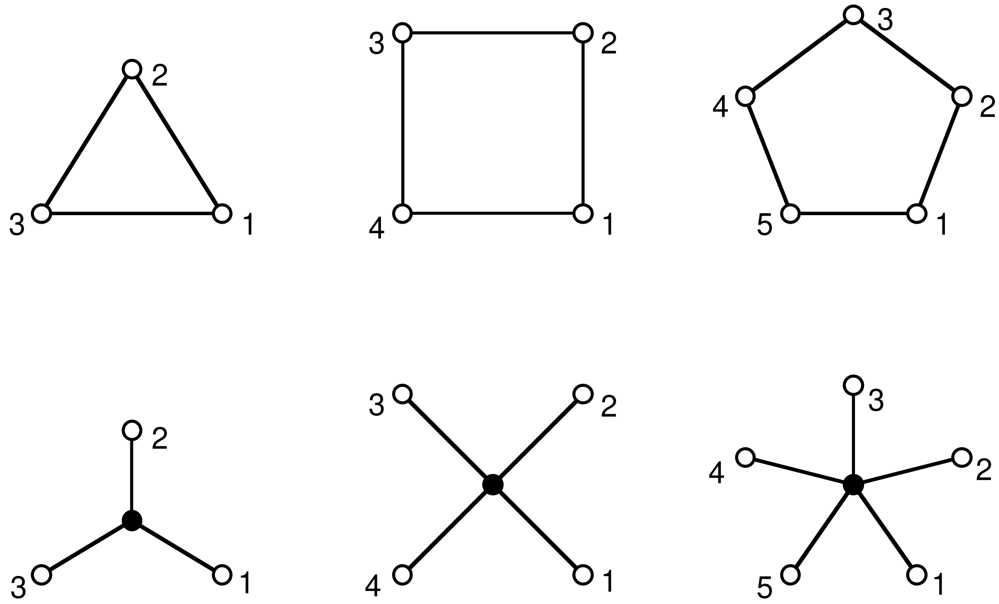

Figure 1: Building blocks for a cactus

A *cactus* of degree n is a tree-like figure glued of n copies of the polygon (all the n polygons have the same number k of sides). "Tree-like" means that there is no cycles other than the polygons themselves. If we glue a cactus not of polygons but of stars, what we get is a plane tree, but not an arbitrary one: all its black vertices have the same degree k (while colored vertices may be of arbitrary degrees). In Figure 2 the same cactus is represented in two graphical forms. If we superimpose both images, each black vertex will be inside a triangle, and the colored vertices will coincide with one another. We will use one of the two ways of drawing cacti, whichever will fit the best.

The cyclic order of branches around each vertex is important: changing this order we may obtain another cactus. Therefore it is convenient to code cacti by permutations. Let all the polygons of a cactus be marked by numbers $1, 2, \ldots, n$. Let the permutation g_i, $i = 1, 2, \ldots, k$ act on the set of the n polygons in the following way: it sends a polygon to the "next" one in the counterclockwise direction around its vertex of color i. For example, if we mark the triangles of the cactus of Figure 2 as is shown on Figure 3, we get the permutations

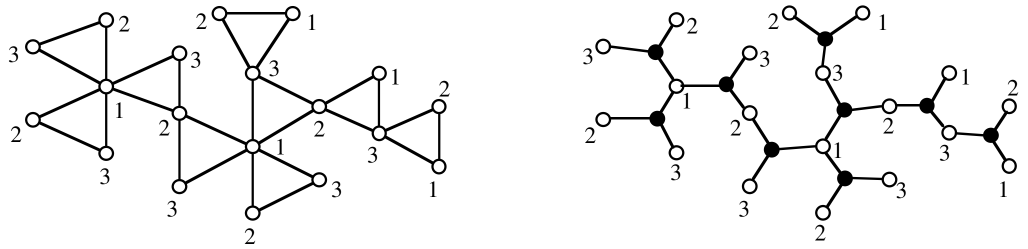

Figure 2: A cactus of degree $n = 9$ ($k = 3$)

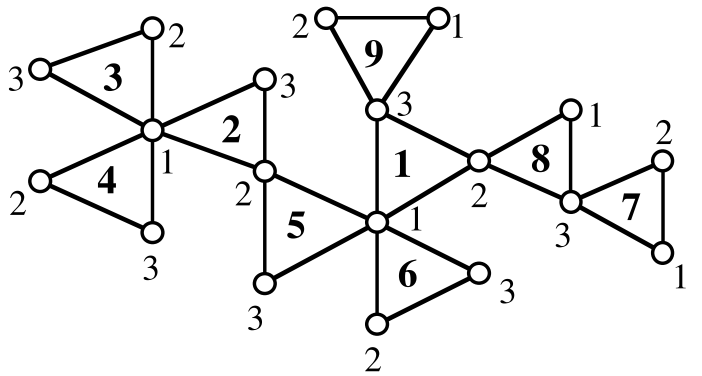

Figure 3: A marking of the above cactus

$$\begin{aligned} g_1 &= (1,5,6)(2,3,4), \\ g_2 &= (1,8)(2,5), \\ g_3 &= (1,9)(7,8). \end{aligned}$$

Remark 2.1 The product $\gamma = \prod_{i=1}^{k} g_i$ is a circular permutation. In a way, it encodes a circular path around the whole figure. In the above example

$$\gamma = g_1 g_2 g_3 = (1,2,3,4,5,6,7,8,9).$$

Let us fix once and for all

$$\gamma = (1,2,3,\ldots,n).$$

Now we are in a position to give a more formal definition of a cactus.

Definition 2.2 A *rooted cactus* C of degree n is a k-tuple of permutations $C = [g_1, g_2, \ldots, g_k]$ on the set $\{1, 2, \ldots, n\}$ which satisfies the condition

$$\prod_{i=1}^{k} g_i = \gamma = (1, 2, \ldots, n).$$

The *root* of a cactus is the polygon number 1.

Two rooted cacti $C_1 = [g_1, g_2, \ldots, g_k]$ and $C_2 = [h_1, h_2, \ldots, h_k]$ are *isomorphic* if there exists a permutation φ such that

$$h_1 = \varphi^{-1} g_1 \varphi, \quad h_2 = \varphi^{-1} g_2 \varphi, \quad \ldots, \quad h_k = \varphi^{-1} g_k \varphi.$$

As φ must preserve γ, it is equal to a power of γ: $\varphi = \gamma^m$. An equivalence class of isomorphic cacti is called a *non-rooted cactus*.

Remark 2.3 It is clear that an asymmetric non-rooted cactus produces n different rooted ones. The only possible automorphism group of a non-rooted cactus is a cyclic group of order s, where s divides n, and the group itself is generated by the γ^m, $m = \frac{n}{s}$. Thus a symmetric non-rooted cactus produces $m = \frac{n}{s}$ different rooted ones.

Attention: the cactus of Figure 4 is *not* symmetric! The center of symmetry must be a vertex.

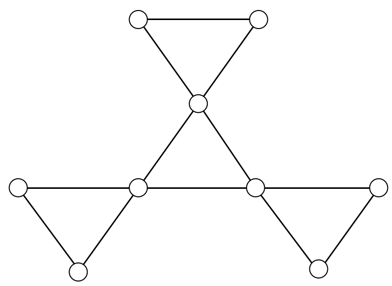

Figure 4: This cactus is not symmetric

To each permutation $g \in S_n$ there corresponds its *cyclic structure*, which is partition $\lambda = (d_1, d_2, \ldots, d_p)$ of the number n. Following [17] we introduce the definition:

Definition 2.4 The list $\pi = [\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}]$ of k partitions of the number n that represent cyclic structures of the permutations of a cactus is called the *passport* of this cactus. The set of all the cacti having the same passport is called a *family* of cacti.

