On alternating products of graph relations

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Abstract

It is well-known that one can give an elegant version of the Kuratowski-type theorem for the projective plane by means of the five elementary relations $R_i, i = 0, 1, \ldots, 4$, on the set Γ of all finite, undirected graphs without loops and multiple edges. Furthermore, these five relations play an interesting role in didactics of mathematics. Following a theory given in [2], C.Thies investigates them in [3]. In order to show that $R_0, R_1, \ldots, R_4$ are an appropriate curriculum he has to deal with so-called alternating products

$$RT(R_i) \circ RT(R_i^{-1}) \circ RT(R_i) \circ RT(R_i^{-1}) \circ \ldots$$

or

$$RT(R_i^{-1}) \circ RT(R_i) \circ RT(R_i^{-1}) \circ RT(R_i) \circ \ldots,$$

$i = 0, 1, \ldots, 4$, where $RT(R_i)$ denotes the reflexive transitive closure of R_i and $RT(R_i^{-1}) = RT(R_i)^{-1}$ the reflexive, transitive closure of R_i^{-1}. Here, it is shown that, in case of $i = 0$, there exists exactly one alternating product in the set of all alternating products of R_0, in case of $i = 3$ and $i = 4$ the sets of all alternating products of R_3 and R_4 are infinite sets.

1 Introduction

In the theory of embedding graphs into orientable and nonorientable surfaces, the five elementary relations $R_0, R_1, \ldots, R_4$, introduced in [1], play an important role because it is possible to give elegant and precise definitions of the subgraph relation and the subdivision relation and the minor relation on the set Γ of all finite, undirected graphs without loops and multiple edges. Observing that an ordered pair $(G, H) \in \Gamma \times \Gamma$ belongs to R_0 iff H arises from G by removing an edge $e \in E(G)$ or an isolated vertex $v \in V(G)$ the subgraph relation is equal to the reflexive, transitive closure $RT(R_0)$ of R_0.

An ordered pair $(G, H) \in \Gamma \times \Gamma$ belongs to R_1 iff H arises from G by contracting an edge $e = \{v, v^{'}\} \in E(G)$, provided at least one of the two endpoints $v, v^{'}$ has got degree 1 or 2, and $v, v^{'}$ do not have a common neighbour in G. If $U_1 = R_0 \cup R_1$, then the well-known subdivision relation is equal to the reflexive, transitive closure $RT(U_1)$ of $U_1 = R_0 \cup R_1$.

To give a precise definition of the famous minor relation on Γ we need R_0, R_1 and R_2. An ordered pair $(G, H) \in \Gamma \times \Gamma$ belongs to R_2 iff H arises from G by the contraction of an edge $e = \{v, v^{'}\} \in E(G)$ provided $v, v^{'}$ do not have a common neighbour in G, and both v and $v^{'}$ have got a degree ≥ 3.

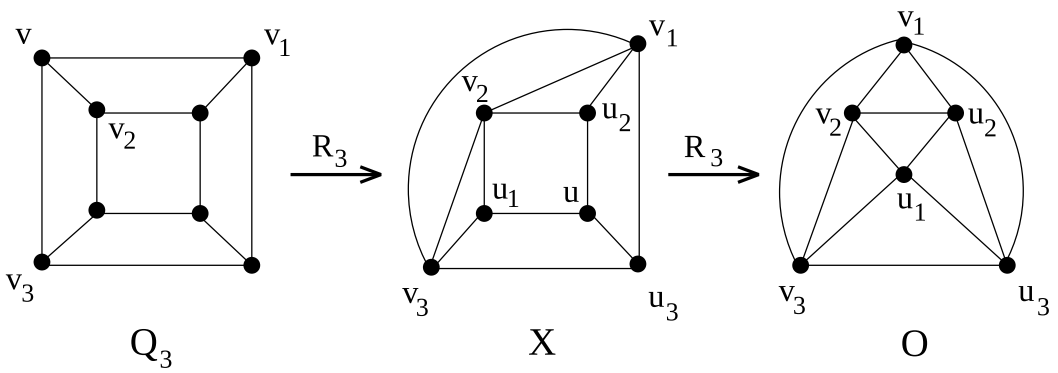


Figure 1:

If U_2 denotes the union $U_2 = R_0 \cup R_1 \cup R_2$, then the minor relation on Γ is equal to the reflexive, transitive closure $RT(U_2)$ of $U_2 = R_0 \cup R_1 \cup R_2$.

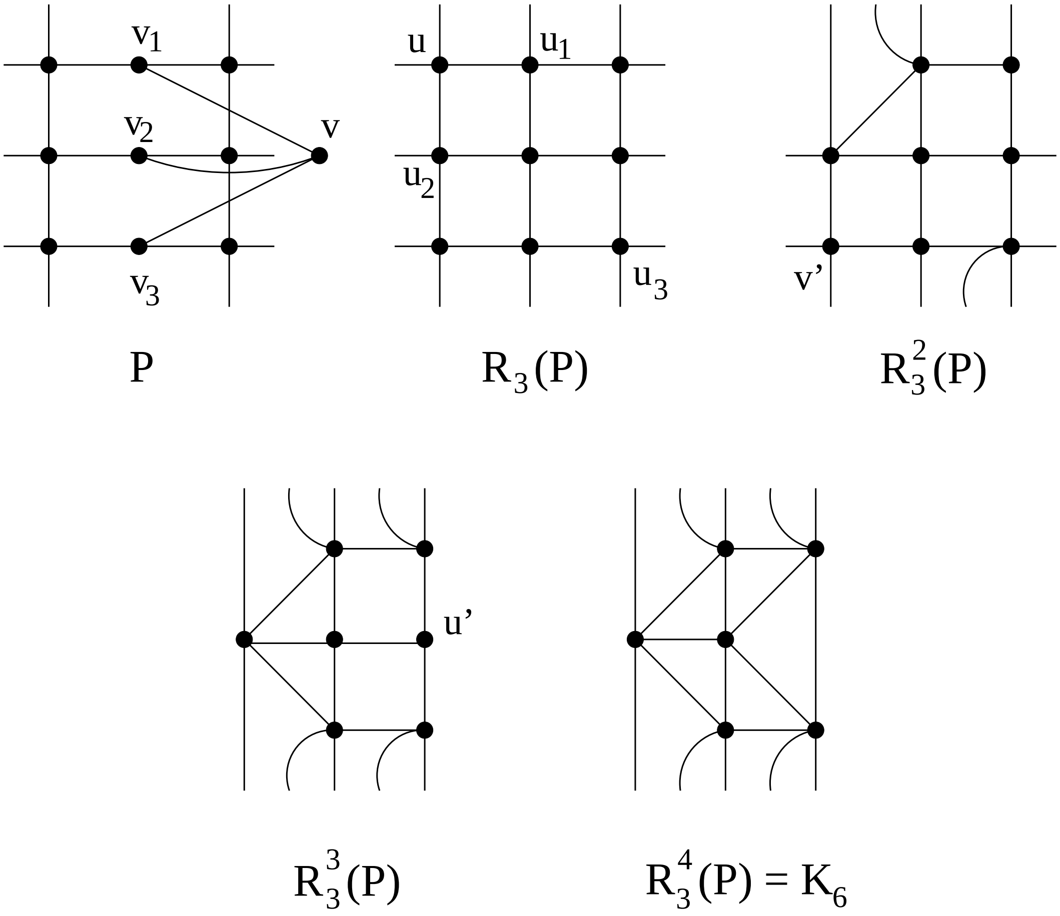


Figure 2:

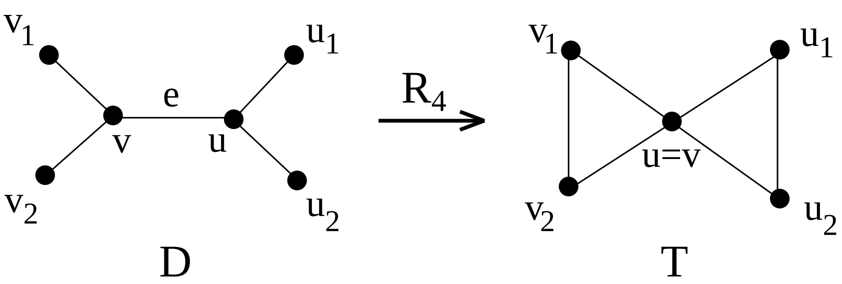


Figure 3:

It is almost superflous to mention that $RT(R_0), RT(U_1), RT(U_2)$ are partial ordering relations on Γ while R_0, R_1, R_2 are not partial ordering relations on Γ.

