Some Properties of the singular words of the Fibonacci word

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Abstract

The combinatorial properties of the Fibonacci infinite word are of great interest in some aspects of mathematics and physics, such as number theory, fractal geometry, formal language, computational complexity, quasicrystals etc.

In this note, we introduce the singular words of the Fibonacci infinite word and discuss their properties. We establish two decompositions of the Fibonacci word in singular words and their consequences. By using these results, we discuss the local isomorphism of the Fibonacci word and the overlap properties of the factors. Moreover, we also give new proofs for the results on special words and the power of the factors.

The combinatorial properties of the Fibonacci infinite word are of great interest in some aspects of mathematics and physics, such as number theory, fractal geometry, formal language, computational complexity, quasicrystals etc. See [1, 3, 8, 9, 11]. Moreover, the properties of the subwords of the Fibonacci infinite word have been studied extensively by many authors [2, 4, 5, 6, 9, 10]. In this note, we shall present some new properties of the subwords of the Fibonacci word: as we shall see, the most striking of these properties is that the adjacent singular words of the same order are positively separated.

This note is organized as follows. After recalling some preliminary remarks on the Fibonacci word, we introduce the singular words and discuss their elementary properties. Then we establish two decompositions of the Fibonacci word in singular words (Theorems 1 and 2) and their consequences. By using these results, we discuss the local isomorphism of the Fibonacci word (Theorem 4) and the overlap properties of the factors (Theorem 6). Moreover, we also give new proofs for the results on special words (Theorem 5) and the power of the factors (Theorem 3).

In this note, we use the following definitions and terminology.

Let $A = \{a, b\}$ be an alphabet of two letters, let A^* be the free monoid on A, and let F be the free group generated by A. The elements of A^* are called words. The neutral element of A^* is called the empty word which we denote by ε. Let w be a word: we denote by $|w|$ the length of w, we denote by $|w|_a$

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(resp. $|w|_b$) the number of letters of a (resp. b) appearing in w, we denote by $L(w)$ the vector $(|w|_a, |w|_b)$.

An infinite word on A is a mapping $\boldsymbol{x} : \mathbb{N} \longrightarrow A$, and we write $\boldsymbol{x} = x_1 x_2 \cdots x_n \cdots$, where $x_i \in A$. The set of infinite words is denoted by A^ω.

A word v is a factor of word w and we write $v \prec w$, if there exist $u, u' \in A^*$, such that $w = uvu'$. We say that v is a left (resp. right) factor of a word w and we note that $v \triangleleft w$ (resp. $v \triangleright w$), if there exists $u \in A^*$ such that $w = vu$ (resp. $w = uv$). The notions of left factors and right factors are extended in a natural way to A^ω.

Let $w = x_1 x_2 \cdots x_n$. We denote by $\overline{w}$ the mirror image of w; that is, $\overline{w} = x_n \cdots x_2 x_1$. If $w = \overline{w}$, the word will be called a palindrome, and the set of the palindromes is denoted by $\mathcal{P}$. A word $w \in A^*$ is called primitive if $u = v^p,\ v \in A^*,\ p > 0$, implies $u = v$.

Let $w = x_1 x_2 \cdots x_n \in A^*$, and let $1 \leq k \leq n$. We define $C_k(w) = x_{k+1} \cdots x_n x_1 \cdots x_k$, the k-th conjugation of the word w, and we note that $C(w) = \{C_k(w);\ 1 \leq k \leq |w|\}$. By convention, $C_{-k}(w) = C_{|w|-k}(w)$.

Now let $\sigma : A \longrightarrow A^*$ be a morphism defined by $\sigma(a) = ab$, $\sigma(b) = a$. We define the n-th iteration of σ by $\sigma^n(a) = \sigma(\sigma^{n-1}),\ n \geq 2$ and we denote $F_n = \sigma^n(a)$ (by convention, we define $\sigma^0(a) = a$, $\sigma^0(b) = b$). Then the Fibonacci word F_∞ is obtained by iterating σ with the letter a (see [2]).

Let w be a word. We denote by $\Omega_n(w)$ the set of factors of w of length n, where $|w| \geq n$, and we simply note that $\Omega_n := \Omega_n(F_\infty)$.