Geometrically, the partition $\lambda^{(i)}$ is the list of degrees of the cactus vertices of color i. For example, the passport of the cactus of Figure 3 is

$$\pi = [1^3 3^2, 1^5 2^2, 1^5 2^2].$$

Lemma 2.5 (Planarity condition) *The total number of vertices of a k-gonal cactus of degree n is $(k-1)n+1$.*

This lemma is an immediate consequence of the Euler theorem.

The total number of vertices is also the total number of elements of all the k partitions. Hence the planarity condition may be rewritten in the following way. Let

$$\lambda^{(1)} = (d_{11}, d_{12}, \ldots, d_{1p_1}), \quad \ldots, \quad \lambda^{(k)} = (d_{k1}, d_{k2}, \ldots, d_{kp_k}).$$

Then

$$\sum_{i=1}^{k} \sum_{j=1}^{p_i} (d_{ij} - 1) = n - 1. \tag{1}$$

Indeed,

$$\sum_{i,j} d_{ij} = kn, \quad \text{while} \quad \sum_{i,j} 1 = (k-1)n + 1.$$

A passport π which satisfies the condition (1) will be called *valuable.*

Lemma 2.6 (Existence of cacti) *For any valuable passport π, there exists at least one cactus with passport π.*

Proof The lemma immediately follows from the enumerative formula given below. But it is so simple that we prefer to give an independent proof. The following is true:

> In each partition of a valuable passport except probably one, there exists a part equal to 1.

Indeed, if all the parts of a partition are ≥ 2, then their number is $\leq n/2$. If this occurs in two or more partitions, we will be unable to get the total number $(k-1)n+1$ of parts. Thus we can make the following operation ("cutting off a leaf"):

- remove a part equal to 1 from all partitions except one;
- in the remaining partition diminish by 1 a part that is ≥ 2;
- if one of the partitions becomes equal to 1^{n-1}, remove it from the passport.

The result of this operation is a valuable passport of degree $n-1$, and we may proceed by induction. $\square$

In order to introduce the enumerative formula we need some notation. Let a partition $\lambda \vdash n$ be presented in the "power notation":

$$\lambda = 1^{p_1} 2^{p_2} \ldots n^{p_n}, \quad \text{where} \quad \sum_{i=1}^{n} p_i = p, \quad \sum_{i=1}^{n} i p_i = n.$$

Then denote

$$N(\lambda) = \frac{(p-1)!}{p_1! p_2! \ldots p_n!} = \frac{1}{p} \begin{pmatrix} & & p & \\ p_1 & p_2 & \ldots & p_n \end{pmatrix}.$$

The following result is borrowed from [14].

Theorem 2.7 (Enumeration of cacti) *The number of rooted cacti for a given passport* $\pi = [\lambda^{(1)}, \ldots, \lambda^{(k)}]$ *is equal to*

$$n^{k-1} \prod_{i=1}^{k} N(\lambda^{(i)}).$$

It is more convenient to divide the above formula by n. Together with Remark 2.3 this gives us what in physics is usually called a *mass-formula*:

$$\sum_C \frac{1}{|\mathrm{Aut}(C)|} = n^{k-2} \prod_{i=1}^{k} N(\lambda^{(i)}), \tag{2}$$

where the sum is taken over all the *non*-rooted cacti with the passport $[\lambda^{(1)}, \ldots, \lambda^{(k)}]$, and $|\mathrm{Aut}(C)|$ is the order of the automorphism group of a non-rooted cactus.

The reader will see that the last formula will be of a primordial importance for our study. In the most frequent case, when all the cacti are asymmetric, it gives the exact number of non-rooted cacti. We will call the number given by formula (2) *Goulden–Jackson number*.

Example 2.8 The number of cacti with the passport $[1^4 2^1 3^1, 1^3 2^3, 1^5 2^2]$ (here $n = 9$ and $k = 3$) is equal to

$$9 \times \frac{5!}{4!1!1!} \times \frac{5!}{3!3!} \times \frac{6!}{5!2!} = 450.$$

As for the passport $[1^3 2^3, 1^3 2^3, 1^6 3^1]$ the formula gives

$$9 \times \frac{5!}{3!3!} \times \frac{5!}{3!3!} \times \frac{6!}{6!1!} = 100.$$

But in fact there are 102 cacti with this passport; 99 of them are asymmetric, and the remaining 3 are symmetric with the symmetry of order 3, so we have $99 + 3 \times \frac{1}{3} = 100$.

Remark 2.9 For a given passport it is very easy to see if the corresponding family contains any symmetric cacti. Let us write down the lists of vertex degrees of the above passport $[1^3 2^3, 1^3 2^3, 1^6 3^1]$:

$$222111 \qquad 222111 \qquad \underline{3}111111$$

The distinguished vertex *may* serve as a center of the symmetry of order 3, because it satisfies two conditions:

1. its degree is divisible by 3;
2. for the remaining vertices, all the vertices of the same color and degree may always be divided into 3 equal parts.

3 Cacti and complex polynomials

Let $P \in \mathbb{C}[z]$ be a polynomial of degree n. For a given $w \in \mathbb{C}$, the equation $P(z) = w$ usually has n distinct solutions. The exceptional values of w, for which the equation $P(z) = w$ has multiple roots, are called *critical values* of P. The multiplicities of the roots of $P(z) = w$ form a partition $\lambda \vdash n$, $\lambda = (d_1, d_2, \ldots, d_p)$. (For a *non*-critical value w the corresponding partition is 1^n.) A root of multiplicity $d_i \geq 2$ is called *critical point* of P. Sometimes by abuse of language we will call simple roots that are preimages of critical values, critical points of multiplicity 1.

Let polynomial P have k critical values $w_1, w_2, \ldots, w_k$. The list of the corresponding partitions $\pi = [\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}]$ is called the *passport* of P.

If we draw on the plane of the variable w any image which avoids critical values (that is, does not touch them, and does not have any of critical values inside), then its preimage via P will consist of n disjoint homeomorphic copies of this image. If, on the contrary, the image contains a critical value on its boundary, then several of its copies on the z-plane will glue together at critical points. Now let us draw on the w-plane a polygon with k sides, whose vertices are $w_1, w_2, \ldots, w_k$. Of course, only the topological form of the polygon is important: it may well be curvilinear, not convex, etc. However, necessary condition is that the vertices $w_1, w_2, \ldots, w_k$ must be numbered in exactly this order in the counterclockwise direction. Or, alternatively, let us take a non-critical value w_0 and draw a star by joining w_0 with $w_1, w_2, \ldots, w_k$ by non intersecting arcs, respecting the order of the ends taken in the counterclockwise direction. Then *the preimage of the polygon (or of the star) is a cactus.*

Thus a cactus is just a convenient form of representing a polynomial. Let us translate two characteristic properties of cacti into the language of polynomials.

(1) The property that the product $\prod_{i=1}^{k} g_i$ is a circular permutation means that a polynomial of degree n has a critical point of multiplicity n at ∞ (the corresponding critical value also being ∞). This property also implies the connectivity of the cactus and the absence of cycles other than polygons themselves. Working only with polynomials, we will often forget about this additional critical value.