Now, it remains to give the definition of R_3 and R_4. The great advantage of R_3 and R_4 and their reflexive, transitive closures $RT(U_3), RT(U_4), U_i = R_0 \cup R_1 \cup \ldots \cup R_i, i = 3, 4$, is that we need essentially less graphs in order to characterize the set of all graphs not embeddable into an orientable or nonorientable surface.

Due to [1], an ordered pair $(G, H) \in \Gamma \times \Gamma$ belongs to R_3 iff H arises from G by substituting a star $S_3 = \{v\} * \{v_1, v_2, v_3\}, v, v_1, v_2, v_3 \in V(G)$, by the triangle $K_3 = (v_1, v_2, v_3)$ provided $\{v_1, v_2, v_3\}$ is an independent set in V.

Figure 1 shows that the octahedron O arises from the cube Q_3 by applying R_3 twice. In other words, we can say that the ordered pair $(Q_3, O) \in R_3 \circ R_3$, or equivalently $(Q_3, O) \in RT(R_3)$.

Figure 2 shows a representation of the Petersen graph P and four other graphs $R_3(P), R_3^2(P), R_3^3(P), R_3^4(P)$ in the projective plane such that it is clear that the ordered pair $(P, K_6) \in R_3 \circ R_3 \circ R_3 \circ R_3$, or equivalently $(P, K_6) \in RT(R_3)$.

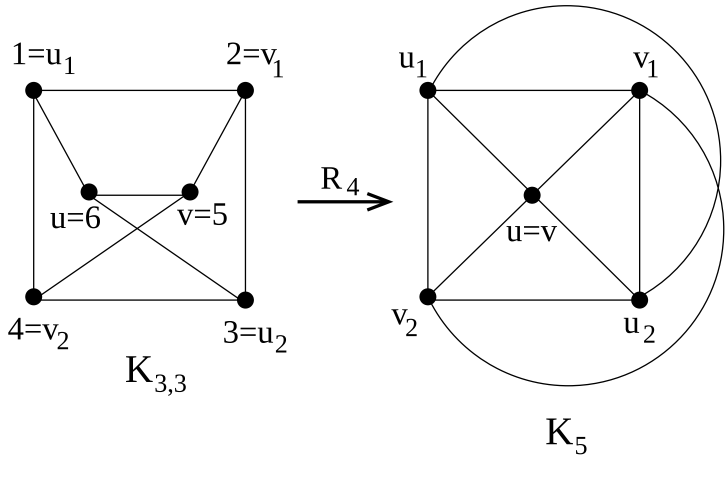


Figure 4:

Due to [1] an ordered pair $(G, H) \in \Gamma \times \Gamma$ belongs to R_4 iff H arises from G by substituting a double star $D \subseteq G$ by a double triangle T (Figure 3) provided the edges $\{v_1, v_2\}$ and $\{u_1, u_2\}$ do not belong to G.

Figure 4 shows the decisive advantage of R_4 because the two Kuratowski graphs K_5 and $K_{3,3}$ have got the property that the ordered pair $(K_{3,3}, K_5) \in R_4$.

2 Alternating products

In order to be able to define the concept of an alternating product it makes sense to introduce the converse relations of $R_i, i = 0, 1, 2, 3, 4$. Since we are going to restrict ourselves to the converse relations R_0^{-1}, R_3^{-1} and R_4^{-1} we omit the definitions of R_1^{-1} and R_2^{-1}.

Observing the definitions of R_0, R_3 and R_4 we obtain

a) an ordered pair $(G, H) \in \Gamma \times \Gamma$ belongs to R_0^{-1} iff H arises from G by adding an edge $e \in \mathfrak{P}_2(V) - E$ or a new isolated vertex $v \notin V$,

b) an ordered pair $(G, H) \in \Gamma \times \Gamma$ belongs to R_3^{-1} iff H arises from G by substituting a triangle $C_3 = (v_1, v_2, v_3)$ by the star $v * \{v_1, v_2, v_3\}$ such that v is a new vertex , not belonging to G,

c) an ordered pair $(G, H) \in \Gamma \times \Gamma$ belongs to R_4^{-1} iff H arises from G by substituting a double triangle T (1-amalgamation of two disjoint triangles) by a double trihedral D (double star) (Figure 3).

Observing $RT(R_i^{-1}) = RT(R_i)^{-1}, i = 0, 3, 4$, we are able to introduce alternating products $\mathfrak{A}_i(n)$ and $\mathfrak{B}_i(n), i = 0, 3, 4$, of $n \in \mathbb{N}, n \geq 2$, factors as follows

$$\mathfrak{A}_i(n) = \begin{cases} RT(R_i) \circ RT(R_i)^{-1} \circ \ldots \circ RT(R_i) \circ RT(R_i)^{-1} & \text{if } n \equiv 0 \bmod 2, \\ RT(R_i) \circ RT(R_i)^{-1} \circ \ldots \circ RT(R_i)^{-1} \circ RT(R_i) & \text{if } n \equiv 1 \bmod 2, \end{cases}$$

$$\mathfrak{B}_i(n) = \begin{cases} RT(R_i)^{-1} \circ RT(R_i) \circ \ldots \circ RT(R_i)^{-1} \circ RT(R_i) & \text{if } n \equiv 0 \bmod 2, \\ RT(R_i)^{-1} \circ RT(R_i) \circ \ldots \circ RT(R_i) \circ RT(R_i)^{-1} & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

If we denote the sets of all alternating products of $R_i, i = 0, 3, 4$, by $\mathcal{A}_i$ and $\mathcal{B}_i$, defined as follows

$$\mathcal{A}_i = \{\mathfrak{A}_i(n) | n \in \mathbb{N} - \{1\}\}, \mathcal{B}_i = \{\mathfrak{B}_i(n) | n \in \mathbb{N} - \{1\}\}, \tag{1}$$

we are going to investigate the question whether the sets $\mathcal{A}_i$ and $\mathcal{B}_i, i = 0, 3, 4$, are finite or infinite.

3 The finiteness of $\mathcal{A}_0$ and $\mathcal{B}_0$

Since first investigations lead to suppose that

$$\mathcal{A}_0 = \{\mathfrak{A}_0(2), \mathfrak{A}_0(3), \mathfrak{A}_0(4), \ldots\} \tag{2}$$

and

$$\mathcal{B}_0 = \{\mathfrak{B}_0(2), \mathfrak{B}_0(3), \mathfrak{B}_0(4), \ldots\} \tag{3}$$

are finite we first restrict ourselves to $\mathfrak{A}_0(2)$ and $\mathfrak{B}_0(2)$, and prove

Theorem 1

$$\begin{aligned} &\mathfrak{A}_0(2) = \mathfrak{B}_0(2) = \Gamma \times \Gamma \Longleftrightarrow \\ &\quad RT(R_0) \circ RT(R_0)^{-1} = RT(R_0)^{-1} \circ RT(R_0) = \Gamma \times \Gamma. \end{aligned} \tag{4}$$

Proof: Because of $RT(R_0) \circ RT(R_0)^{-1} \subseteq \Gamma \times \Gamma$ and $RT(R_0)^{-1} \circ RT(R_0) \subseteq \Gamma \times \Gamma$ it suffices to show the two inclusions (a) and (b) by

(a) $RT(R_0) \circ RT(R_0)^{-1} \supseteq \Gamma \times \Gamma$

and

(b) $RT(R_0)^{-1} \circ RT(R_0) \supseteq \Gamma \times \Gamma$.

Ad (a) : Observing the definitions of a subset and a composition of relations the inclusion (a) is equivalent to the proposition

$$\bigwedge_{(G,H) \in \Gamma \times \Gamma} \bigvee_{X \in \Gamma} [(G, X) \in RT(R_0) \wedge (X, H) \in RT(R_0)^{-1}]. \tag{5}$$

Assume (G, H) is any ordered pair in $\Gamma \times \Gamma$. As the intersection graph $G \cap H$ is a subgraph of both G and H we obtain $(G, G \cap H) \in RT(R_0)$ and $(H, G \cap H) \in RT(R_0)$ such that $(G \cap H, H) \in RT(R_0)^{-1}$. Putting $X = G \cap H$ we obtain $(G, G \cap H) \in RT(R_0)$ and $(G \cap H, H) \in RT(R_0)^{-1}$ such that (5) is true. This fact proves the inclusion (a).