By a natural embedding, we can regard the A^* as a subset of F, and if we say that w is in F, this means that is reduced (see [7] and therein). Let $w = x_1 x_2 \cdots x_n \in A^*$. We denote by w^{-1} the inverse word of w, that is $w^{-1} = x_n^{-1} \cdots x_2^{-1} x_1^{-1}$. Let $w = uv$, $u, v \in A^*$, then $wv^{-1} := u$ and $u^{-1}w := v$ by convention.

One of the motivations of this note is as follows: we know that the Fibonacci word is related closely to the Fibonacci numbers (the Fibonacci number is defined by the recurrence formula $f_{n+2} = f_{n+1} + f_n$ with the initial condition $f_{-1} = f_0 = 1$). Consider the following decomposition of the Fibonacci word

$$\underline{a}\ \underline{b}\ \underline{aa}\ \underline{bab}\ \underline{aabaa}\ \underline{babaabab}\ \underline{aabaababaabaa}\ \underline{babaababaabaababaabab}\ \cdots$$

that is, the length of the n-th block in the decomposition is f_n, $n \geq -1$. Then a question is posed naturally: What are these blocks? As we shall see, Theorem 1 will answer this question completely.

In this note, we shall use the following known facts which can be found in [2, 4, 8, 9].

Property 1 *1) $|\sigma^n(a)| = f_n$ and $|C_n(F_n)| = f_n$, where f_n is the n-th Fibonacci number. That is, all conjugations of F_n are different each other. In particular, for any $w \in C(F_n)$, we have*

$$L(w) = L(F_n) = (f_{n-1}, f_{n-2}),$$

and moreover,

$$C(F_n) = \{\overline{w};\ w \in C(F_n)\}.$$

2) $F_{n+1} = F_n F_{n-1}$.
3) For any $k \geq 1$, $\sigma^k(F_\infty) = F_\infty$, *that is*

$$F_\infty = F_k F_{k-1} F_k F_k F_{k-1} \cdots.$$

4) ab *is a suffix of* F_n *for odd positive* n, *and* ba *is a suffix of* F_n *for even positive* n.
5) $b^2 \not\prec F_\infty,\ a^3 \not\prec F_\infty$.
6) Any factor of F_∞ *will appear infinitely many times in* F_∞.
7) $w \prec F_\infty$ *if and only if* $\overline{w} \prec F_\infty$.

Remark 1 *In this note, we shall only use Property 1 and not the other known results of the Fibonacci word. In particular, we shall again prove Theorems 3 and 5 by using singular words.*

Let α and β be two words. Note that, by Property 1.4), if $\alpha\beta \rhd F_n$, then $\alpha \neq \beta$.

Lemma 1 *Let* $n \geq 2$, *and assume that* $\alpha\beta$ *is a suffix of* F_n. *Then*

$$\begin{aligned} F_n &= F_{n-2}F_{n-1}\alpha^{-1}\beta^{-1}\alpha\beta, \\ F_{n-2}F_{n-1} &= F_n\beta^{-1}\alpha^{-1}\beta\alpha. \end{aligned}$$

Proof. Note that $\alpha\beta \rhd F_n$, so $\beta\alpha \rhd F_{n-1}$ by property 1.4). It is easy to check the case of $n = 2$ directly. Suppose that the lemma is true for n. Then, by the hypothesis of the induction, we obtain

$$\begin{aligned} F_{n+1} &= F_nF_{n-1} = F_{n-1}F_{n-2}F_{n-1} = F_{n-1}F_n\alpha^{-1}\beta^{-1}\alpha\beta, \\ F_{n-1}F_n &= F_{n-1}F_{n-2}F_{n-1}\beta^{-1}\alpha^{-1}\beta\alpha = F_{n+1}\beta^{-1}\alpha^{-1}\beta\alpha. \end{aligned}$$

Now let $|w| = f_n$. Then by Property 1.3), w will be a factor of the following words: F_nF_n, $F_nF_{n-1}F_n$, F_nF_{n-1} and $F_{n-1}F_n$. If $w = uF_{n-1}v$ with $u \rhd F_n$, $v \lhd F_n$ and $|v| \leq f_{n-2}$, then $w \prec F_nF_{n-1}F_{n-2} = F_nF_n$. On the other hand, evidently, $F_nF_{n-1} \prec F_nF_n$, and thus the four cases above will be reduced to the cases F_nF_n and $F_{n-1}F_n$.