(2) Each root of multiplicity d of an equation $P(z) = w$, $w \in \mathbb{C}$, is also a root of multiplicity $d-1$ of the equation $P'(z) = 0$ (this is also true of simple

roots of $P(z) = w$). Hence the property $\sum_{i=1}^{k} \sum_{j=1}^{p_k} (d_{ij} - 1) = n - 1$ means that the derivative $P'(z)$, being a polynomial of degree $n-1$, has $n-1$ roots, with multiplicities taken into acount.

To any complex polynomial there corresponds a cactus. The following theorem answers in the affirmative the opposite question: is it true that to any cactus there corresponds a polynomial? This theorem is a particular case of the well-known Riemann's existence theorem for Riemann surfaces (see, for example, [13]).

Theorem 3.1 (Riemann's existence theorem) *Let the following data be given:*

- *a k-tuple of permutations $[g_1, g_2, \ldots, g_k]$, $g_i \in S_n$, defining a k-gonal cactus of degree n;*
- *arbitrary complex numbers $w_1, w_2, \ldots, w_k$.*

Then there exists a polynomial $P(z)$ of degree n, with k critical values equal to $w_1, w_2, \ldots, w_k$, and with the corresponding cactus $[g_1, g_2, \ldots, g_k]$. This polynomial is unique, up to an affine change of variables $z \mapsto az+b$, $a, b \in \mathbb{C}$, $a \neq 0$.

In the language of Riemann surfaces the polygons of a cactus "represent" sheets of the Riemann surface of P, points w_i are ramification points, and permutations g_i prescribe the passage from a sheet to another one when we go around w_i.

The above theorem, together with Lemma 2.6, gives the following

Proposition 3.2 (Existence of polynomials) *For any valuable passport π, there exists at least one polynomial P with passport π.*

This simple though important proposition was proved in [22], [10] and [17].

4 Braid group action

Up to what extent does a cactus represent a class of the topological equivalence of polynomials?

It is easy to see that the homeomorphisms h_1 and h_2 of the diagram on page 2 must bring critical points to critical points, and critical values to critical values. Thus, the number of critical values, and the *unordered* passport are both invariants of the topological equivalence.

It is also clear that if two polynomials P_1 and P_2 are represented by the same combinatorial cactus, they are topologically equivalent. The opposite is not true: two different cacti may represent topologically equivalent polynomials. The following fundamental theorem may be found in [23], [17].

Theorem 4.1 (Braid orbits) *Let $\sigma_1, \ldots, \sigma_{k-1}$ be the operations that act on k-gonal cacti in the following way:*

$$\sigma_i : \begin{array}{lcccl} g_i & \mapsto & g'_i & = & g_{i+1} \\ g_{i+1} & \mapsto & g'_{i+1} & = & g_{i+1}^{-1} g_i g_{i+1} \\ g_j & \mapsto & g'_j & = & g_j \text{ for } j \neq i, i+1 \end{array}$$

Then

(1) *This is an action of the braid group B_k; that is, the operations σ_i, $i = 1, \ldots, k-1$ satisfy the following relations*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i-j| \geq 2,$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

(2) *The classes of the topological equivalence of complex polynomials are in one to one correspondence with the orbits of this action.*

Remark 4.2 The action, as it is introduced, is on the set of the *rooted* cacti (see Section 2). However, it is obvious that the operations σ_i preserve the product $\gamma = \prod_{i=1}^{k} g_i$, and thus the action may be (and is) considered on the set of *non rooted* cacti. It is the orbits of the action on non rooted cacti that we had in mind in statement (2) of the above theorem.

As we have already mentioned in the introduction, the action itself was introduced by Hurwitz in 1891 [15]. In fact, it is an action of the *sphere* braid group, also called Hurwitz braid group (to be distinguished from the Artin braid group, which is the group of the braids on the plane). The action is not faithful: there are some relations satisfied by the operations σ_i (on non rooted cacti!) that do not exist in the braid group.

We see that g'_i is equal to g_{i+1}, and g'_{i+1} is conjugate to g_i. Thus the operation σ_i permutes the partitions $\lambda^{(i)}$ and $\lambda^{(i+1)}$ of the passport.

Let us explain the geometrical meaning of the operation σ_i. Suppose a star is drawn on the w-plane, with its leaves at the critical values $w_1, w_2, \ldots, w_k$. Fix the positions of all the w_j, $j \neq i, i+1$, and move the points w_i and w_{i+1} continuously until they exchange their places. Figure 5 shows the image of the initial star under this homeomorphism.

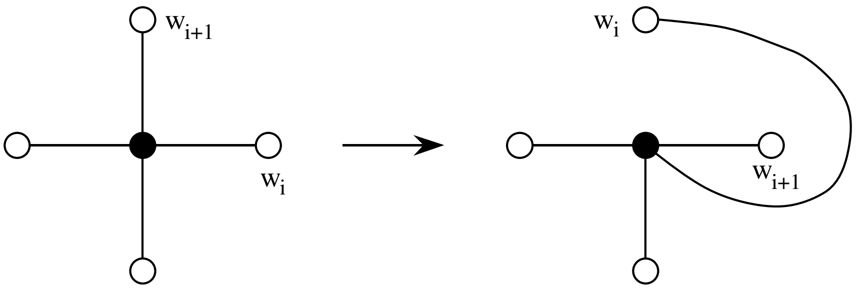

Figure 5: Homeomorphism that exchanges w_i with w_{i+1}

For this new star, its preimage is *combinatorially* the same cactus as before, because it is described by the same list of permutations.

The next stage would be to change labels of critical values, and from now on to call w_i by the new name w'_{i+1}, and w_{i+1}, by w'_i. But we must not forget that the colors $1, 2, \ldots, k$ must encircle the center of the star in exactly this order. Hence we must erase the "curvilinear" ray that goes from the center to the new point w'_{i+1}, and draw another one after w'_i, see Figure 6.

Note that this last operation does not mean any cutting or pasting of the Riemann surface. The surface and the covering remain the same; the only thing that changes is the picture drawn on the surface.

Now we must understand how the new cactus, that is, the preimage of the new star, is described in terms of permutations. Let us draw a path on the w-plane that goes counterclockwise from the star center to itself around the point w_i. Its preimage on the z-plane consists of n paths, each going from a star number m to the star number $g_i(m)$, $m = 1, 2, \ldots, n$. It is therefore

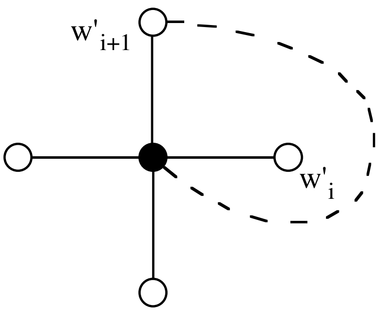

Figure 6: New star

sufficient to represent the paths going around w'_i and w'_{i+1} in terms of the "initial" paths going around w_i and w_{i+1}.

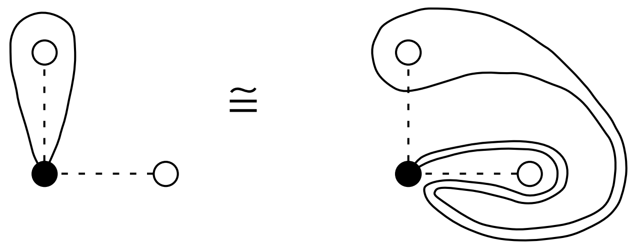

Figure 7: Permutation g'_{i+1}

These initial paths may be regarded as being very close to the rays of the star going from its center to the points w_i and w_{i+1}. During the first stage (the homeomorphism) the paths remain close to the rays. While changing the labels, the former g_{i+1} becomes the new g'_i. Now look at Figure 7: in the left-hand picture a path goes counterclockwise around w'_{i+1}. It is clear that it is homotopically equivalent to the path shown in the right-hand picture. This one goes first around w'_i (clockwise!), than around w_i (following the dashed ray of Figure 6), and than once more around w'_i (this time counterclockwise). This "long way" is $g_{i+1}^{-1} g_i g_{i+1}$.

