Combinatorics in Bordeaux

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1 The main domains

The historical and central domain of research in our group, conducted by Xavier Viennot, is undoubtly **enumerative combinatorics**. The problems that we study are often motivated by their connections with algorithmics or statistical physics. However, during the last few years, we have diversified our fields, as shown by the following list of (more or less new) topics:

- enumerative combinatorics,
- algorithmics,
- "dessins d'enfants",
- knot theory,
- umbral calculus,
- free Lie algebras and factorizations in the free monoid,
- covering codes,
- random generation,
- software.

We present here a few recent results, beginning with enumerative combinatorics.

2 Enumerative combinatorics

2.1 Animals

An *animal* on a graph G is a finite connected set of vertices of G (Figure 1). This simple notion can be fruitfully generalized into a notion of *directed animals* living on *oriented* graphs. Two main statistics are defined on animals: the *area* (number of vertices) and the *(site) perimeter*. The enumeration of animals according to these parameters is related to percolation models: more precisely, if one could enumerate a class of animals according to their perimeter and area, the corresponding percolation model would be automatically solved. Let us mention that no such model has been solved yet... Directed animals are also related to other models known in statistical physics as *hard particle models*.

We have also extensively studied *self-avoiding polygons* (*polygons* for short). They correspond to animals of the square lattice having no hole. The main statistics are in this case the area and the (usual) perimeter. Essentially, we are now able to enumerate according to these parameters any class of polygons having a *convexity property* [6, 7, 11]. For instance, here is the perimeter and area generating function of column-convex polygons [5]:

$$V(z,q) = z\frac{(1-z)X}{1+W+zX}$$

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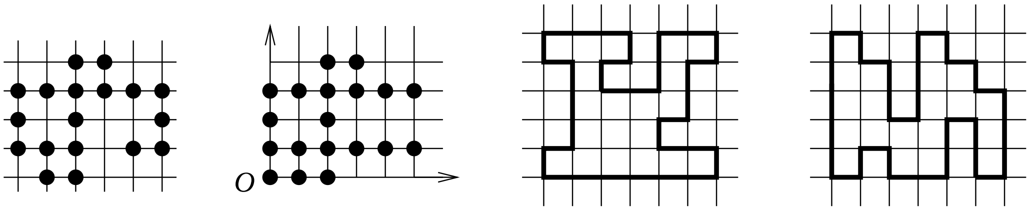

Figure 1: An animal, a directed animal, a polygon and a column-convex polygon.

where

$$X = \frac{zq}{(1-z)(1-zq)} + \sum_{n\geq 2} \frac{(-1)^{n+1} z^n (1-z)^{2n-4} q^{\binom{n+1}{2}} (z^2q)_{2n-2}}{(q)_{n-1}(zq)_{n-2}(zq)^2_{n-1}(zq)_n(z^2q)_{n-1}}$$

and

$$W = \sum_{n\geq 1} \frac{(-1)^n z^n (1-z)^{2n-3} q^{\binom{n+1}{2}} (z^2q)_{2n-1}}{(q)_n (zq)^3_{n-1}(zq)_n(z^2q)_{n-1}}.$$

As far as general animals are concerned, we have given several combinatorial explanations for the algebraicity of the area generating function of directed animals on the square and triangular lattices [2, 30]. More recently, we have refined these generating functions by taking into account an additional statistic [6]. Although the generating functions that we obtain are simple quadratic functions, we have not been able to prove these results in a combinatorial way. Our proof extends the link between animals and hard particle models.

2.2 Permutations with forbidden subsequences

One of our achievements in this domain is a combinatorial proof of a conjecture of West [31]: the number of *two-stack sortable permutations* of S_n is

$$T(n) = 2\frac{(3n)!}{(n+1)!(2n+1)!}.$$

This has been proved by describing a (complicated) bijection between these permutations and *non-separable rooted planar maps* having $n+1$ edges [15, 16, 21]. The number of such maps is known to be $T(n)$ since the works of Tutte [29]. West's conjecture had first been proved analytically by Zeilberger, using more than 10 hours of CPU time [32].

2.3 And also...

Here are a few other achievements, which we only mention without giving any details:

- a complete combinatorial theory of Padé approximants [26],
- a unified and simplified method for the enumeration of Baxter's permutations and Baxter's alternating permutations [17],
- enumeration of pairs of permutations according to various statistics [18, 19, 20].

3 Algorithmics in biology

A RNA molecule can be seen as a word on the four letter alphabet $\{A, C, G, U\}$, each letter denoting one of the four bases: Adenine, Cytosine, Guanine, Uracile. Such a sequence is actually fold onto itself due to the presence of hydrogen bonds that connect the bases. This structure is called the *secondary structure* of the sequence and can be encoded by a tree. We study several algorithmic problems posed by biologists. Two new algorithms have been designed: the first one allows to draw secondary structures in a nice way, the other aligns two primary or secondary structures according to large common factors [12, 13].

