Hamiltonicity exponent of digraphs

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0. Initiated by Sekanina [6], in the sixties and more intensively in the seventies there were considered powers of undirected graphs with special respect to their Hamiltonian behaviour. These investigations have resulted in a lot of interesting and partly very profound propositions; we only remind of the (simple) result of Sekanina [6] that the cube of every finite connected graph is Hamiltonian connected, or of the famous theorem of Fleischner [1] that the square of any nontrivial block is Hamiltonian. For directed graphs (digraphs), the situation is completely different; till not long ago nobody was seriously engaged in studying the analogous problem for digraphs (see [4]). The main reason for this situation is that for digraphs, these questions become much more complicated than in the undirected case. This paper intends to discuss some of the difficulties arising in the case of digraphs and to present some beginning results. All digraphs are supposed to be finite.

1. Let $G = (V, E)$ be a simple digraph with vertex set $V(G) = V$ and edge set $E(G) = E$; then every (directed) edge can be written as an ordered pair of uniquely determined different vertices, and (directed) paths and (directed) cycles are denoted by the associated sequences of their vertices. Now, such properties of G as to be *traceable, Hamiltonian, r-Hamiltonian* $(r = 1, 2, ...)$, *Hamiltonian connected* are defined in the same way as for undirected graphs by taking paths and cycles as directed ones, and this can also be done regarding the construction of higher powers of G, namely: The *k-th power* G^k of G is the digraph $G^k = (V_k, E_k)$ with

$$V_k := V \text{ and } E_k := \{(x, y) : x, y \in V \wedge 1 \leq d_G(x, y) \leq k\},$$

$k = 1, 2, ...$, where $d_G(x, y)$ denotes the distance from x to y in G, that is the length of a shortest path from x to y in G if there is any and ∞ otherwise.

In spite of these analogous definitions the Hamiltonian properties of powers of undirected and directed graphs are essentially different. A first difference is rather evident: For an undirected connected graph G on $n \geq 3$ vertices, G^3 is Hamiltonian. For a connected digraph G, however, it may happen that

$d_G(x,y) = \infty$ for some $x, y \in V$; then it follows that $d_{G^k}(x,y) = \infty$ for every $k = 1, 2, \ldots$, and no power G^k can become a Hamiltonian digraph. Therefore, if we want to improve the Hamiltonian behaviour of a digraph G by raising G to higher powers and to obtain Hamiltonicity we have to suppose G to be *strongly connected.* Notwithstanding, even under this suppositon it is not valid that G^3 (or any other power G^k) is Hamiltonian for every such G, but it holds:

For every $k \geq 1$ and every $n \geq 2k+4$ there is a strongly connected digraph G on n vertices such that G^k is not traceable.

For the proof, we consider the digraph G consisting of the path $(x_0, x_1, \ldots, x_r, y_0)$ with $r \geq k$ and of the t paths $(y_0, y_i, x_0), i = 1, \ldots, t$, with $t \geq k+2$; obviously, G is a strongly connected digraph on $r + t + 2 \geq 2k + 4$ vertices and G^k does not contain any Hamiltonian path.
This fact makes clear that for digraphs, we cannot expect to get results being satisfactory or simple in a similar way as for undirected graphs. On the other hand, for every strongly connected digraph G there is a k such that G^k is the complete digraph and therefore, has the Hamiltonian properties mentioned above. Thus it is sensible to define the following invariants for strongly connected digraphs G :
The *Hamiltonicity exponent* $e_H(G)$ by

$$e_H(G) := min\{k \geq 1 : G^k \text{ is Hamiltonian } \} \text{ (if } \quad |V| \geq 2),$$

and analogously, the *traceability exponent* $e_T(G)$, the *r-Hamiltonicity exponent* $e_{r-H}(G), r = 1, \ldots, |V| - 2$, and the *Hamiltonian connectedness exponent* $e_{HC}(G)$.
Clearly, $e_T(G) \leq e_H(G) \leq e_{HC}(G) \leq |V| - 1$ and $e_{r-H}(G) \leq |V| - 1$, i.e. we have trivial upper bounds for these invariants. Now a first task will be to find nontrivial upper bounds for these exponents, and animated by the satisfactory results for powers of undirected graphs we are especially interested in such upper bounds not depending on the vertex number. We have just seen that the latter kind of bounds does not exist for the whole class of all strongly connected digraphs, and therefore, we have to look for suitable subclasses. A very simple subclass - and the only one having been studied in this connection hitherto - is the class of directed cacti. Before changing over into considering such cacti besides, we point out to the following fact:
For the strong connectivity $c_s(G)$ of a digraph G it holds (cf. [4])