5 Combinatorial invariants

The goal of this section is to propose a certain number of combinatorial invariants of the braid group action which could help us to prove that there is only one orbit, or otherwise to explain why there are several orbits. Of course, the first and the most important invariant of this kind is the unordered passport.

5.1 What was known before

The following theorem was proved by the classics of the XIX-th century [18], [8], [15]. A critical value w is called *simple*, if the equation $P(z) = w$ has only one multiple root, and that of multiplicity 2. The corresponding partition in the passport is $1^{n-2}2^1$.

Theorem 5.1 (Simple critical values) *If all the critical values of a polynomial are simple, then there is only one class of the topological equivalence of such polynomials.*

The two theorems that follow generalize this result in two directions. They are proved in [17]. The idea of the proof is a construction of more or less explicit homeomorphisms using Lagrange interpolation.

Definition 5.2 The *defect* $d(\pi)$ of a passport π is the sum of all the parts $d_{ij} \geq 2$ that correspond to non-simple critical values.

Theorem 5.3 (Defect) *Let π be a passport of degree n consisting of k partitions. If its defect $d(\pi) \leq n+1$, then there is only one class of the topological equivalence of polynomials with the passport π.*

Simple arithmetic considerations lead to the following

Corollary 5.4 (Many critical values) *Let a polynomial of degree n have $k \geq 3n/4$ critical values. Then there is only one class of the topological equivalence of the polynomials with the same passport.*

A critical value w is called *quasi-simple*, if the eqaution $P(z) = w$ has only one multiple root, of multiplicity $d \geq 2$. The corresponding partition is $1^{n-d}d^1$.

Theorem 5.5 (Quasi-simple critical values) *If all the critical values of a polynomial are quasi-simple, then there is only one class of the topological equivalence of the polynomials with the same passport.*

Example 5.6 Consider the following passport of degree $n = 9$:

$$\underline{52}11 \qquad \underline{3}111111 \qquad 21111111.$$

The first two critical values are non-simple; the parts of the partitions to be taken into account for computing the defect are underlined. We have the defect $d = 5 + 2 + 3 = 10 \le n + 1$, hence there is only one topological class of the polynomials with this passport.

The defect of the passport

$$\underline{5}1111 \qquad \underline{3}111111 \qquad \underline{3}111111$$

is equal to 11, so Theorem 5.3 does not work. But this time all the critical values are quasi-simple, and hence Theorem 5.5 works: there is also only one class.

For the passport

$$\underline{42}111 \qquad \underline{3}111111 \qquad \underline{22}11111$$

the defect is equal to 13, and two of the three critical values are not quasi-simple, so neither of two theorems works. In fact, there is also only one class of polynomials with this passport. But in order to establish this result we need some additional tools.

Theorems 5.3 and 5.5 suffice in order to classify all the polynomials of degree $n \le 6$ except the polynomials with two critical values, and that of the passport $[1^2 2^2, 1^2 2^2, 1^4 2^1]$. For these cases the braid group orbits were computed explicitly in [23] and [17]. For the above passport there are two orbits. The case of two critical values will be considered in the next subsection.

5.2 Two critical values

Polynomials with two critical values are specially interesting because of their relations to Galois theory (see [21]). Concerning their topological classification, this case is rather trivial. First of all, in graphical representations of

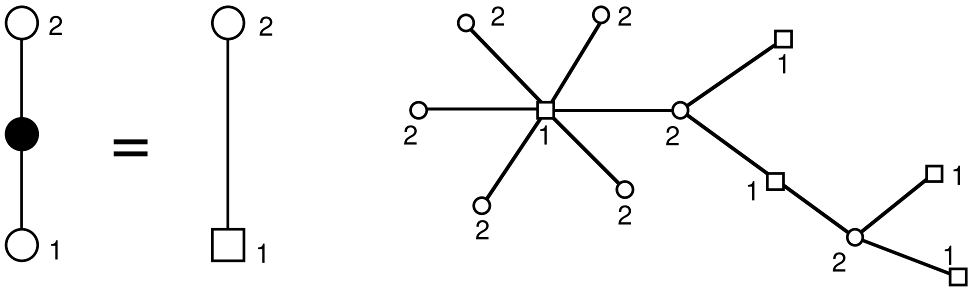

Figure 8: A 2-cactus is a bicolored plane tree

the corresponding cacti we may eliminate the "black vertices" and draw just bicolored plane trees, see Figure 8.

Second, we have only one operation σ_1, which acts on a couple of permutations $[g_1, g_2]$ in the following way:

$$g_1' = g_2 = g_2^{-1} g_2 g_2, \qquad g_2' = g_2^{-1} g_1 g_2.$$

Thus the couple $[g_1', g_2']$ is conjugate to the couple $[g_2, g_1]$: the action consists in exchanging the colors, and in changing the labels of the tree edges, while the tree itself is preserved. There are as many orbits as there are plane trees with the given passport. The enumerative results of Section 2 give us the possibility to calculate the number of orbits without computing the action itself.

The main point of our interest (cf. also [17]) is, what are the passports that uniquely determine the topological class of corresponding polynomials? The following theorem gives the complete answer in the case of two critical values. This result belongs to N. Adrianov (1989). It was never properly published (see however [2]) and gradually became a part of the folklore.

Theorem 5.7 (Unique trees) *The following is the complete list of passports of polynomials with one or two critical values for which there exists only one class of the topological equivalence of polynomials:*

1. $[n^1]$ (one critical value);
2. $[1^{n-p}p^1, 1^{n-q}q^1]$;
3. $[2^p, 1^2 2^{p-1}]$ (n even), or $[1^1 2^p, 1^1 2^p]$ (n odd);
4. $[1^{n-m}m^1, p^{m-1}q^1]$ (here $n = (m-1)p + q$);
5. $[1^{n-(p+q)}p^1 q^1, 2^{n/2}]$ (n even);
6. $[1^{n-2p}p^2, 3^{n/3}]$ (n is divisible by 3);
7. $[1^5 3^3, 2^7]$ ($n = 14$).