Ad (b): Similarly to (a) the inclusion (b) is equivalent to the proposition (6) by

$$\bigwedge_{(G,H) \in \Gamma \times \Gamma} \bigvee_{Y \in \Gamma} [(G, Y) \in RT(R_0)^{-1} \wedge (Y, H) \in RT(R_0)]. \tag{6}$$

The proof of (6) is done by choosing $Y = G \cup H$(= union graph of G and H) for any graphs $G, H \in \Gamma$ for G and H are subgraphs of $Y = G \cup H$ implying $(G \cup H, G) \in RT(R_0), (G \cup H, H) \in RT(R_0)$ and $(G, G \cup H) \in RT(R_0)^{-1}$. This proves (b).

The answer of the question asked above concerning the finiteness of $\mathcal{A}_0$ and $\mathcal{B}_0$ is an immediate consequence of theorem 1 and says as follows

Theorem 2 *$\mathcal{A}_0$ and $\mathcal{B}_0$ are equal, finite sets and consist of exactly one element $x = \Gamma \times \Gamma$, i.e. $\mathcal{A}_0 = B_0 = \{\Gamma \times \Gamma\}$.*

Proof: Observing Theorem l and the fact that the composition of relations on the sets of all relations on Γ is an associative binary operation, the statement of Theorem 2 has been proved iff the two inclusions (7) and (8) by

$$\Gamma \times \Gamma \subseteq (\Gamma \times \Gamma) \circ RT(R_0) \tag{7}$$

and

$$\Gamma \times \Gamma \subseteq (\Gamma \times \Gamma) \circ RT(R_0)^{-1} \tag{8}$$

are true. It is obvious that (7) is true iff the statement

$$\bigwedge_{(G,H)\in\Gamma\times\Gamma} \bigvee_{X\in\Gamma} [(G, X) \in \Gamma \times \Gamma \wedge (X, H) \in RT(R_0)] \tag{9}$$

holds. Choosing $X = H$ for any $(G, H) \in \Gamma \times \Gamma$ we recognize immediately that (7) is true.

The proof of (8) is quite the same as the one of (7) such that we are allowed to omit details. This proves Theorem 2.

After this beautiful proof we are going to turn to $\mathcal{A}_3$ and $\mathcal{B}_3$ and want to investigate the question whether these two sets are also finite.

4 The infinity of $\mathcal{A}_3$ and $\mathcal{B}_3$

Figure 5 shows the five graphs G, G_1, G_2, G_3 and H that can be characterized by means of R_3 and R_3^{-1} represented in Figure 6. It shows that the ordered pair (G, H) belongs to $\mathfrak{A}_3(4) = RT(R_3) \circ RT(R_3)^{-1} \circ RT(R_3) \circ RT(R_3)^{-1}$

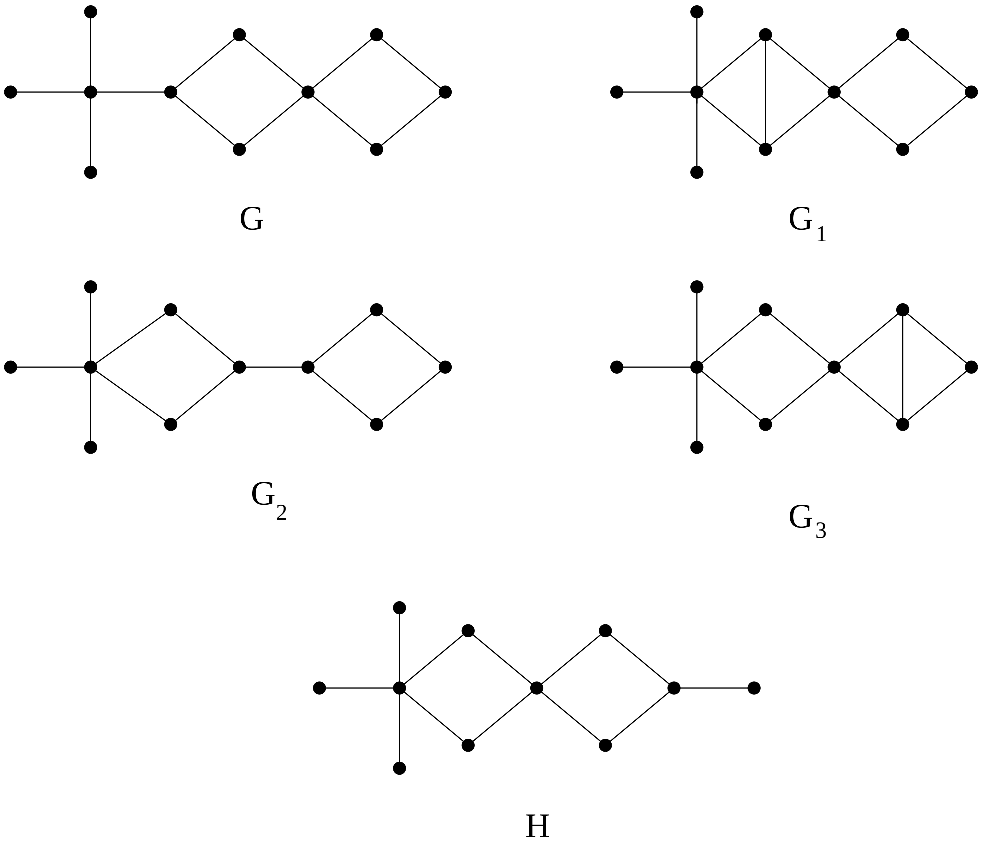


Figure 5:

but it is neither an element of $\mathfrak{A}_3(3)$ nor of $\mathfrak{A}_3(2)$. This fact implies that $\mathfrak{A}_3(4) \neq \mathfrak{A}_3(i), i = 2, 3$.

Therefore, it is obvious that $\mathcal{A}_3$ has at least the three elements $\mathfrak{A}_3(2), \mathfrak{A}_3(3)$ and $\mathfrak{A}_3(4)$.

This fact implies that the situation of $\mathcal{A}_3$ and $\mathcal{B}_3$ is quite different from the situation of $\mathcal{A}_0$ and $\mathcal{B}_0$ such that we suppose that are $\mathcal{A}_3$ and $\mathcal{B}_3$ are even infinite sets. In order to give a proof of this assumption we generalize the example of figure 5 and prove

Theorem 3 *$\mathcal{A}_3$ and $\mathcal{B}_3$ are infinite sets.*

Proof: (1) In order to show that $\mathcal{A}_3$ is infinite we suppose to argue by con-

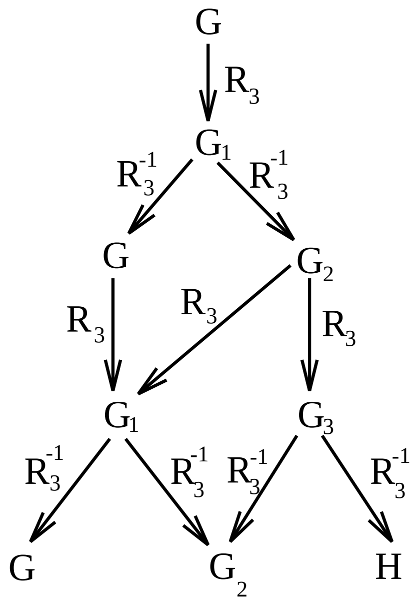


Figure 6:

tradiction that $\mathcal{A}_3$ is finite. Then there necessarily exists a natural number $n \in \mathbb{N}, n \geq 3$, such that $\mathcal{A}_3 = \{\mathfrak{A}_3(2), \mathfrak{A}_3(4), \ldots, \mathfrak{A}_3(n-1)\}$ with the property that there is an element $k \in \{2, 3, \ldots, n-1\}$ for which the equality $\mathfrak{A}_3(k) = \mathfrak{A}_3(n)$ holds. This is a contradiction to statement (10) by

$$\bigwedge_{i \in \{2,3,\ldots,n-1\}} \mathfrak{A}_3(n) \not\subseteq \mathfrak{A}_3(i). \tag{10}$$