On the other hand, by Property 1.1), $\Omega_{f_n}(F_nF_n) = C(F_n)$. Therefore it is sufficient to determine the factors of $F_{n-1}F_n$.

Let α and β be two words such that $\alpha\beta$ is a suffix of F_n. We denote the word $\alpha F_n \beta^{-1}$ by w_n.

Lemma 2 *We have the following:*
1) $w_n \notin C(F_n)$;
2) $\Omega_{f_n}(F_{n-1}F_n) = w_n \bigcup \{C_k(F_n);\ 0 \leq k \leq f_{n-2} - 2\}$. *In particular, as a factor,* w_n *appears only once in* $F_{n-1}F_n$.

Proof. 1) Since $\alpha \neq \beta$, $L(w_n) \neq L(F_n)$, which yields 1).
2) By Lemma 1: if $\alpha\beta \triangleright F_n$, then we have $f_{n-1}F_n = F_nF_{n-1}\alpha^{-1}\beta^{-1}\alpha\beta$. Since $F_{n-1} \triangleleft F_n$, the first f_{n-1} factors of length f_n of the word $F_{n-1}F_n$ are exactly $C_k(F_n)$, $1 \leq k \leq f_{n-1}-2$, and the last factor is $F_n = C_{f_n}(F_n)$, the $(f_{n-1}-1)$-th factor is $\alpha F_n \beta^{-1} = w_n$.

As we have seen, for any $n \geq 1$, the set Ω_{f_n} consists of two parts: the first part consists exactly of all conjugations of F_n, the other is w_n. As we shall see, w_n possesses some interesting properties, which play an important role in the studies of the factors of F_∞.

The word w_n is called the n-th singular word of the Fibonacci word F_∞. For convenience, we define $w_{-2} = \varepsilon$, $w_{-1} = a$ and $w_0 = b$, and we denote by $\mathcal{S}$ the set of singular words of F_∞.

Now we discuss the properties of the singular words:

Property 2 *We have the following:*
1) If $n \geq 1$, then

$$L(w_n) = \begin{cases} (f_{n-1}+1, f_{n-2}-1) & \text{if } n \text{ is odd;} \\ (f_{n-1}-1, f_{n-2}+1) & \text{if } n \text{ is even.} \end{cases}$$

2) $w_n \not\prec w_{n+1}$.
3) If $\alpha \triangleright w_{n+1}$, then $w_{n+2} = w_nw_{n+1}\alpha^{-1}\beta$.
4) $w_n = w_{n-2}w_{n-3}w_{n-2}$, $n \geq 1$.
5) For $n \geq 1$,

$$\begin{aligned} C_{f_{n-1}-1}(F_n) &= w_{n-2}w_{n-1}, \\ C_{f_n-1}(F_n) &= w_{n-1}w_{n-2}; \end{aligned}$$

In particular,

$$\begin{aligned} w_{n-2} \prec C_k(F_n) & \quad \text{if and only if} \quad 0 \leq k \leq f_{n-1}-1; \\ w_{n-1} \prec C_k(F_n) & \quad \text{if and only if} \quad f_{n-1}-1 \leq k \leq f_n - 1. \end{aligned}$$

6) For $n \geq 1$, $w_{2n-1} = aauaa$, $w_{2n} = bvb$, where $u, v \in A^*$.
7) For $n \geq 2$, $1 < k < f_n$, no proper conjugate of w_n is a subword of F_∞.
8) For $n \geq 0$, $w_n^2 \not\prec F_\infty$.
9) For $n \geq -1$, w_n is a palindrome.
10) w_n is not the product of two palindromes for $n \geq 2$.
11) If $n \geq 2$, then w_n is primitive.
12) For $n \geq 1$, we have

$$w_n = w_n^* \left(\prod_{j=-1}^{n-2} w_j \right) = \left(\prod_{j=-1}^{n-2} w_{n-j-3} \right) w_n^*, \text{ where } w_n^* = \begin{cases} a & \text{if } n \text{ is odd,} \\ b & \text{if } n \text{ is even;} \end{cases}$$

13) $w_n \not\prec \prod_{j=-1}^{n-1} w_j$;

14) Let $k \geq -1$ and $p \geq 1$, and let $u = \prod_{j=k}^{k+p} w_j$. Then $u \notin \mathcal{S}$.