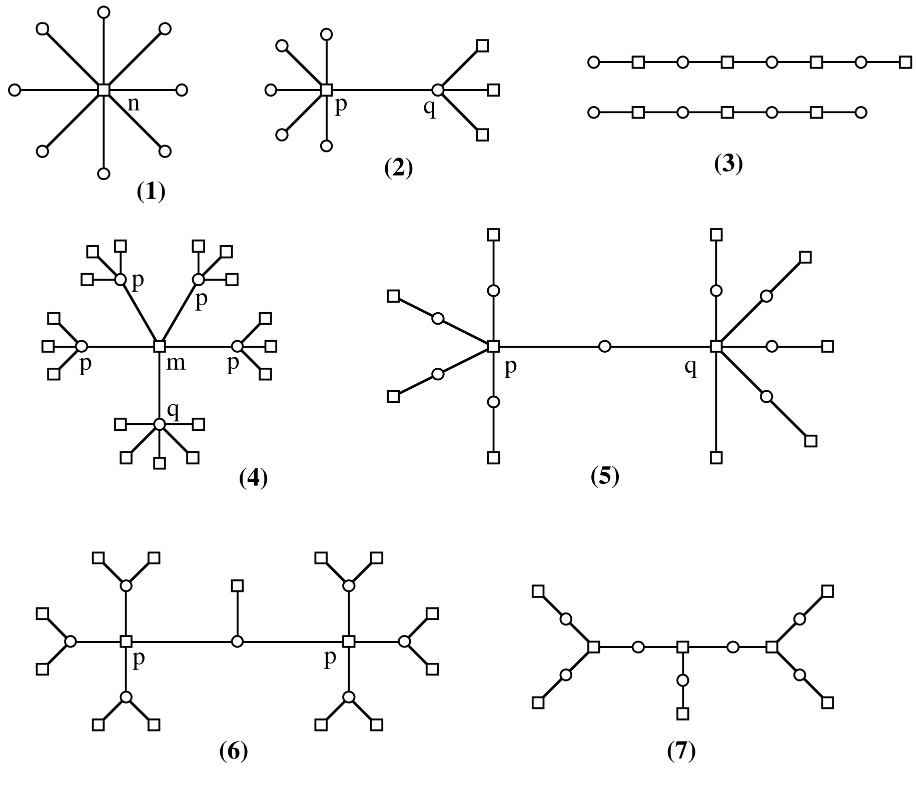

Figure 9: Trees uniquely determined by their passports

The corresponding trees are shown in Figure 9. We see here 6 infinite series (depending on one, two or three parameters), and a "sporadic" tree (number 7) that does not belong to any series. In fact, the series number 1 and 2 are particular cases of the series number 4. But we give them separately, because these are the only cases covered by Theorem 5.3. For example, for the sporadic tree we have $n = 14$, while the defect $d = 23$. For the series number 3 (the chain-tree) the defect attains its maximum: $d = 2n - 2$; and still there is only one orbit.

5.3 Monodromy group

Definition 5.8 The *monodromy group* of a cactus $[g_1, g_2, \ldots, g_k]$, and also that of the corresponding polynomial, is the permutation group

$$G = <g_1, g_2, \ldots, g_k> \leq S_n,$$

generated by the permutations $g_1, g_2, \ldots, g_k$.

The next proposition follows immediately from the formulas for the braid group action given in Theorem 4.1.

Proposition 5.9 (Invariance of monodromy) *The monodromy group is an invariant of the braid group action.*

Corollary 5.10 (Symmetry) *If for a given passport there exist both symmetric and asymmetric cacti, they necessarily belong to different orbits.*

The proof is obvious: the monodromy group of a symmetric cactus possesses a non-trivial centralizer in S_n, while the monodromy group of an asymmetric cactus does not.

Example 5.11 For the passport $[1^2 2^2, 1^2 2^2, 1^4 2^1]$ of degree 6 mentioned at the end of Section 5.1, the Goulden–Jackson number is equal to

$$6 \times \frac{3!}{2!2!} \times \frac{3!}{2!2!} \times \frac{4!}{4!1!} = \frac{27}{2}.$$

The presence of fractions in the answer assures us of the presence of symmetric cacti. Hence for this passport there are at least two orbits (explicit computations show that there are exactly two).

Example 5.12 Consider the following passport: $[1^3 2^2, 1^3 2^2, 1^3 2^2]$ ($n = 7$). Information given in the catalogue [7] allows us to construct two different groups generated by permutations of this cyclic structure.

(1) Take

$$g_1 = (1,7)(2,4), \quad g_2 = (2,7)(5,6), \quad g_3 = (3,4)(5,7).$$

Then the monodromy group is $G = PSL_3(2) \cong PSL_2(7)$, a group of order 168.

(2) Take

$$g_1 = (2,7)(3,5), \quad g_2 = (3,7)(4,5), \quad g_3 = (1,2)(6,7).$$

Then the monodromy group is $G = A_7$, the group of order 2520.

Thus in this case we must have at least two different orbits. In fact, the total number of 56 cacti having this passport splits into 4 different orbits. Two of them, of size 7, have the monodromy group $PSL_3(2)$, and the other two, of size 21, have the monodromy group A_7. We did not find a combinatorial invariant that would be responsible for this "additional" splitting. Our only observation is that the seven cacti of one $PSL_3(2)$-orbit are axially symmetric to those of the other $PSL_3(2)$-orbit, and the same is true of the A_7-orbits.

In fact, the monodromy group is not a very powerful invariant, as the following very powerful theorem shows.

Theorem 5.13 (Exceptional groups) *Let P be a polynomial of degree n with $k \geq 3$ critical values and with* primitive *monodromy group not equal to A_n ar S_n. Then $k = 3$, and there are only three cases possible:*

1. $n = 7$: $\pi = [1^3 2^2, 1^3 2^2, 1^3 2^2] = [2A, 2A, 2A]$, $G = PSL_3(2) \cong PSL_2(7)$;

2. $n = 13$: $\pi = [1^5 2^4, 1^5 2^4, 1^5 2^4] = [2A, 2A, 2A]$, $G = PSL_3(3)$;

3. $n = 15$: $\pi = [1^3 2^6, 1^7 2^4, 1^7 2^4] = [2B, 2A, 2A]$, $G = PSL_4(2) \cong A_8$.

Here $2A$ and $2B$ is the notation of conjugacy classes used in the Atlas [9].

A proof of this theorem, based on some previous results of [12], may be found in [19]. We have learned this fact from N. Adrianov [1], who found the result independently of [19]. It is also N. Adrianov who provided us with the following two examples (not to be found in [19]).

Example 5.14 Let $n = 13$, and

$$\begin{aligned} g_1 &= (1,13)(2,12)(3,7)(8,10), \\ g_2 &= (2,13)(3,11)(4,5)(6,7), \\ g_3 &= (3,12)(4,6)(8,11)(9,10). \end{aligned}$$

Then $G = PSL_3(3)$. This group is of order 5616, and the orbit consists of 13 cacti.

Example 5.15 Let $n = 15$, and

$$\begin{aligned} g_1 &= (1,15)(2,3)(5,14)(7,10)(8,9)(11,13), \\ g_2 &= (2,5)(6,11)(8,10)(12,13), \\ g_3 &= (2,15)(4,5)(6,14)(7,11). \end{aligned}$$

Then $G = PSL_4(2) \cong A_8$. This group is of order 20160, and the orbit consists of 15 cacti.

The results concerning the group orders may be verified by means of MAPLE (for example).

5.4 Conjugacy classes

The following proposition is trivial:

Proposition 5.16 (Set of conjugacy classes) *The set of conjugacy classes $\{H_1, H_2, \ldots, H_k\}$ of permutations $g_1, g_2, \ldots, g_k$, $g_i \in H_i$ in the monodromy group is an invariant of the braid group action.*

It is clear that σ_i only permutes classes H_i and H_{i+1}.

For the group S_n there is nothing new in this proposition, as conjugacy classes in S_n are completely determined by the cyclic structure of their elements. For the group A_n this is not so. The following lemma may be found in [16] (Lemma 1.2.10):

Lemma 5.17 (Splitting types in A_n) *A set of permutations of a given cyclic structure splits into two conjugacy classes in A_n if and only if the lengths of all cycles are odd and different.*

Note that the parity of the monodromy group is easily seen from the passport: the monodromy group G is a subgroup of A_n if and only if the sum $\sum_{j=1}^{p_i}(d_{ij}-1)$ is even for all $i=1,2,\ldots,k$.