In order to show that (10) is true we argue as follows. (10) is obviously shown, if and only if, we give an ordered pair $(G, H) \in \mathfrak{A}_3(n)$ not belonging to $\mathfrak{A}_3(i)$ for any $i \in \{2, 3, \ldots, n-1\}$. It turns out that it is even possible to show that there is an ordered pair $(G, H) \in \mathfrak{A}_3(n)$ not belonging to $\mathfrak{A}_3(i)$ for every $i \in \{2, 3, \ldots, n-1\}$. If $n = 2k, k \geq 1$, is even then we generalize the graph G from Figure 5. Thus we consider the graphs $G = G(k)$ and $H = G_n$ depicted in Figure 7. Generalizing the diagram of Figure 6 we recognize that the ordered pair $(G, H) = (G(k), G_n)$ is an element of

$$\mathfrak{A}_3(n) = RT(R_3) \circ RT(R_3)^{-1} \circ \ldots \circ RT(R_3) \circ RT(R_3)^{-1} \tag{11}$$

but

$$(G, H) = (G(k), G_n) \notin \mathfrak{A}_3(i)$$

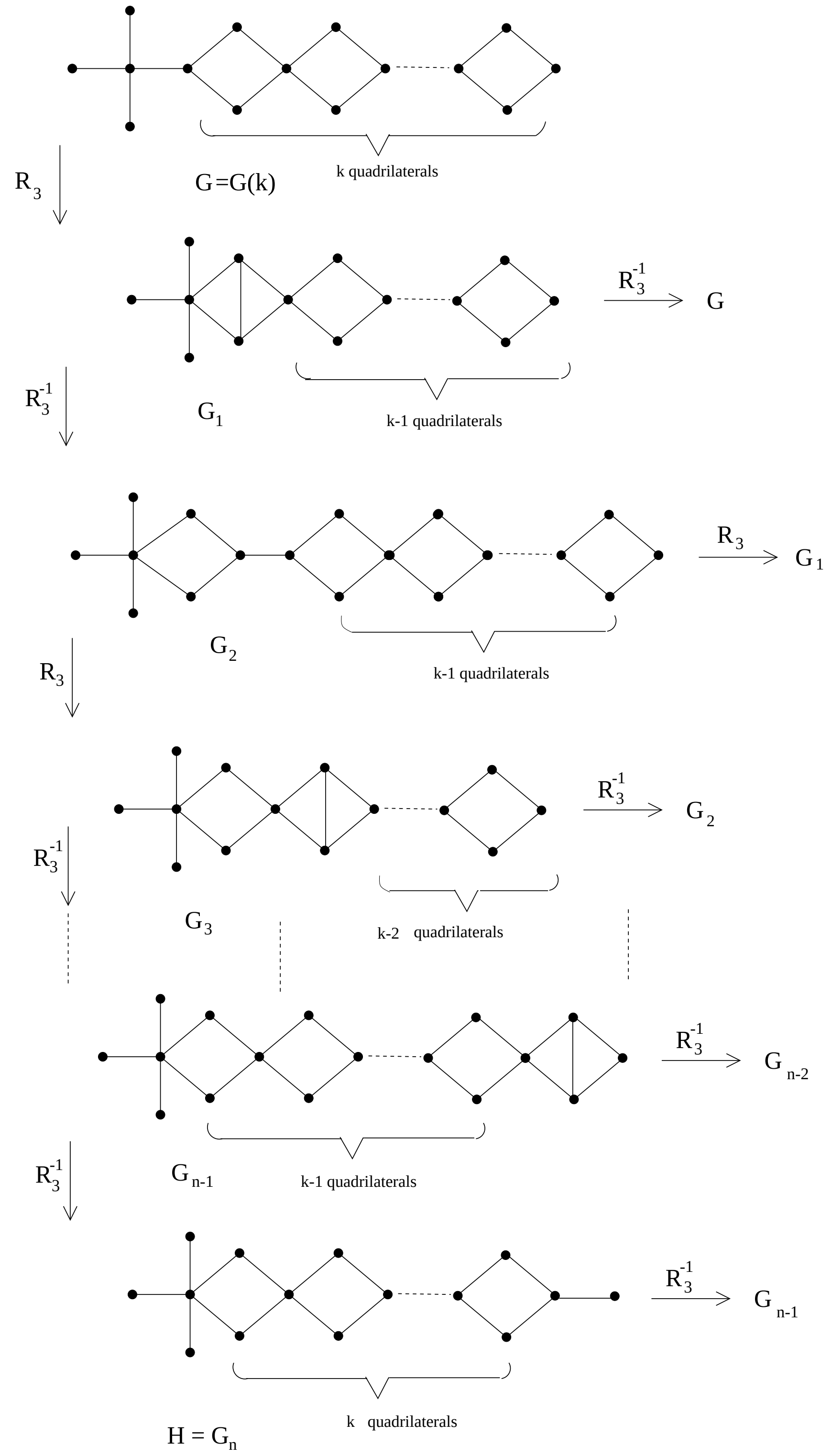

k quadrilaterals
R_3
G = G(k)
R_3^{-1}
G
R_3^{-1}
G_1
k-1 quadrilaterals
R_3
G_1
G_2
k-1 quadrilaterals
R_3
R_3^{-1}
G_2
R_3^{-1}
G_3
k-2 quadrilaterals
R_3^{-1}
G_{n-2}
R_3^{-1}
G_{n-1}
k-1 quadrilaterals
R_3^{-1}
G_{n-1}
k quadrilaterals
$H = G_n$

Figure 7:

for each $i = 2, 3, \ldots, n-1$.

Since the proof of these two statements is a matter of routine checking, we are allowed to omit details. In the case of odd $n = 2k+1, k \geq 1$, we can argue by observing Figure 8 and choosing $(G, H) = (G'(k), G'_n)$. A routine checking shows that $(G, H) \in \mathfrak{A}_3(n)$, but $(G, H) \notin \mathfrak{A}_3(i)$ for each $i = 2, 3, \ldots, n-1$. After all, (10) is proved. This implies that our assumption is false and $\mathcal{A}_3$ is infinite.

(2) To argue by contradiction we assume that $\mathcal{B}_3$ is finite. Then there exists an integer $n \geq 3$ such that $\mathcal{B}_3 = \{\mathfrak{B}_3(2), \mathfrak{B}_3(3), \ldots, \mathfrak{B}_3(n-1)\}$ with the property that there is an element $k \in \{2, 3, \ldots, n-1\}$ for which $\mathfrak{A}_3(n)$ is equal to $\mathfrak{A}_3(k)$. This equality is a contradiction to statement (12) by

$$\bigwedge_{i \in \{2,3,\ldots,n-1\}} \mathfrak{B}_3(n) \not\subseteq \mathfrak{B}_3(i). \tag{12}$$

We prove (12) by showing the existence of an ordered pair $(G, H) \in \mathfrak{B}_3(n)$ not belonging to $\mathfrak{B}_3(i)$ for every $i \in \{2, 3, \ldots, n-1\}$.

It is useful to distinguish the two cases

(a) $\quad n \equiv 0 \mod 2$ and
(b) $\quad n \equiv 1 \mod 2$.