Proof. 1) If n is odd, then $a \triangleright F_{n-1},\ b \triangleright F_n$. Thus

$$L(w_n) = L(aF_n b^{-1}) = (f_{n-1}+1, f_{n-2}-1),$$

and the case of n being even can be proved in the same way.
2) Let $\alpha \triangleright F_n$: then $\beta \triangleright F_{n+1}$. By the definition of w_n, it is easi to see that $w_n \not\prec F_{n+1}$, so $w_n \not\prec F_{n+1}\beta^{-1}$. On the other hand $w_n = \beta F_n \alpha^{-1} \neq \alpha F_n \beta^{-1}$. Since $w_{n+1} = \alpha F_{n+1}\alpha^{-1}$, thus $w_n \not\prec w_{n+1}$.
3) By definition, $w_{n+2} = \beta F_n \alpha^{-1}$. Then by Lemma 1, we have

$$w_{n+2} = \beta F_n F_{n+1} \beta^{-1}\alpha^{-1}\beta = w_n w_{n+1}\alpha^{-1}\beta.$$

4) Let $\alpha \triangleright F_n$: then $\alpha \triangleright F_{n-2}$ and $\beta \triangleright F_{n+1}$, $\beta \triangleright F_{n-1}$. Thus

$$\begin{aligned} w_{n+1} &= \alpha F_{n+1}\beta^{-1} = \alpha F_{n-1}F_{n-2}F_{n-1}\beta^{-1} \\ &= (\alpha F_{n-1}\beta^{-1})(\beta F_{n-2}\alpha^{-1})(\alpha F_{n-1}\beta^{-1}) \\ &= w_{n-1}w_{n-2}w_{n-1}. \end{aligned}$$

5) Let $\alpha \triangleright F_n$: then

$$F_n = F_{n-1}F_{n-2} = (F_{n-1}\alpha^{-1})(\alpha F_{n-2}\beta^{-1})\beta,$$

and so the results follow from the definitions of singular word and conjugation of word.
6) This follows immediately from the definition of w_n and 9).
7) and 8) follow from Property 1.5) and Property 2.9).
9) We prove by induction. It is checked directly that the conclusion is true for $n \le 2$. Now suppose that the conclusion is true for $k \le n$. Then, by 5),

$$\overline{w_{n+1}} = \overline{w_{n-1}w_{n-2}w_{n-1}} = \overline{w_{n-1}}\,\overline{w_{n-2}}\,\overline{w_{n-1}} = w_{n-1}w_{n-2}w_{n-1} = w_{n+1},$$

that is, $w_{n+1} \in \mathcal{P}$.
10) Let $w_n = uv$, where $u, v \in \mathcal{P}$. Since w_n is a palindrome, so

$$w_n = \overline{w_n} = \overline{uv} = uv = vu.$$

Therefore, the $|u|$-th conjugation of w_n will be a factor of F_∞. Then, by 6), if $n \ge 2$, we have $a^4 \prec F_\infty$, or $b^2 \prec F_\infty$, which will contradict Property 1.5).
11) Let $w_n = u^p$, with $u \in A^*$, and $p \ge 2$. Since $w_n \in \mathcal{P}$, so do u and u^{p-1}, and hence $w_n = u^p = uu^{p-1}$ will be a product of two palindromes; but, by 10), that is impossible.
12) It is easy to verify that $F_n = abF_0F_1\cdots F_{n-3}F_{n-2}$. If n is odd, then $b \triangleright F_n$. Therefore

$$\begin{aligned} w_n &= aF_nb^{-1} = aab(aF_1b^{-1})(bF_2a^{-1})\cdots(bF_{n-3}a^{-1})(aF_{n-2}b^{-1}) \\ &= aw_{-1}w_0w_1\cdots w_{n-3}w_{n-2}, \end{aligned}$$

and the case of n being even may be proved in the same manner.