$$c_s(G^k) \geq min(kc_s(G), |V| - 1), \quad k = 1, 2, \ldots;$$

in view of this formula it may be expected that the strong connectivity of G will have effects also on the Hamiltonian properties of higher powers of G. As an example for first results in this direction we mention that

$$e_H(G) \leq \left\lceil \frac{|V|}{2r} \right\rceil$$

can be proved (cf. [7]) for every strongly r-connected digraph G with $|V| \geq 2, r \geq 1$.

2. A *directed cactus (*or *dicactus)* is defined to be a strongly connected finite simple digraph G every edge of which is contained in at most one (and therefore, in exactly one) cycle in G. Among the properties of dicacti we emphasize the following one which is important for the proofs of our main results.

Lemma 2.1: *For every (directed) cycle c in a dicactus G, after deleting the edges of c each component of the resulting digraph G' contains exactly one vertex in common with c and is a (possibly trivial) dicactus itself.*

P r o o f: Let H be a component of $G', x \in V(c) \cap V(H)$ and $e' = (x', x) \in E(c), e" = (x, x") \in E(c)$. Consider a maximal strongly connected subgraph H' of H with $x \in V(H')$. Then H' cannot contain any $y \in V(c)$ with $y \neq x$, because otherwise there would be an x, y-path in H' and therefore, a cycle $c' \neq c$ in G containing e'.
Assume $H' \neq H$; then there is an edge $(a', a) \in E(H)$ with $a' \in V(H'), a \notin V(H')$ or an edge $(b, b') \in E(H)$ with $b' \in V(H'), b \notin V(H')$, and in G there exists a path p from a to x (resp., from x to b). Assume that not all edges of p belong to H, and let $e_1 = (y, z)$ be the first (resp., $e_1 = (z, y)$ the last) edge on p not belonging to H. Then $e_1 \in E(c)$, and $y, z \in V(c)$ and $y \neq x$. Further we have a subpath p' of p in H with $p' = (a, ..., y)$ (resp., $p' = (y, ..., b)$) and therefore, a path $p_1 = (x, ..., a', a, ..., y)$ (resp., $p_1 = (y, ..., b, b', ..., x)$) in H because H' is strongly connected. Hence it follows that there exists a cycle $c' \neq c$ in G containing e' (resp., a cycle $c" \neq c$ in G containing $e"$); this is a contradiction in either case.
Thus, p is a path in H, and the digraph $H' \cup p \cup e^*$ with $e^* = (a', a)$ (resp., $e^* = (b, b')$) is a strongly connected subgraph of H containing x, in contradiction to the maximality of H'. This yields $H' = H$, and the asserted properties of H are proved.□
Using Lemma 2.1 and induction to the number of cycles in G we can prove (cf. [4])

Lemma 2.2: *Let G be a dicactus; then for every $e = (x, y) \in E(G)$ there is a Hamiltonian path $h = (y, y', ..., x', x)$ in G^3 with $d_G(y, y'), d_G(x', x) \in \{1, 2\}$.*

As immediate consequences of Lemma 2.2 we obtain

Corollary 2.3 (cf. [4]): *For every dicactus G on at least two vertices, $e_H(G) \leq 3$,*

Corollary 2.4 (cf. [5]): *For every dicactus G on at least three vertices, $e_{1-H}(G) \leq 3$.*

Lemma 2.2 is also an important tool for proving the following statement which improves Corollary 2.3.

Theorem 2.5 (cf.[4]): *For every dicactus* $G, e_{HC}(G) \leq 3$.

Obviously, the upper bound 3 is best possible in Corollary 2.3 and therefore, in Corollary 2.4 and Theorem 2.5. (It is sufficient to take the dicactus arising from an undirected 3-star by replacing at first every edge by an undirected path of length 2 and then every edge by a directed cycle of length 2.)