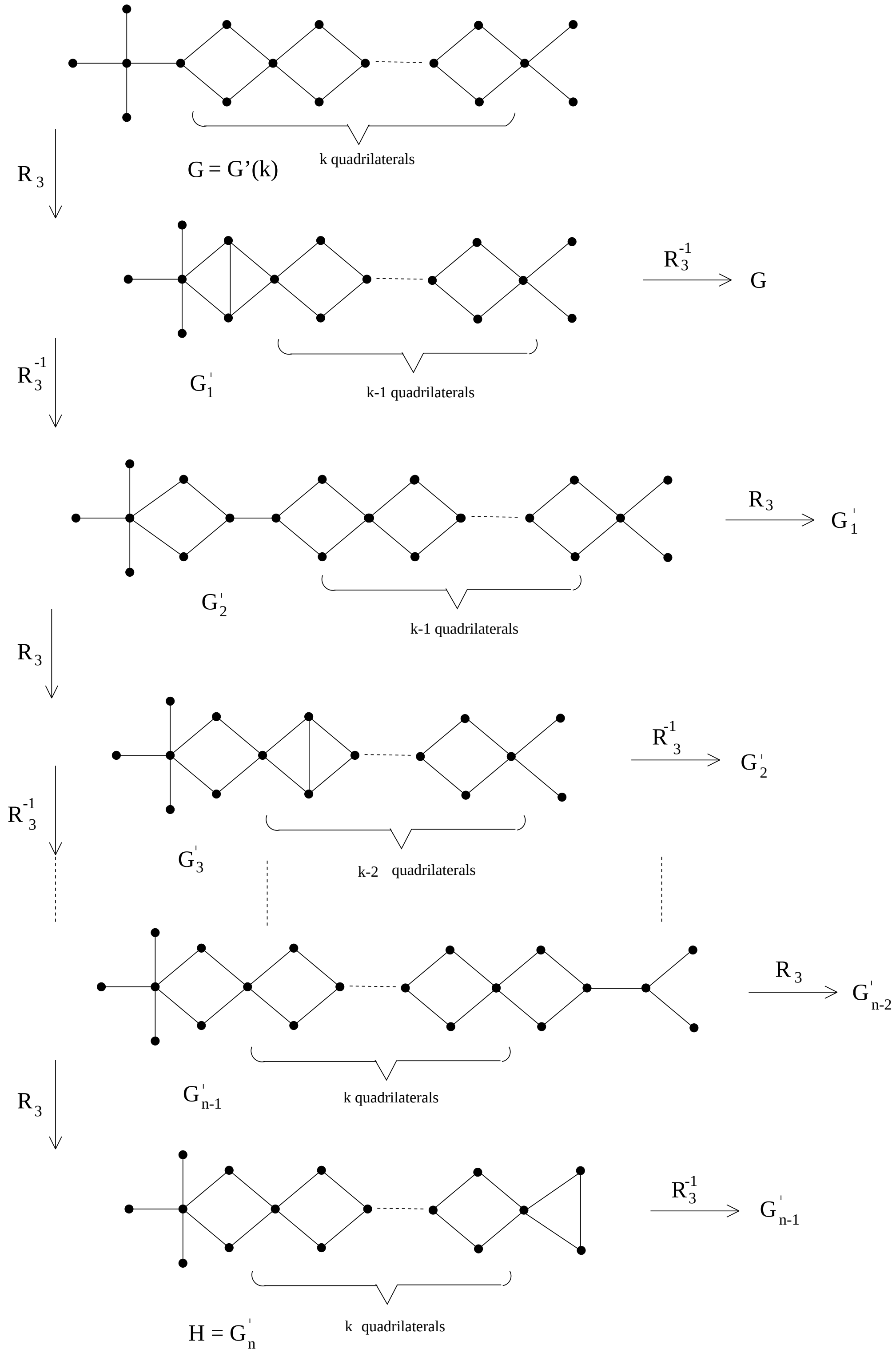
In (a) we take $G = X(k), n = 2k+2, k \geq 1, k = \frac{n-2}{2}$ and $H = X_{2k}$ represented in Figure 9. Observing Figure 9 the proof of $(G, H) \in \mathfrak{B}_3(n)$ and $(G, H) \notin \mathfrak{B}_3(i)$ for each $i = 2, 3, \ldots, n-1$ is a matter of routine checking.

In case (b), n is odd such that $n = 2k+1, k \geq 1$. Then we choose for G, H the graphs $Y(k)$ and Y_{2k+1} depicted in Figure 10. It is easy to see that $(G, H) \in \mathfrak{B}_3(n)$ and $(G, H) \notin \mathfrak{B}_3(i)$ for each $i = 2, 3, \ldots, n-1$. Hence, the assumption is not true. That means that $\mathcal{B}_3$ is infinite.

5 The infinity of $\mathcal{A}_4$ and $\mathcal{B}_4$

In this section we turn toward the elementary relation R_1. After having seen that $\mathcal{A}_3$ and $\mathcal{B}_3$ are infinite sets it is not difficult to guess that $\mathcal{A}_4$ and $\mathcal{B}_4$ are also infinite sets. The confirmation of this guess is given in

Theorem 4 *$\mathcal{A}_4$ and $\mathcal{B}_4$ are infinite sets.*



k quadrilaterals
G = G'(k)
R_3
R_3^{-1}
G
k-1 quadrilaterals
G'_1
R_3^{-1}
R_3
G'_1
G'_2
k-1 quadrilaterals
R_3
R_3^{-1}
G'_2
G'_3
k-2 quadrilaterals
R_3^{-1}
R_3
G'_{n-2}
G'_{n-1}
k quadrilaterals
R_3
R_3^{-1}
G'_{n-1}
H = G'_n
k quadrilaterals

Figure 8:

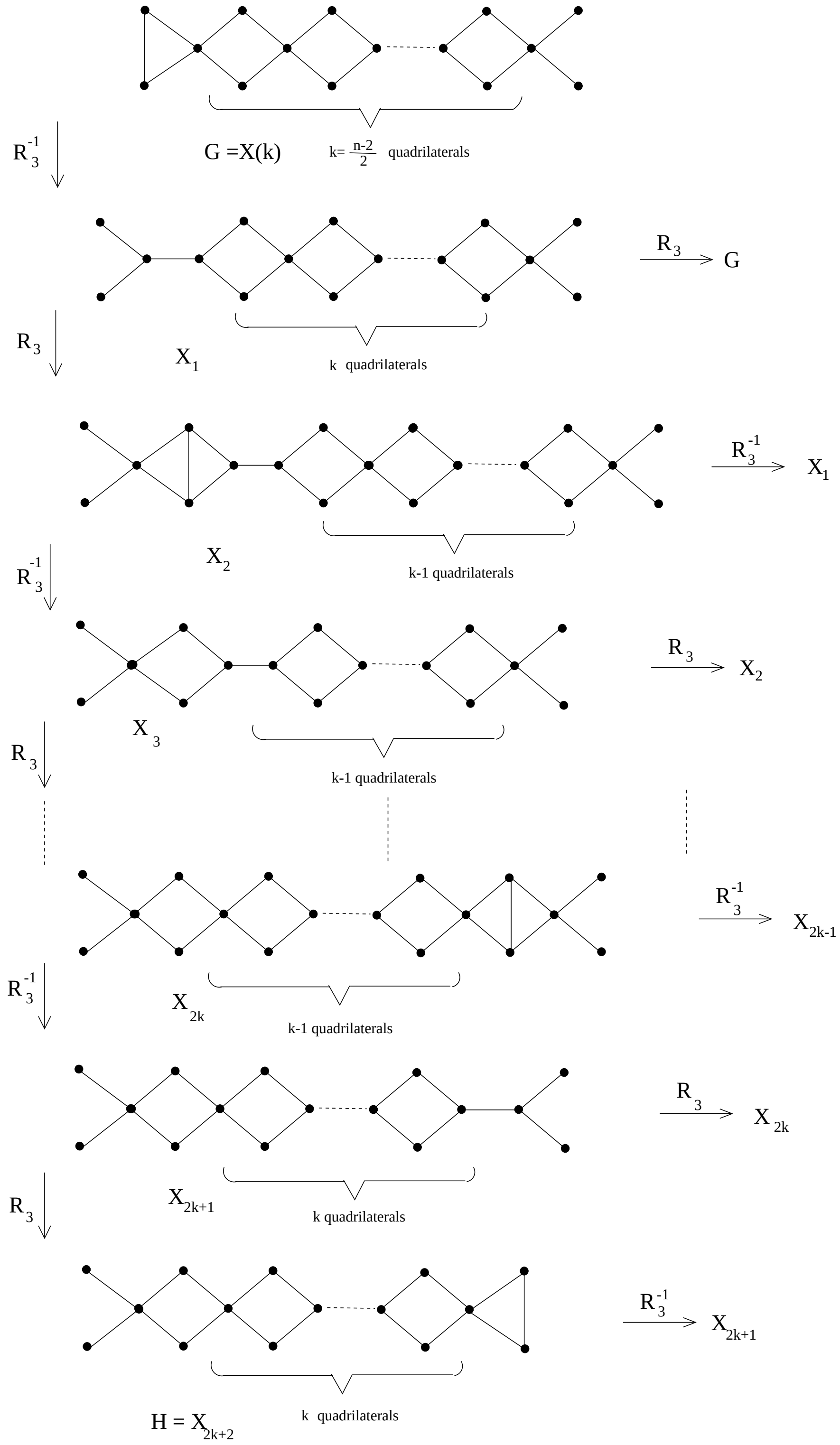

G =X(k)
k= (n-2)/2 quadrilaterals
R_3^{-1}
R_3
G
X_1
k quadrilaterals
R_3
R_3^{-1}
X_1
X_2
k-1 quadrilaterals
R_3^{-1}
R_3
X_2
X_3
k-1 quadrilaterals
R_3
R_3^{-1}
X_{2k-1}
X_{2k}
k-1 quadrilaterals
R_3^{-1}
R_3
X_{2k}
X_{2k+1}
k quadrilaterals
R_3
R_3^{-1}
X_{2k+1}
H = X_{2k+2}
k quadrilaterals