13) If $w_n \prec \prod_{j=-1}^{n-1} w_j$, then by 12), $w_n \prec w_n^* \prod_{j=-1}^{n-1} w_j = w_{n+1}$, which will contradict 2).

14) Assume that $u = \prod_{j=k}^{k+p} w_j = w_m$ for some $m \geq 0$. Since w_{k+p} is a factor, $m > k+p$. On the other hand, by 12), $w_m \prec w_{k+p}^* \prod_{j=-1}^{k+p} w_j = w_{k+p+2}$, so $m = k+p+1$. By 13), this is impossible.

By an argument analogous to that of Property 2.12), we obtain the following result, which answers the question posed in the introduction.

Theorem 1 $F_\infty = \prod_{j=-1}^{\infty} w_j$.

Proof. The proof is similar that of Property 2.12).

Now we are going to introduce another decomposition of F_∞ which will show the positively separate property of the singular words. To this end, we first establish some lemmas.

Lemma 3 *Let $w_n w_{n+1} = u_1 u_2 u_3$ (or $w_{n+1} w_n = u_1 u_2 u_3$) with $0 < |u_1| < f_n$ and $0 < |u_3| < f_{n+1}$, then $u_2 \notin \mathcal{S}$.*

Proof. i) By the condition of the lemma, $2 < |u_2| < f_{n+2} - 2$, so $u_2 \neq w_{n+2}$.

ii) Let $\alpha \triangleright F_n$, then $w_n w_{n+1} = \beta F_n F_{n-1} \beta^{-1}$. By Lemma 2, $w_{n+1} = \alpha F_{n+1} \beta^{-1}$ appears only once in $F_n F_{n+1}$. Note that $|u_3| \geq 1$, we obtain $u_2 \neq w_{n+1}$.

iii) Let $|u_2| = f_n$ and $F_{n+1} = F_n F_{n-1}$, $u_2 \prec F_n F_n$. But by lemma 2, $w_n \not\prec F_n F_n$, and thus $u_2 \neq w_n$.

iv) Let $|u_2| = f_{n-1}$. Since $w_n w_{n+1} = w_n w_{n-1} w_{n-2} w_{n-1}$, then we must have $u_2 \prec \alpha F_n F_{n-1} \alpha^{-1}$. By using lemma 2, a discussion as in ii) yields $u_2 \neq w_{n-1}$.

The other cases will be reduced to one of the four cases above, so by repeating this argument, we prove that, for any $k \geq 1$, $u_2 \neq w_k$, that is, $u_2 \notin \mathcal{S}$.

Now let $n \geq 0$ be fixed. We define a substitution $\phi_n : A \longrightarrow A^*$ by $\phi_n(a) = w_{n+1}$, $\phi_n(b) = w_{n-1}$. Let $\Sigma_n = \{w_{n+1}, w_{n-1}\}$. For a singular word w_k, we say $W_k(\Sigma_n) := \phi_n(w_k)$ (if there is no confusion, we simply write W_k), the k-th singular word over Σ_n.

Lemma 4 *Let $n \geq 0$ and $k \geq 1$. Then we have*

$$\begin{aligned} w_{n+2k} &= w_n x_1 w_n x_2 \cdots w_n x_{f_{2k-2}} w_n, \\ w_{2k+1} &= y_1 w_n y_2 w_n \cdots y_{f_{2k-1}-1} w_n y_{f_{2k-1}}, \end{aligned}$$

where $x_j, y_j \in \Sigma_n$. Moreover,

$$x_1 x_2 \cdots x_{f_{2k-2}} = W_{2k-2} \text{ and } y_1 y_2 \cdots y_{f_{2k-1}} = W_{2k-1}$$

are, respectively, the $(2k-2)$-th and $(2k-1)$-th singular words over Σ_n.

