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# Plethysm and the Non-Compact Groups Sp(2n,R)

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**ABSTRACT**

Some preliminary results on the plethysms for the non-compact group $Sp(2n,R)$ are presented. Complete results are given for power 2 plethysms of the two fundamental irreducible representations of $Sp(2n,R)$. Several new $S-$function identities arise from this work. The stabilisation properties of the plethysms are briefly considered and some remarkable conjugacy mappings observed.

## Introduction

The plethysm of $S-$functions has been the subject of much research ever since its introduction by Littlewood[1]. Many applications have been made to the classical compact Lie groups by expressing the characters of the irreducible representations of the group in terms of $S-$functions[2−6]. To date rather scant attention has been paid to application of plethysm to non-compact Lie groups[7−9].

The non-compact group $Sp(2n,R)$ is of special interest in physics as it is the dynamical group[10] of the $n-$dimensional isotropic harmonic oscillator which finds important applications in symplectic models of nuclei[11] and in the mesoscopic physics of quantum dots[12,13]. The non-trivial unitary irreducible representations of $Sp(2n,R)$ are all of infinite dimension[14,15]. An extensive outline of notation, characters, Kronecker products and branching rules is developed in reference 15. In other matters we follow the notation of Macdonald[16] Arbitrary positive discrete harmonic series irreducible representations of $Sp(2n,R)$ will be labelled as $<\frac{k}{2};(\lambda)>$ or equivalently as $<s\kappa;(\lambda)>$ where $\kappa$ and $s$ are the integer and residue parts of $\frac{k}{2}$.

The infinite set of states of a harmonic oscillator span the pair of infinite- dimensional fundamental unitary irreducible representations of $Sp(2n,R)$ which we shall designate[15] as $<\frac{1}{2};(0)>$ and $<\frac{1}{2};(1)>$. Our central problem is to resolve symmetrised powers of these two irreducible representations which amounts to evaluating the plethysms

$$s_\lambda(<s;(0)>) \qquad \text{and} \qquad s_\lambda(<s;(1)>) \tag{1}$$

To proceed with the plethysm problem for $Sp(2n,R)$ we first consider the $Sp(2n,R)\to U(n)$ decompositions where $U(n)$ is the unitary group in $n-$dimensions and then show how these can be used to build up $Sp(2n,R)$ plethysms up to some finite cutoff and present some complete results for $\lambda\vdash 2$. We are then led to some new $S-$function identities and after some comments on the stability of $Sp(2n,R)$ plethysms we end with a remarkable observation of the existence of a mapping between the two types of plethysms.

## $\mathbf{Sp(2n,R) \to U(n)}$ decompositions

Under the restriction[15] $Sp(2n,R) \to U(n)$ a given irreducible representation of $Sp(2n,R)$ decomposes into an infinite set of finite dimension irreducible representations of the unitary group $U(n)$. In the case of the two fundamental irreducible representations of $Sp(2n,R)$ we have[15]

$$< s;(0) > \to \varepsilon^{\frac{1}{2}} M_+ \tag{2a}$$

$$< s;(1) > \to \varepsilon^{\frac{1}{2}} M_- \tag{2b}$$

where $M_+$ and $M_-$ are the *even* and *odd* weight $S-$functions $s_m$ appearing in the infinite series

$$M = \sum_{m=0}^{\infty} s_m \tag{3}$$

In general one has[15]

$$< \frac{k}{2};(\lambda) > \to \varepsilon^{\frac{k}{2}} \cdot ((s_{\lambda_s})^k_N \cdot D_N)_N \tag{4}$$

where $N = min(n,k)$ and $D$ is the infinite $S-$function series

$$D = \sum_{\delta} s_\delta \tag{5}$$

where the $\delta$ are partitions involving only *even* parts. The subscript $N$ means that all terms involving partitions into more than $N$ parts are to be discarded. The first $\cdot$ indicates a product in $U(n)$ and the second $\cdot$ a product in $U(N)$. $(s_{\lambda_s})^k$ is a *signed sequence*[14,15] of terms $\pm s_\rho$ such that $\pm s_\rho$ is equivalent to $s_\lambda$ under the modification rules of the orthogonal group $O(k)$.

## Plethysms in $\mathbf{Sp(2n,R)}$

We are primarily interested in plethysms of the form $s_\lambda(< s;(0) >)$ and $s_\lambda(< s;(1) >)$. No general procedure seems to be known for evaluating $Sp(2n,R)$ plethysms. Here we evaluate the terms, up to a given weight, by first decomposing the $Sp(2n,R)$ irreducible representation into those of $U(n)$, performing the plethysm at the $U(n)$ level and then inverting to get irreducible representations of $Sp(2n,R)$. This has been done for all $\lambda \vdash 4$ and in some cases to $\lambda \vdash 6$. Tables of the relevant plethysms are located at http://www.phys.uni.torun.pl/~bgw/. In the case of $\lambda \vdash 2$ it is possible to obtain completely general results as follows

$$\begin{aligned}
s_2(< s;(0) >) &= \sum_{i=0}^{\infty} < 1;(0+4i) > \\
s_{1^2}(< s;(0) >) &= \sum_{i=0}^{\infty} < 1;(2+4i) > \\
s_2(< s;(1) >) &= \sum_{i=0}^{\infty} < 1;(2+4i) > \\
s_{1^2}(< s;(1) >) &= < 1;(1^2) > + \sum_{i=0}^{\infty} < 1;(4+4i) >
\end{aligned} \tag{6}$$

These results imply that the following $S-$function identity must hold

$$s_{1^2}(M_+) = s_2(M_-) \tag{7}$$

as indeed may be shown to be the case[17].