4 Dessins d'enfants

In his famous – though unpublisehd – manuscript "Esquisse d'un programme", A. Grothendieck has indicated the existence of a correspondence between combinatorial maps drawn on a surface and *Belyi pairs* formed with a Riemann surface S and a meromorphic function on S with only 3 critical values. The specialization to the case of maps of genus 0 having a single face gives essentially a one-to-one correspondence between bicolored plane trees and *Shabat polynomials*: a polynomial $P(z) \in \mathbb{C}[z]$ is a *Shabat polynomial* if there exist c_0 and $c_1 \in \mathbb{C}$ such that

$$P'(z) = 0 \Rightarrow P(z) \in \{c_0, c_1\}.$$

The polynomial associated with a tree has *algebraic* coefficients, which allows to associate an algebraic number field to every combinatorial tree. The absolute Galois group, that is, the group of automorphisms of the field $\overline{Q}$ of algebraic numbers, acts on bicolored plane trees. We study various combinatorial invariants of this action, such that the bicolored set of valencies of vertices, the symmetry group of a tree, the cartographic group, etc. A new operation of composition of plane trees is introduced; it corresponds to the composition of Shabat polynomials. The phenomenon of the "combinatorial factorization" of the discriminants of number fields in question is discovered. The work is carried out on algorithms that compute Shabat polynomials, and on visualization and geometric representation of trees and maps [3, 28].

5 Knot theory

The theory of knots is now very "hot" and extensively studied. Figure 2 shows a planar Gauss curve, a knot, and the corresponding *chord diagram*, that describes the sequence of crossings of the curve. It is important to note that some chord diagrams do *not* correspond to any Gauss curve. We have first enumerated (all) chord diagrams. Then, we have given an algorithm that generates them (this is not the difficult part) and then decides whether a chord diagram corresponds to a Gauss curve (and hence to a knot) or not — and gives the curve if the answer is "yes" [1].

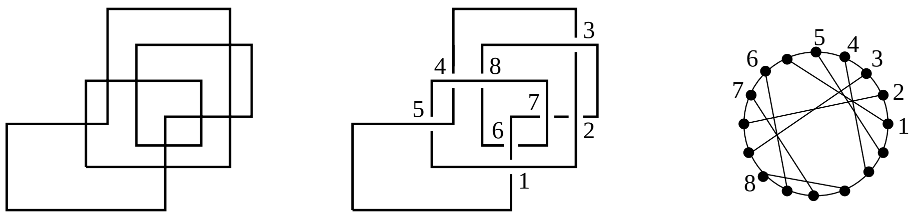

Figure 2: A Gauss curve, a knot and their chord diagram.

6 Combinatorics of bases of the free Lie algebra and factorization of free monoids

The free Lie algebra naturally appears as a key concept in nonlinear control theory; it has also recently been studied from the representation theory point of view [10]. Using rewriting system techniques, we gave an identity expressing the Fliess series as an exponential Lie series [25], turning it into a more useful tool in control theory. Bases of the free Lie algebras are closely linked to factorizations of the free monoids. We study the Spitzer-Foata factorization and exploit Duval's ideas to produce linear time algorithms for generating Spitzer-Foata words, and factorize words over the Spitzer-Foata factorization. This approach helps in studying combinatorial properties of orders associated to Spitzer-Foata factorizations [24].

7 Covering codes

Let A be a q-letter alphabet, and let u and $v \in A^n$ be two words of A^* having the same length. The *Hamming distance* between u and v is

$$d(u, v) = |\{i, 1 \leq i \leq n, u_i \neq v_i\}|.$$

A subset C of A^n is said to be a *covering code of radius* r if

$$\forall u \in A^n, \quad \exists v \in C \quad \text{s.t.} \quad d(u, v) \leq r.$$

Given q, n and r, the problem is to find the minimal size $K_q(n, r)$ of such a code. We have improved many lower bounds for $K_q(n, r)$, especially in the case $r = 1$ [22, 23].

8 Random generation and software

Random generation of combinatorial objects provides a useful tool for testing conjectures, or discovering asymptotic results. It also allows to test algorithms that handle these objects. We work on such questions, combining several methods: recursive methods, rejection methods, as well as a method of quasi-uniform generation based on the convergence of a Markov chain. We also make a great use of classical encodings of objects by words: this is not exactly a method of random generation, but words are usually simpler to generate that the objects they encode [9]. The new formalism of *object grammars* can also be used for random generation [14].

We have developed several efficient algorithms that generate animals (linear time), planar maps (time $O(n^2)$), paths, etc... They have been implemented on *CalICo*, which is a software environment that allows to compute, visualize classical combinatorial objets and handle them. It is formed of several specialized workshops (including MAPLE) that communicate with one another [8, 27].

A new MAPLE package that implements Rota's umbral calculus and some of its generalizations has also been developed [4].

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