Proof. For any fixed n, we prove the lemma by induction. We have, by Property 2.4),

$$\begin{aligned} w_{n+2} &= w_n w_{n-1} w_n, \\ w_{n+3} &= w_{n+1} w_n w_{n+1}, \\ w_{n+4} &= w_n w_{n-1} w_n w_{n+1} w_n w_{n-1} w_n, \\ w_{n+5} &= w_{n+1} w_n w_{n+1} w_n w_{n-1} w_n w_{n+1} w_n w_{n+1}, \end{aligned}$$

and hence the conclusion is true for $k = 1, 2$. Now suppose that the conclusion is true for $k-1$ and k. Then

$$\begin{aligned} w_{n+2(k+1)} &= w_{n+2k} w_{n+2k-1} w_{n+2k} \\ &= w_n x_1 \cdots w_n x_{f_{2k-2}} w_n y_1 w_n \cdots w_n y_{f_{2k-3}} w_n \cdots w_n x_{f_{2k-2}}, \end{aligned}$$

since $x_1 x_2 \cdots x_{f_{2k-2}}$ and $y_1 y_2 \cdots y_{f_{2k-3}}$ are, respectively, the $(2k-2)$-th and $(2k-3)$-th singular words W_{2k-2} and W_{2k-3} on Σ_n by the assumption of the induction. So, by Property 2.4),

$$x_1 x_2 \cdots x_{f_{2k-2}} y_1 y_2 \cdots y_{f_{2k-3}} x_1 x_2 \cdots x_{f_{2k-2}} = W_{2k-2} W_{2k-3} W_{2k-2} = W_{2k}$$

is the $(2k)$-th singular word. The same discussion gives the proof for w_{n+2k+3}. From Lemmas 3 and 4, we immediately obtain the following.

Corollary 1 *Let $m \geq n+2$. Then there are exactly $m-n-2$ factors w_n appearing in w_n which are separated by w_{n-1} and w_{n+1} as in Lemma 4.*

Let n be fixed, then by Property 1.6), the word w_n will appear in F_∞ infinitely many times. We arrange these words as a sequence $w_{n,k}$ the k-th singular word of the order n.

Lemma 5 *Let $F_\infty = \prod_{j=-1}^{\infty} w_j$ be the decomposition as in Theorem 1. Let u be any singular word of order n (that is, $u = w_{n,k}$ for some k). Then u must be contained completely in some w_m, where $m \geq n$.*

Proof. i) From Property 2.13), $w_n \not\prec \prod_{j=-1}^{n-1} w_j$.

ii) If $u \prec \prod_{j=-1}^{n} w_j$, then by Property 2.12),

$$u \prec \left(w_{n-1}^{*} \prod_{j=-1}^{n-1} w_j \right) w_n = w_{n+1} w_n,$$

so by Lemma 3, u must be w_n.

From i) and ii), we only need to consider $u \prec \prod_{j=n}^{\infty} w_j$. Since $|u| = |w_n|$, there exists m, $m \geq n$, such that, either $u \prec w_m$, or $u \prec w_m w_{m+1}$ with $u \not\prec w_n$ and $u \not\prec w_{n+1}$. But by lemma 3, the later case is impossible.

We thus finish the proof from the discussions above.

Now we can state our main result of this note.

Theorem 2 *For any $n \geq 0$, we have*

$$F_\infty = \left(\prod_{j=-1}^{n-1} w_j \right) w_{n,1} z_1 w_{n,2} z_2 \cdots w_{n,k} z_k w_{n,k+1} \cdots$$

where $\boldsymbol{z} = z_1 z_2 \cdots z_n \cdots$ is the Fibonacci word over Σ_n.