If $L_+$ and $L_-$ are respectively the positive and negative terms of the series

$$L = \sum_{m=0}^{\infty} (-1)^m s_m \tag{7}$$

then one finds

$$s_{1^2}(L_+) = s_2(L_-) \tag{8}$$

Still further identities arise for the infinite $S-$function series defined by

$$\begin{array}{ll} A_\pm = L_\pm(s_{1^2}) & B_\pm = M_\pm(s_{1^2}) \\ C_\pm = L_\pm(s_2) & D_\pm = M_\pm(s_2) \end{array} \tag{8}$$

Use of the associativety property of plethysms leads directly to

$$s_{1^2}(Z_+) = s_2(Z_-) \tag{9}$$

for $Z = A, B, C, D$. Furthermore

$$s_2(Z) = ZZ_+ \qquad \text{and} \qquad s_{1^2} = ZZ_- \tag{10}$$

The study of plethysms within the group $Sp(2n, R)$ leads to still further identities. The observation that

$$s_{21^2}(< s; (0) >) = s_{31}(< s; (1) >) \tag{11}$$

leads to the remarkable $S-$function identity

$$s_{21^2}(M_+) = s_{31}(M_-) \tag{12}$$

which generalises to

$$s_\sigma(s_{1^2}(M_+)) = s_\sigma(s_2(M_-)) \tag{13}$$

Again these identities extend to the series $Z$ defined earlier.

**Stability of Kronecker products and plethysms**

A given plethysm, Kronecker product or decomposition will be said to be *stable* if at the stable value of $n = n_s$ there is a one-to-one mapping between the resultant list of irreducible representations obtained at the stable value $n_s$ and those obtained for all values of $n > n_s$. The $Sp(2n, R)$ Kronecker product[15]

$$< \frac{k}{2}(\lambda) > \times < \frac{\ell}{2}(\nu) > = < \frac{(k+\ell)}{2}; ((s_{\lambda_s})^k \cdot (s_{\nu_s})^\ell \cdot D)_{k+\ell,n} > \tag{14}$$

is certainly stable for all $n \geq (k+\ell)$. We say *certainly* because in some cases *premature stability* may occur for values of $n < (k+\ell)$.

One observes that the power 3 plethysms for the two fundamental irreducible representations stabilise at $n = 3$ which is consistent with the stabilsation of the products $< s;(0) > \times < 1;(\mu) >$ and $< s;(1) > \times < 1;(\mu) >$ at $n = 3$ and for similar reasons stabilsation of power $N$ plethysms must occur at $n = N$ as observed. Again, premature stabilisation for individual plethysms may occur for $n < N$. Thus foe $N = 3$ all the plethysms stabilise at $n = 2$ except for $s_{1^3}(< s;(1) >)$ which stabilises at $n = 3$. Stabilisation for arbitrary $N$ occurs at $n = N - 1$ except for $s_{1^N}(< s;(1) >)$ which stabilises at $n = N$.

### Plethysms and conjugacy mappings

Below we give two short examples of plethysms with terms kept to weight 10.

$$
\begin{aligned}
s_4(< s;(0) >) = \; & < 2;(0) > \quad + < 2;(4) > \quad + < 2;(4^2) > \quad + < 2;(6) > \\
& + < 2;(62) > \quad + < 2;(73) > \quad +2 < 2;(8) > \quad + < 2;(91) > \\
& + < 2;(10 \;\; >
\end{aligned}
$$

$$
\begin{aligned}
s_{1^4}(< s;(1) >) = \; & < 2;(1^4) > \quad + < 2;(41^2) > \quad + < 2;(4^2) > \quad + < 2;(61^2) > \\
& + < 2;(62) > \quad + < 2;(73) > \quad +2 < 2;(81^2) > \quad + < 2;(91) >
\end{aligned}
$$

Looking at the above results one cannot help but be struck by the apparent simple mapping between them. Indeed looking at much more extensive tabulations one observes that the terms in $s_\lambda(< s;(0) >)$ are simply related to those of $s_{\tilde{\lambda}}(< s;(1) >)$ by a one-to-one mapping subject to the following simple rules:-

| | | | | | |
|---|---|---|---|---|---|
| $\lambda \vdash 2$ | $(0) \to (1^2)$ | | | | |
| $\lambda \vdash 3$ | $(0) \to (1^3)$ | $(a) \to (a1)$ | $(a1) \to (a)$ | | |
| $\lambda \vdash 4$ | $(0) \to (1^4)$ | $(a) \to (a1^2)$ | $(a1^2) \to (a)$ | | |
| $\lambda \vdash 5$ | $(0) \to (1^5)$ | $(a) \to (a1^3)$ | $(ab) \to (ab1)$ | $(ab1) \to (ab)$ | |
| $\lambda \vdash 6$ | $(0) \to (1^6)$ | $(a) \to (a1^4)$ | $(a1^4) \to (a)$ | $(ab) \to (ab1^2)$ | $(ab1^2) \to (ab)$ |

The explanation of such simple results remains unknown and deserves further study.

### Concluding remarks

The study of plethysms for the non-compact group $Sp(2n,R)$ throws up many surprises that could be of interest to combinatorialists. The study of plethysms for other non-compact groups, such as $SO(4,2)$ which plays a key role in Coulomb systems, is completely unknown. I hope in these notes I might stimulate others to consider some of the problems raised herein.

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