Figure 9:

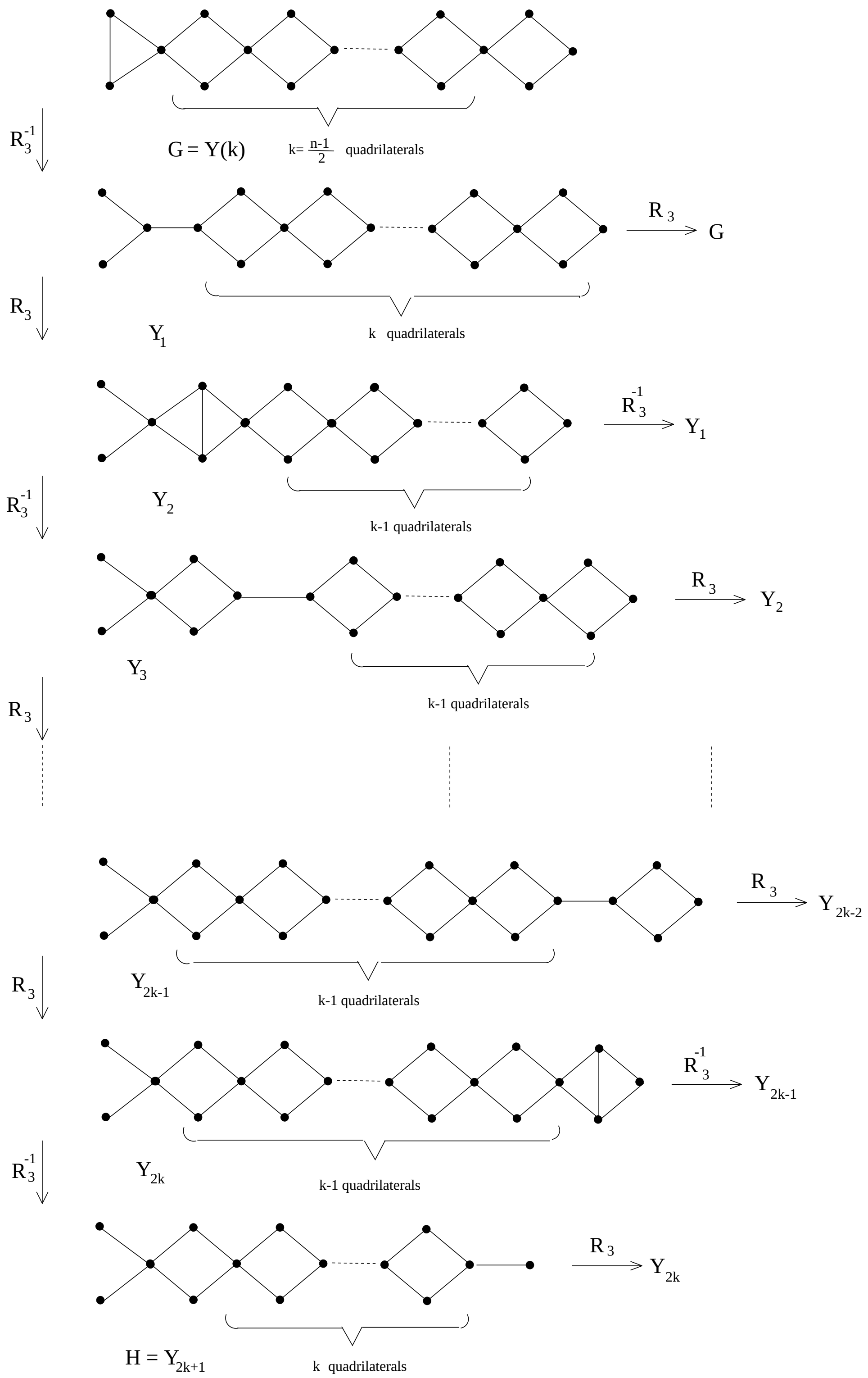

G = Y(k)
k= n-1/2 quadrilaterals
R3-1
Y1
k quadrilaterals
R3
G
R3
Y2
k-1 quadrilaterals
R3-1
Y1
R3-1
Y3
k-1 quadrilaterals
R3
Y2
R3
Y2k-1
k-1 quadrilaterals
R3
Y2k-2
R3
Y2k
k-1 quadrilaterals
R3-1
Y2k-1
R3-1
H = Y2k+1
k quadrilaterals
R3
Y2k

Figure 10:

Proof: (1) It is evident that the main idea of this proof is similar to the idea of the proof of (1) in Theorem 3. Therefore, we omit details and restrict ourselves to the proof of statement (13) by

$$\bigwedge_{i\in\{2,3,\ldots,n-1\}} \mathfrak{A}_4(n) \not\subseteq \mathfrak{A}_4(i), \tag{13}$$

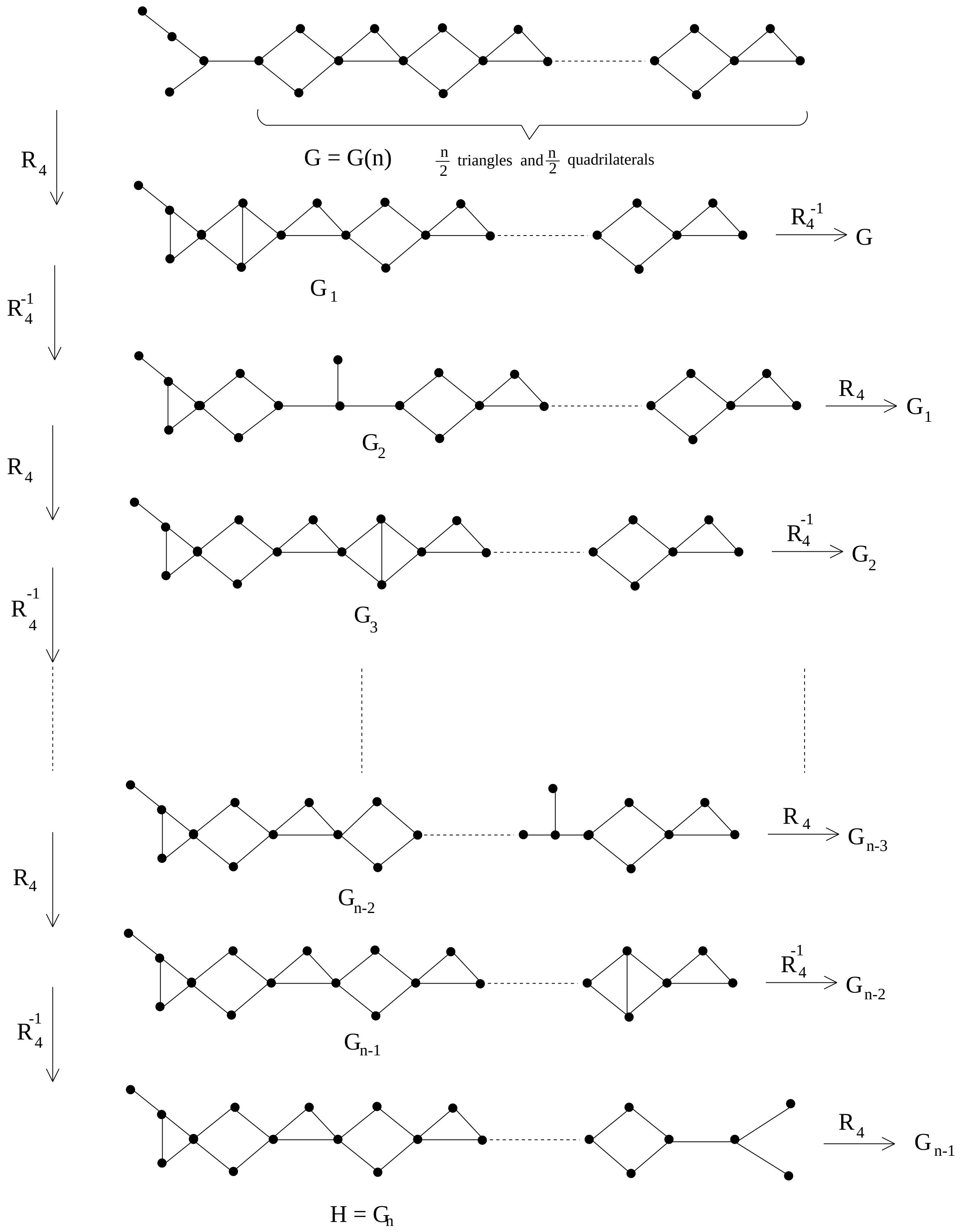
where n is assumed to be a positive integer ≥ 3 with the property that $\mathcal{A}_4$ consists of the elements $\mathfrak{A}_4(2), \mathfrak{A}_4(3), \ldots, \mathfrak{A}_4(n-1)$. In order to find an ordered pair $(G, H) \in \mathfrak{A}_4(n)$ not belonging to each $\mathfrak{A}_4(i), i = 2, 3, \ldots, n-1$, we distinguish the two cases $n \equiv 0 \mod 2$ and $n \equiv 1 \mod 2$. In the first case $n \equiv 0 \mod 2$, Figure 11 shows the graphs $G = G(n)$ and $H = G_n$ satisfying the properties mentioned above. The proof is not quite easy. In spite of his fact we cannot deal with details. In the second case $n \equiv 1 \mod 2$, Figure 12 depicts the corresponding graphs $G = H(n)$ and $H = H_n$. It is clear by means of Figure 12 that $(G, H) \in \mathfrak{A}_4(n)$ and $(G, H) \notin \mathfrak{A}_4(i)$ for each $i = 2, 3, \ldots, n-1$ are true. This completes the proof of (1).