Proof. From theorem 1 and lemma 4, we get

$$\begin{aligned} F_\infty &= \left(\prod_{j=-1}^{n-1} w_j \right) w_n w_{n+1} \left(\prod_{j=n+2}^{\infty} w_j \right) \\ &= \left(\prod_{j=-1}^{n-1} w_j \right) w_n w_{n+1} (w_n w_{n-1} w_n)(w_{n+1} w_n w_{n+1}) \cdots \\ &\quad (w_n x_1 w_n \cdots x_{f_{2k-2}} w_n)(y_1 w_n y_2 \cdots w_n y_{f_{2k-1}}) \cdots \end{aligned}$$

Note that

i) by lemma 4, lemma 5 and corollary 1, all factor w_n of F_∞ (or the sequence $w_{n,k}$, $k \geq 1$) appear in the formula above;

ii) by lemma 4,

$$\begin{aligned} x_1 \cdots x_{f_{2k-2}} &= W_{2k-2}, \\ y_1 \cdots y_{f_{2k-1}} &= W_{2k-1}, \end{aligned}$$

thus $\prod_{j=1}^{\infty} z_j = \prod_{j=-1}^{\infty} W_j$ is the Fibonacci word on Σ_n.

i) and ii) follow the theorem.

The following example illustrates the decomposition of F_∞ of the words w_1, w_2:

$abaa(bab)\underline{aabaa}(bab)\underline{aa}(bab)\underline{aabaa}(bab)\underline{aabaa}(bab)\underline{aa}(bab)\underline{aabaa}(bab)\underline{aa}(bab)\underline{aabaa}\cdots$

Let $\boldsymbol{y} = y_1 y_2 \cdots y_n \cdots$ be an infinite word over $\{a, b\}$. Let $u, v \prec \boldsymbol{y}$, $u = y_k y_{k+1} \cdots y_{k+p}$ and $v = y_l y_{l+1} \cdots y_{l+m}$, where $l \geq k$, then the distance of the words u and v define by

$$d(u,v) = \begin{cases} l-k-p & \text{if } l > k-p; \\ 0 & \text{otherwise.} \end{cases}$$

If $d(u, v) > 0$, we say that the words u and v are positively separate.

The theorem 2 has the following direct consequences:

Corollary 2 *The adjacent singular words of the same order are positively separate. More precisely, for any n and k, we have*

$$d(w_{n,k}, w_{n,k+1}) \in \{f_{n+1}, f_{n-1}\}.$$

Moreover, one of $d(w_{n,k}, w_{n,k+1})$ and $d(w_{n,k+1}, w_{n,k+2})$ is f_{n+1}.

Corollary 3 *The left and the right adjacent word of the length f_{n-2k} of the singular word w_{n+1} are exactly w_{n-2k}.*

Let $w = x_k x_{k+1} \cdots x_{k+p}$ $(k, p \geq 1)$ be a factor of F_∞. If there is an integer l, $1 \leq l \leq p$, such that $w = x_{k+l} x_{k+l+1} \cdots x_{k+l+p}$, then we say that w has overlap with $p - l$ as length of overlap. The above definition is equivalent to the following assertion: Let $u \prec F_\infty$, if there exist words x, y and z such that $u = xy = yz$ and $u(y) := uz = xyz \prec F_\infty$. From corollary 2, we obtain immediately

Corollary 4 *For $n \geq 1$, w_n has no overlap.*

Corollary 5 *Let $u \prec F_\infty$ and let $f_n < |u| \leq f_{n+1}$, let w be one of the largest singular words contained in u (in the sense of order), then w appears only once in u, moreover, w must be one of the three following singular words: w_{n-1}, w_n and w_{n+1}.*

Proof. Suppose that the conclusion is not true. Then there will be another singular word of the same order contained in u which is adjacent to w and we denote by w'. Thus there is a word v, such that $wvw' \prec u$ (or $w'vw \prec u$). By theorem 2, either v, or wvw', will be a singular word which has higher order than w, this is in contradiction with the hypothesis of w.

The second conclusion of the corollary follows from directly the property 2.5).

As applications of singular word, in particular, the positively separate property of the singular words, we are going to illustrate some examples in the following. Although some results are known (example 1 and example 3), but the proofs are new. Moreover, these proofs show that the singular words play an important role in the studies of the factor of the Fibonacci word.