3. The results just mentioned give rise to the question which subclasses of dicacti have a Hamiltonian (or 1-Hamiltonian or Hamiltonian connected) square. For this end we consider dicacti with the following property:
A dicactus G is said to fulfil the *2-degree-condition* iff every cycle in G contains a vertex of degreee (= valency) 2 in G (i.e. a vertex x with $v(x : G) = v^+(x : G) + v^-(x : G) = 2$).
By induction to the number of edges we obtain (cf. [5])

Lemma 3.1: *For every dicactus* G *fulfilling the 2-degree-condition and any edge* $e = (x, y) \in E(G)$ *there is a Hamiltonian path* $h = (y, ..., x)$ *in* G^2.

This yields immediately

Corollary 3.2: *For every dicactus* G *on at least two vertices fulfilling the 2-degree-condition,* $e_H(G) \leq 2$.

However, the suppositions in Corollary 3.2 are not sufficient for G to have a 1-Hamiltonian square or a Hamiltonian connected square; this can be easily seen by simple examples. (For instance, let G_r be the cycle $(0, 1, ..., 4r - 1, 0)$ of length $4r$, $r = 1, 2, ...$; then there is no Hamiltonian path in G_r^2 from 0 to $2r$. Similarly, for the dicactus G consisting of the three cycles (x, y, z, x), (y, a, y), (z, b, z), the digraph $G^2 - x$ is not Hamiltonian.) By a more detailed proof exhausting the suppositions on G most accurately one can improve Lemma 3.1 and obtain further statements (cf. [5]); after all, many problems concerning dicacti with 2-degree-condition remain open.

4. In this final section let us consider dicacti of the simplest shape, namely those having at most one cycle of length greater than 2. If all cycles in G are of length 2 (Case 1) it follows that G is a bidirected simple connected digraph without cycles of length > 2. Replacing every cycle of length 2 in G by an undirected edge we get the associated undirected graph G_u which is obviously a tree. Of course, if $|V(G)| \geq 3$ (the case $|V(G)| = 2$ is trivial), G^2 is Hamiltonian iff G_u^2 is Hamiltonian, and this occurs iff the tree G_u is a caterpillar, according a result in [2]. Thus, we obtain a simple characterization of the dicacti of Case 1 satisfying $e_H(G) \leq 2$.

A dicactus G containing exactly one cycle of length ≥ 3 (Case 2) is called a *unicyclic dicactus*. Denote (in such a G) this cycle by $c = (x_0, x_1, ..., x_l = x_0)$, $l \geq 3$, and the l components of the digraph $G - E(c)$ by $G_1, ..., G_l$ where $x_j \in V(G_j), j = 1, ..., l$. Then every G_j is a dicactus of Case 1, and using these denotations we can show (cf.[4]) the following theorem characterizing all unicyclic dicacti fulfilling $e_H(G) \leq 2$.

Theorem 4.1: *For a unicyclic dicactus G, it holds $e_H(G) \leq 2$ iff the following conditions are satisfied:*
(1) Every undirected graph $(G_j)_u, j = 1, ..., l$, is a caterpillar (with at least 3 vertices), a path of length 1 or a trivial graph (i.e. a point graph);
(2) if $(G_j)_u - x_j$ for some $j \in \{1, ..., l\}$ has two nontrivial components then there is an x_i on the cycle c, $i \in \{1, ..., l\}$, such that $G_i - x_i$ is empty (i.e. without any vertex) and for every vertex $x_t \neq x_j, t \in \{1, ..., l\}$, belonging to the directed path on c from x_j to x_i at most one component of $(G_t)_u - x_t$ is nontrivial;
(3) if $(G_j)_u - x_j$ for some $j \in \{1, ..., l\}$ has a nontrivial component then x_j either belongs to the path g_j arising from $(G_j)_u$ by deleting all vertices of degree 1, or it has degree 1 in $(G_j)_u$ and is a neighbour of an endvertex of this path g_j.

The sufficiency is proved by construction: Every $(G_j)_u, j = 1, ..., l$, is a caterpillar (possibly degenerated), and we can construct suitable Hamiltonian paths in $(G_j)_u^2$ by Neuman's Theorem on trees [3] and so in G_j^2; these paths are composed to a Hamiltonian cycle in G^2. The proof of the necessity is straightforward, indeed, it is rather complicated in the details.
One of the most essential tasks of further investigations on this subject will be to find other suitable subclasses of strongly connected digraphs and good upper bounds for their Hamiltonian exponents.

References

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