The "smallest" example that may be constructed using the idea of the above lemma is the following one:

Example 5.18 Take the passport of degree $n=25$

$$\pi = [1^1 3^1 5^1 7^1 9^1, 1^{22}3^1, 1^{22}3^1]$$

and a corresponding cactus $[g_1, g_2, g_3]$, where

$$g_1 = (1,2,3,4,5,6,7,8,9)(10,11,12,13,14,15,16)(17,18,19,20,21)(22,23,24)(25)$$

and

$$g_2 = (1,10,17), \qquad g_3 = (1,22,25)$$

(verify that the product $g_1 g_2 g_3 = \gamma = (1,2,3,\ldots,25)$!). We now change g_1 at four rightmost places, thus obtaining the permutation

$$h_1 = (1,2,3,4,5,6,7,8,9)(10,11,12,13,14,15,16)(17,18,19,20,21)(22)(23,24,25),$$

and take

$$h_2 = g_2, \qquad h_3 = (1,22,23)$$

(in order to have the same product $h_1 h_2 h_3 = \gamma$). Permutations g_1 and h_1 are conjugate in S_n by an *odd* permutation $(22,23,24,25)$, and hence they are not conjugate in A_n. Therefore the cacti $[g_1, g_2, g_3]$ and $[h_1, h_2, h_3]$ belong to different orbits.

The explicit computations show that in this case there are indeed two orbits, each one of size 900.

5.5 Compositions

If we want to construct different monodromy groups for the same passport, Theorem 5.13 leaves us the only hope, namely, when one of the groups is imprimitive.

Theorem 5.19 (Composition) *The following statements are equivalent:*

1. *The monodromy group of a polynomial is imprimitive.*
2. *The polynomial is a composition of non-linear polynomials of smaller degrees.*
3. *The corresponding cactus is a composition of smaller cacti.*

The equivalence of the statements 1 and 2 is a classical result obtained by Ritt [20] (see also [19]). What remains is to introduce the operation of the composition of cacti in such a way that the theorem would be true: that is, the composition of cacti should be an exact simulation of the composition of polynomials. This operation will be explained in detail elsewhere [11]; in this paper also an algorithm will be presented that verifies, for a given passport, whether or not its representation as a composition is possible (and if yes, it gives the passports of smaller cacti to be composed). Here we give only a few examples; but they are transparent enough to clarify the nature of the operation.

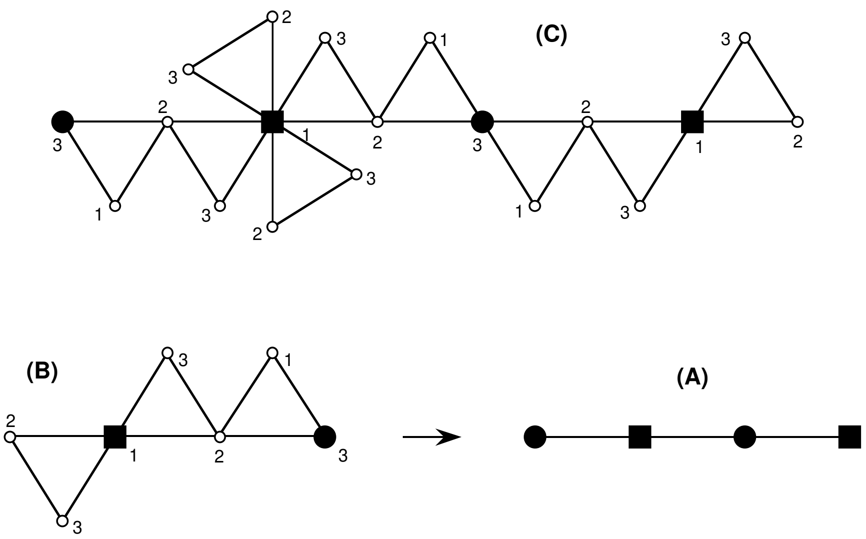

Figure 10: An example of composition

Example 5.20 In Figure 10 a cactus C of degree 9 is shown, which is a composition of two cacti A and B of degree 3. It is constructed in the following way: we take a bicolored tree A (the two colors are now marked by circles and squares); then we take three copies of the cactus B, and replace each edge of A by a copy of B in such a way that distinguished vertices are identified.

Note that in the cactus B the third color is added "artificially": all vertices of color 3 in B are of degree 1, that is, they are non-critical. However, we need this color in order to get a critical point of degree 2 and of color 3 in the cactus C.

Remark 5.21 Symmetry is a particular case of composition. Let a cactus have a symmetry of order s. Put its center to the point $z = 0$ and apply the polynomial z^s: the result is a cactus representing one branch of the initial one (see Figure 11).

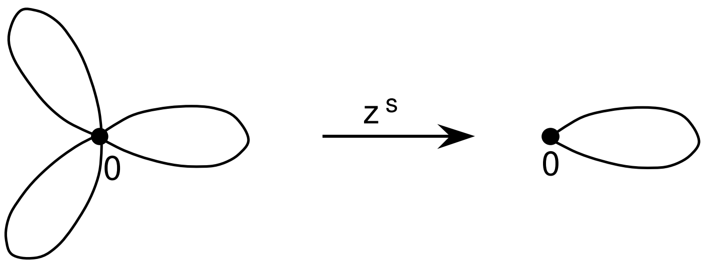

Figure 11: Symmetry is also a composition

Remark 5.22 The reader must not conclude that the presence of a composition leads necessarily to splitting the corresponding family of cacti into several orbits. Let us consider the following passport of degree 8: $\pi = [2^2 4^1, 1^6 2^1, 1^6 2^1]$. On the one hand, the defect $d(\pi) = 8$, and hence according to Theorem 5.3 we must have only one orbit of cacti with this passport. On the other hand, Figure 12 shows, for one of the cacti with this passport, two

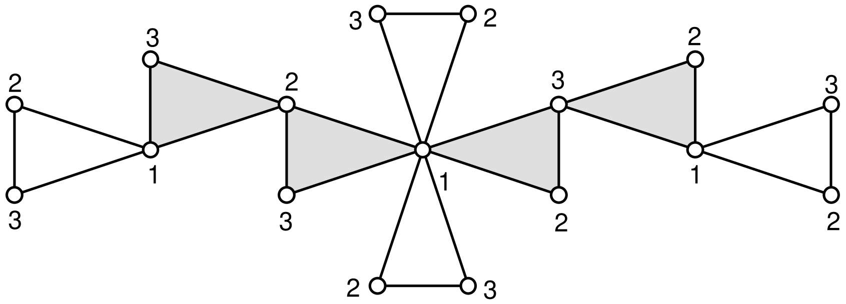

Figure 12: Two blocks of imprimitivity of monodromy group

blocks of imprimitivity of its monodromy group. The group itself is in fact equal to the wreath product $S_4 \wr S_2$.

This seemingly contradicting information is resolved by verifying that *all* the cacti having this passport are compositions, and for all of them the monodromy group is the same: $S_4 \wr S_2$.

Thus, the presence of composition gives us a non-trivial information only in the case when, for the same passport, some of the cacti are decomposable, and others are not.

The next stage of our study involves computers.