Now we conclude by giving a sketch of the proof of (2), where (2) says that $\mathfrak{B}_4$ is infinite. Having assumed that $\mathcal{B}_4$ is finite we have to prove the statement

$$\bigwedge_{i\in\{2,3,\ldots,n-1\}} \mathfrak{B}_4(n) \not\subseteq \mathfrak{B}_4(i). \tag{14}$$

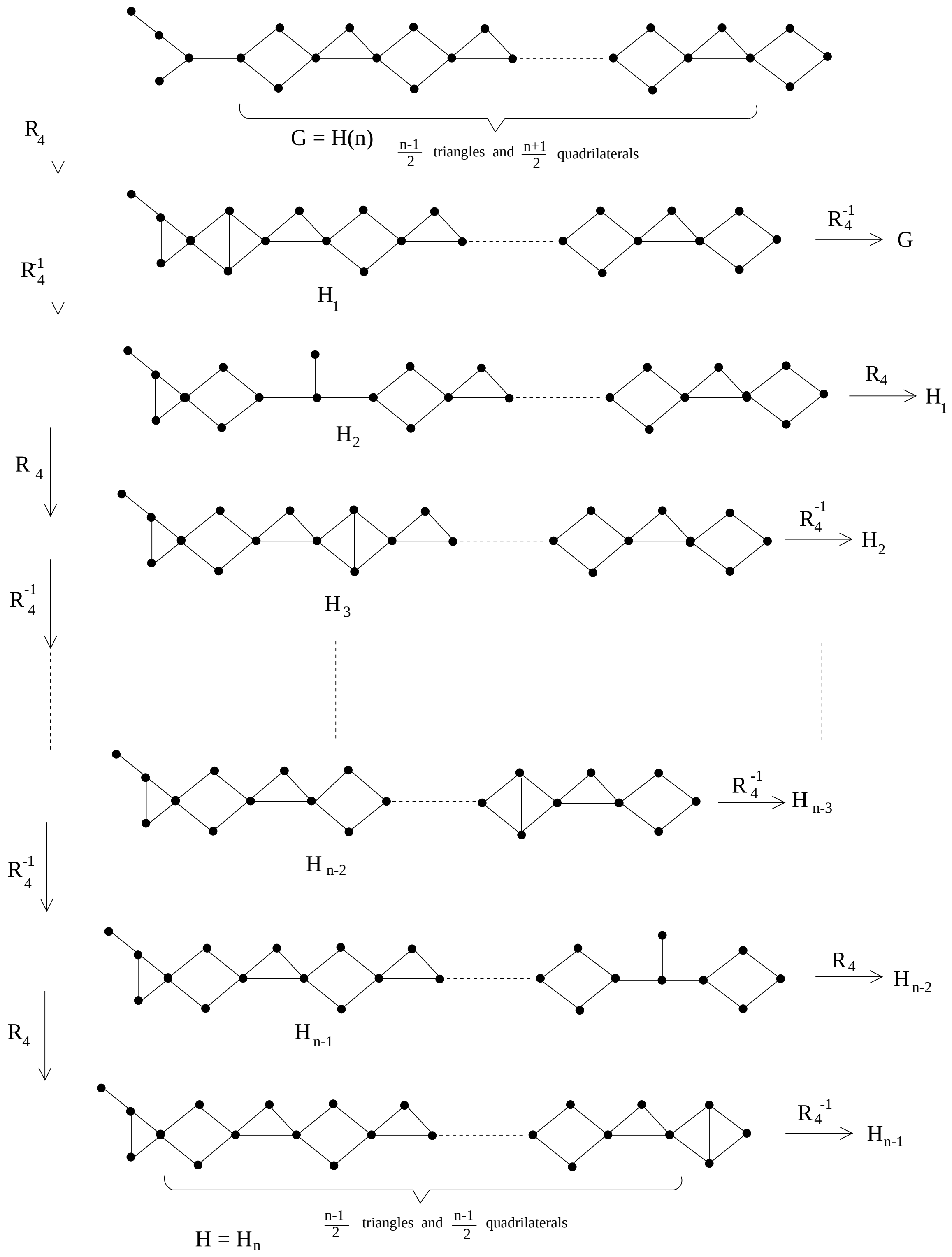
by giving an ordered pair $(G, H) \in \mathfrak{B}_4(n)$ not belonging to each $\mathfrak{B}_4(i), i = 2, 3, \ldots, n-1$, where n is chosen as explained above. If $n \equiv 0 \mod 2$ Figure 13 proves that (14) is true for even n and $G = X(n)$ and $H = X_n$.

If $n \equiv 1 \mod 2$ we choose $G = Y(n)$ and $H = Y_n$. Figure 14 shows that $(G, H) \in \mathfrak{B}_4(n)$ and $(G, H) \notin \mathfrak{B}_4(i)$ for each $i = 2, 3, \ldots, n-1$. This completes the proof in the case (2). That means that $\mathcal{B}_4$ is infinite.



G = G(n)
$\frac{n}{2}$ triangles and $\frac{n}{2}$ quadrilaterals
R_4
G_1
R_4^{-1}
G
R_4^{-1}
G_2
R_4
G_1
R_4
G_3
R_4^{-1}
G_2
R_4^{-1}
G_{n-2}
R_4
G_{n-3}
R_4
G_{n-1}
R_4^{-1}
G_{n-2}
R_4^{-1}
$H = G_n$
R_4
G_{n-1}

Figure 11:



G = H(n)
$\frac{n-1}{2}$ triangles and $\frac{n+1}{2}$ quadrilaterals
R_4
R_4^1
R_4^{-1} G
H_1
R_4 H_1
H_2
R_4
R_4^{-1} H_2
H_3
R_4^{-1}
R_4^{-1} H_{n-3}
H_{n-2}
R_4^{-1}
R_4 H_{n-2}
H_{n-1}
R_4
R_4^{-1} H_{n-1}
$\frac{n-1}{2}$ triangles and $\frac{n-1}{2}$ quadrilaterals
$H = H_n$

Figure 12:

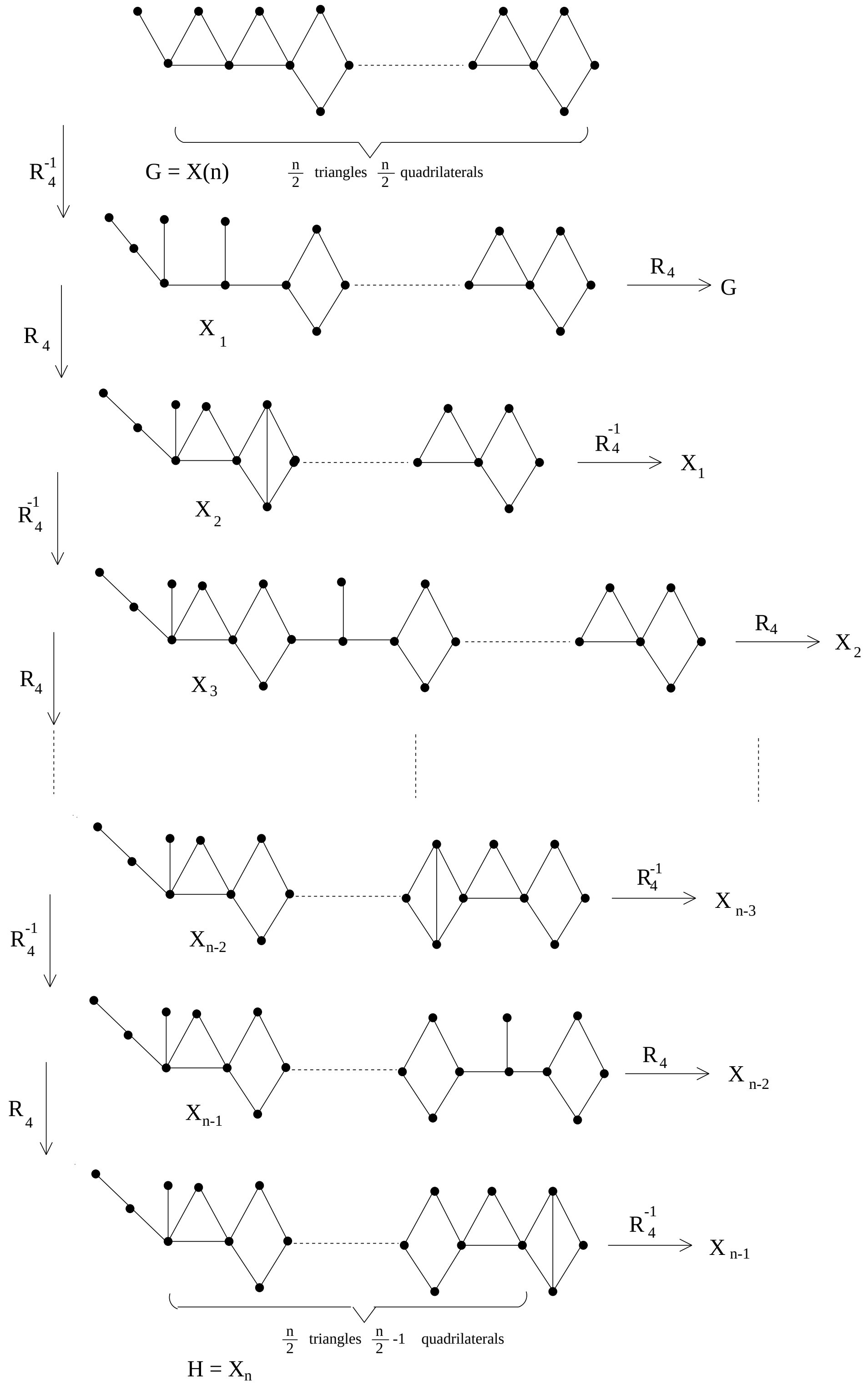
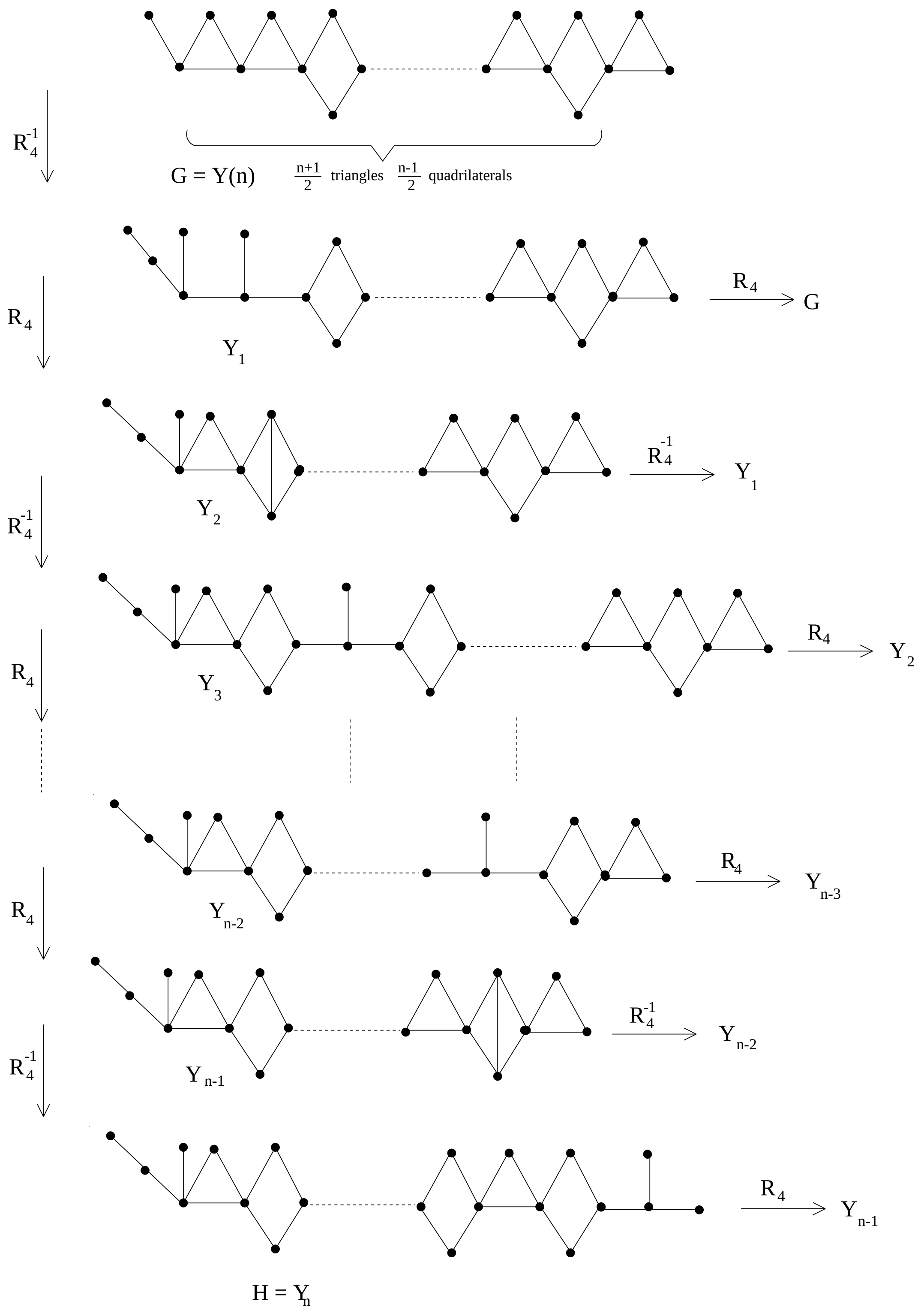

G = X(n)
n/2 triangles n/2 quadrilaterals
R_4^{-1}
X_1
R_4 G
R_4
X_2
R_4^{-1} X_1
R_4^{-1}
X_3
R_4 X_2
R_4
X_{n-2}
R_4^{-1} X_{n-3}
R_4^{-1}
X_{n-1}
R_4 X_{n-2}
R_4
R_4^{-1} X_{n-1}
n/2 triangles n/2 -1 quadrilaterals
H = X_n

Figure 13:



R_4^{-1}
G = Y(n)
$\frac{n+1}{2}$ triangles $\frac{n-1}{2}$ quadrilaterals
R_4
R_4 G
Y_1
R_4^{-1} Y_1
Y_2
R_4^{-1}
R_4 Y_2
Y_3
R_4
R_4 Y_{n-3}
Y_{n-2}
R_4
R_4^{-1} Y_{n-2}
Y_{n-1}
R_4^{-1}
R_4 Y_{n-1}
H = Y_n

Figure 14:

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