6 Computations

Example 6.1 Consider the following passport of degree 9:

$$\pi = [1^3 3^2, 1^4 2^1 3^1, 1^7 2^1].$$

The corresponding Goulden-Jackson number is

$$9 \times \frac{4!}{3!2!} \times \frac{5!}{4!1!1!} \times \frac{7!}{7!1!} = 90.$$

All the three partitions in the passport are different, hence we may permute them in 6 ways, and the total number of cacti corresponding to the *unordered* passport is $90 \times 6 = 540$.

Now let us take an arbitrary cactus with this passport, for example, like that:

$$g_1 = (1,3,5)(7,8,9), \quad g_2 = (1,6,7)(4,5), \quad g_3 = (2,3),$$

and call the program that computes the braid group orbits:

```
Magma V1.03-1    Wed Oct  2 1996 15:40:19   [Seed = 2132753822]
Type ? for help. Type <Ctrl>-D to quit.
> ChangeDirectory("~zipperer/Magma");
> load "exorbit.mag";
Loading "exorbit.mag"
> c:=[Sym(9)!(1,3,5)(7,8,9),Sym(9)!(1,6,7)(4,5),Sym(9)!(2,3)];
> time orbit:=BraidExOrbit(c);
540 elements found
Time: 1089.970
```

We have found an orbit consisting of 540 elements. The question is settled: there is only one orbit.

Example 6.2 Let us take the passport

$$\pi = [1^3 2^3, 1^3 2^3, 1^6 3^1].$$

This is the second passport considered in Example 2.8: there are 99 asymmetric cacti and 3 symmetric ones, the Goulden–Jackson number being $99 + 3 \times \frac{1}{3} = 100$. As the partitions of the passport may be permuted in 3 different ways (because two of them are identical), we have in total 297 asymetric cacti and 9 symmetric ones.

For an asymmetric cactus we may take

$$g_1 = (1,9)(2,8)(3,7), \quad g_2 = (2,9)(3,8)(6,7), \quad g_3 = (4,5,6).$$

Then we have

```
> c:=[Sym(9)!(1,9)(2,8)(3,7),Sym(9)!(2,9)(3,8)(6,7),Sym(9)!(4,5,6)];
> time orbit:=BraidExOrbit(c);
297 elements found
Time: 612.849
```

We take next a symmetric cactus

$$g_1 = (1,9)(3,4)(6,7), \quad g_2 = (2,9)(3,5)(6,8), \quad g_3 = (3,6,9),$$

and compute its orbit:

```
> c:=[Sym(9)!(1,9)(3,4)(6,7),Sym(9)!(2,9)(3,5)(6,8),Sym(9)!(3,6,9)];
> time orbit:=BraidExOrbit(c);
9 elements found
Time: 0.839
```

We may conclude that the cacti having this passport split into two orbits, of 297 and 9 elements respectively.

Example 6.3 Consider the passport

$$\pi = [1^3 2^1 4^1, 1^3 2^3, 1^7 2^1].$$

It is easy to see that there is no symmetric cacti with this passport. The Goulden–Jackson number is

$$9 \times \frac{4!}{3!1!1!} \times \frac{5!}{3!3!} \times \frac{7!}{7!1!} = 120,$$

thus the total number of cacti with the unordered passport is $120 \times 6 = 720$.

Take a cactus

$$g_1 = (1,3,5,7)(8,9), \quad g_2 = (1,8)(2,3)(4,5), \quad g_3 = (6,7),$$

and construct the corresponding orbit:

```
> c:=[Sym(9)!(1,3,5,7)(8,9),Sym(9)!(1,8)(2,3)(4,5),Sym(9)!(6,7)];
> time orbit:=BraidExOrbit(c);
702 elements found
Time: 1796.560
```

Not all of the 720 cacti are found! Where are the remaining ones?

If we turn to Example 5.20, we may see that the cactus-composition C given there has exactly our passport. Let us represent it in the form of permutations

$$g_1 = (2,3,8,9)(5,6), \quad g_2 = (1,2)(4,5)(7,8), \quad g_3 = (4,7),$$

and compute the corresponding orbit:

```
> c:=[Sym(9)!(2,3,8,9)(5,6),Sym(9)!(1,2)(4,5)(7,8),Sym(9)!(4,7)];
> time orbit:=BraidExOrbit(c);
18 elements found
Time: 1.779
```

We may conclude that in this example there are two orbits, of 702 an 18 elements respectively, the reason of splitting being the composition.

The above examples make clear the approach we have taken to classify polynomials. In order to obtain a complete classification for degrees $n \leq 9$, we computed the orbits corresponding to 93 passports (with $k \geq 3$) not covered by Theorems 5.3 and 5.5. The results will be announced in the next section. We finish this section with several remarks concerning the algorithmic part of the work.

Remark 6.4 (Rational and meromorphic functions) The "purely theoretical" part of this work may be generalized from polynomials to rational functions and even to meromorphic functions on Riemann surfaces of arbitary genus [17]. Instead of cacti, we must consider maps that are glued of k-stars or of k-polygons (these maps are planar when we consider rational functions, and they are of higher genus when we consider meromorphic functions defined on Riemann surfaces of higher genera). The action of the braid group is introduced in exactly the same way, and the classes of the topological equivalence of functions are in one to one correspondence with the orbits of this action.

There are, however, two obstacles to a practical approach to the problem. First, there is no enumerative formulas for these kinds of maps; and the information that an orbit has, say, 702 elements, is practically useless, if we don't know the total number of maps in question. Second, we don't even have a reasonable algorithm that would be able to generate an exhaustive list of maps with a given passport. Thus this study remains purely theoretical.

Remark 6.5 (Algorithm) The algorithm of orbits construction, starting from a given cactus C_1, fills in the following table (of course, the number N of its lines is not known in advance):

	σ_1	σ_2	$\ldots$	σ_{k-1}
C_1				
C_2				
C_3				
$\ldots$				
C_N				

Each time after applying an operation σ_i to a cactus, it verifies if the new cactus thus obtained is really new, or already exists in the first column of the table.

The complexity of the algorithm depends quadratically on N: indeed, it must make at most $(k-1)N^2$ comparisons of two cacti. The lower bound is also quadratic: at least N times the algorithm must find a really new cactus, and in order to be sure that it is new, it must be compared with all the cacti found before.

The comparison of the cacti is simplified by the fact that the only permutations that preserve

$$\gamma = \prod_{i=1}^{k} g_i = (1, 2, \ldots, n)$$

are the powers of γ.

Remark 6.6 (Pure braid groups) The *pure braid group* PB_k is a subgroup of B_k consisting of all elements that do not permute points $1, 2, \ldots, k$ (that is, the thread that starts at the point i, returns to the same point); see, for example, [6]. The elements of the pure braid group, while acting on a cactus, do not permute the partitions in the passport.

Let us return to Example 6.1. The orbit of the pure braid group contains 90 elements instead of 540, and the orbit computation becomes much quicker. The following example is even more striking:

$$[1^6 3^1, 1^5 2^2, 1^5 2^2, 1^7 2^1, 1^7 2^1].$$

The Goulden–Jackson number here is 6561, but the partitions may be permuted in 30 different ways, the orbit of the braid group thus becoming of size $30 \times 6561 = 196830$, which is hopeless both from the point of view of memory and of computation time. The pure braid group orbit remains accessible.

Bigger orbits were in fact computed in this way. Computation of the biggest one, having 19683 elements, took about 42 hours of CPU time of a SUN Ultra 1 station, the program being implemented in GAP.

The answer to the following question is a priori not clear: in the case when not all partitions in a passport are different, is it sufficient to act by the pure braid group, or do we also need the elements that permute corresponding points? What is, however, clear, is that if the pure braid group orbit has the number of elements equal to the Goulden–Jackson number, then the question is settled and there is only one orbit. This was always the case in our computations. Two other disadvantages of computing with the pure braid group:

(a) the number of generators in PB_k is bigger than in B_k: $k(k-1)/2$ instead of $k-1$ (see [6]);

(b) each generator of PB_k, expressed as an operation on permutations, is much more complicated than the operations σ_i.

7 Results

We do not touch here the "trivial" case of two critical values. The results of our computations may be summerized as follows:

Theorem 7.1 (Classification for $n \le 9$) *Let a passport π be given, with $n \le 9$ and $k \ge 3$. Then, except for the twelve cases listed below, there is only one class of the topological equivalence of polynomials with this passport.*

In what follows we list all the exceptional cases, giving (a) the passport, (b) the number and the size of orbits (the size of the full braid group orbits is given), and (c) the combinatorial reason for splitting. We use the term "composition" only for the cases of a composition more complicated than symmetry.

Case 1 $n = 6$, $k = 3$: passport

$$2211 \qquad 2211 \qquad \underline{2}1111.$$

Two orbits, of sizes 36 and 6.

Reason: symmetry (the smaller orbit consists of symmetric cacti; the symmetry center is underlined); see Example 5.11.

Case 2 $n = 7$, $k = 3$: passport

$$22111 \qquad 22111 \qquad 22111.$$

Four orbits, of sizes 21, 21, 7, 7.

Reason: different monodromy groups: A_7 versus $PSL_3(2) \cong PSL_2(7)$ (see Example 5.12).

Case 3 $n = 8$, $k = 3$: passport

$$3311 \qquad 221111 \qquad \underline{2}111111.$$

Two orbits, of sizes 168 and 24.
Reason: symmetry.

Case 4 $n = 8$, $k = 3$: passport

$$2222 \qquad 221111 \qquad \underline{2}111111.$$

Two orbits, of sizes 24 and 12.
Reason: symmetry.

Case 5 $n = 8$, $k = 3$: passport

$$\underline{4}1111 \qquad 221111 \qquad 221111.$$

Two orbits, of sizes 144 and 12.
Reason: symmetry (the center of the symmetry is of degree 4, but the symmetry itself is of order 2).

Case 6 $n = 8$, $k = 3$: passport

$$\underline{2}2211 \qquad 221111 \qquad 221111.$$

Two orbits, of sizes 288 and 24.
Reason: symmetry.

Case 7 $n = 8$, $k = 4$: passport

$$221111 \qquad 221111 \qquad 221111 \qquad \underline{2}111111.$$

Two orbits, of sizes 3968 and 64.
Reason: symmetry.

Case 8 $n = 9$, $k = 3$: passport

$$42111 \qquad 222111 \qquad 21111111.$$

Two orbits, of sizes 702 and 18.
Reason: composition (see Example 5.20).

Case 9 $n = 9$, $k = 3$: passport

$$22221 \qquad 222111 \qquad 21111111.$$

Two orbits, of sizes 162 and 18.
Reason: composition (see one of the 18 cacti-compositions in Figure 13).

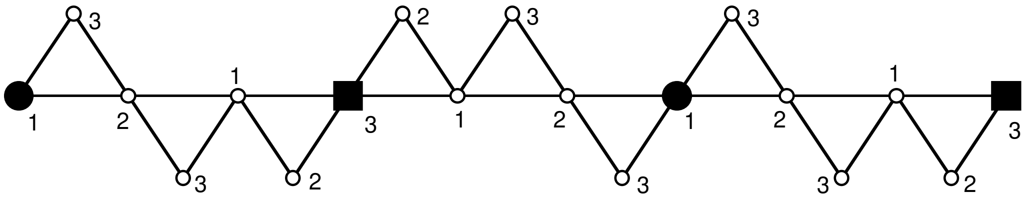

Figure 13: Composition for the Case 9

Case 10 $n = 9$, $k = 3$: passport

$$222111 \qquad 222111 \qquad \underline{3}111111.$$

Two orbits, of sizes 297 and 9.
Reason: symmetry (see Example 6.2).

Case 11 $n = 9$, $k = 3$: passport

$$222111 \qquad 222111 \qquad 2211111.$$

Two orbits, of sizes 891 and 9.
Reason: composition (see one of the 9 cacti-compositions in Figure 14).

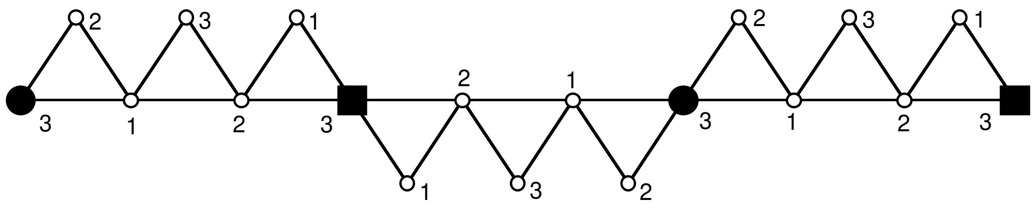

Figure 14: Composition for the Case 11

Case 12 $n = 9$, $k = 4$: passport

222111 222111 21111111 21111111.

Two orbits, of sizes 5346 and 54.
Reason: composition (see one of the 54 cacti-compositions in Figure 15).

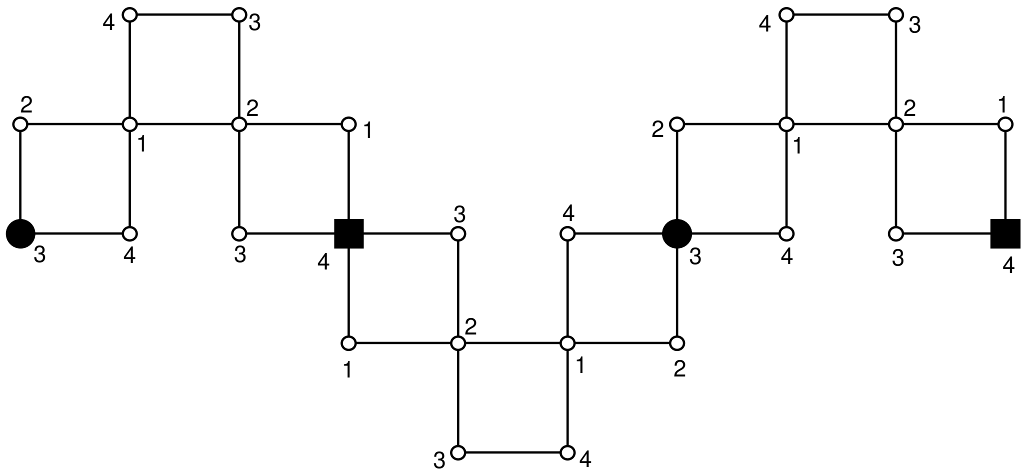

Figure 15: Composition for the Case 12

We may also summarize the mechnisms of splitting of a family of cacti into several orbits that we have observed:

- Composition, and symmetry as its particular case (Section 5.5);
- Exceptional monodromy groups (Theorem 5.13);
- Splitting of conjugacy classes in A_n (Example 5.18);
- A yet unknown mechanism that leads to splitting of the family of Example 5.12 into four orbits instead of two.

Of course, one may combine the above mechanisms in order to get more complicated examples. It would be, however, more interesting to find new mechanisms of splitting that do not reduce to the known ones.

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