Example 1. Power of the factors. [2, 5, 6, 9]

Theorem 3 *We have*
1) For any n, $w_n^2 \not\prec F_\infty$;
2) For $0 \le k \le f_n - 1$, $(C_k(F_n))^2 \prec F_\infty$;
3) If $u \prec F_\infty$ *with* $f_{n-1} < |u| < f_n$, *then* $u^2 \not\prec F_\infty$;
4) If $0 \le k \le f_{n-1} - 2$, *then* $(C_k(F_n))^3 \prec F_\infty$;
5) If $f_{n-1} - 2 < k < f_n$, *then* $(C_k(F_n))^3 \not\prec F_\infty$;
6) For any $u \prec F_\infty$, $u^4 \not\prec F_\infty$.

Proof. 1) It follows from the properties 1.5) and 2.6), $w_n^2 \not\prec F_\infty$;

2) Let $C_k(F_n) = uv$ with $F_n = vu$. Then $u \triangleright F_n$ and $v \triangleleft F_n$. Since

$$(C_k(F_n))^2 = uvuv = uF_nv \prec (F_n)^3,$$

the conclusion $(C_k(F_n))^2 \prec F_\infty$ will follow from $F_n^3 \prec F_\infty$.

3) Suppose that w_k be the largest singular word contained in u as in corollary 5, and let $u = v_1w_kv_2$. Assume that $u^2 = v_1w_kv_2v_1w_kv_2 \prec F_\infty$, then $w_k \not\prec v_2v_1$, otherwise by theorem 2 we shall have either $w_{k+1} \prec v_1$, or $w_{k+1} \prec v_2$, that will be in contradiction with the hypothesis of w_k. Thus two singular words of the order k above are adjacent, so by theorem 2 again, v_2v_1 must be either w_{k+1} or w_{k-1}. By property 2.5), u will be either a conjugation of F_{k+2}, or of F_{k+1}. But these two cases are impossible because of the hypothesis of u.

4) Since $aaba \prec f_\infty$, so dose $F_nF_nF_{n-1}F_n$. Let $\alpha\beta \triangleright F_{n-1}$, then by lemma 1, we have

$$F_n^2F_{n-1}F_n = F_n^2F_{n-1}F_{n-2}F_{n-1}\alpha^{-1}\beta^{-1}\alpha\beta = F_n^3F_{n-1}\alpha^{-1}\beta^{-1}\alpha\beta \prec F_\infty,$$

notice that $F_{n-1} \triangleleft F_n$, hence if $0 \le k \le f_{n-1} - 2$, then

$$(C_k(F_n))^3 \prec F_n^3F_{n-1}\alpha^{-1}\beta^{-1} \prec F_\infty.$$

5) Now suppose that $f_{n-1} - 1 < k < f_n$, then by property 2.5), $w_{n-1} \prec C_k(F_n)$. Let $C_k(F_n) = uw_{n-1}v$, then $vu = w_{n-2}$, thus

$$(C_k(F_n))^3 = uw_{n-1}w_{n-2}w_{n-1}w_{n-2}w_{n-1}v.$$

Hence if $(C_k(F_n))^3 \prec F_\infty$, then the word $w_{n-1}w_{n-2}w_{n-1} = w_{n+1}$ will have overlap, but by corollary 5, this is impossible.

6) The conclusion follows from an analogous argument with 5).

Remark 2 *From theorem 3.2), we see that, any conjugation of* F_n, $n \ge 0$, *is not separated positively. This is an important difference between the conjugations of* F_n *and* w_n.

Example 2. Local isomorphism

Let $u = u_1u_2\cdots u_n\cdots$ and $v = v_1v_2\cdots v_n\cdots$ be two infinite words over the alphabet $\{a, b\}$. We say that u and v are locally isomorphic if any factor (or its mirror image) of u is also factor of v and vice versa (By the property 1.7), for the Fibonacci word, we don't need to consider mirror images of the factors). If u and v are locally isomorphic, we shall write $u \simeq v$. The notion of local isomorphism is very useful in the studies of the energy spectra of one-dimensional quasicrystals [12].

For an infinite word $u = u_1u_2\cdots u_n\cdots$, we define the translation $T(u)$ of u by $Tu = u_2u_3\cdots$ and define $T^k = T(T^{k-1})$ by recurrence. By using the properties of the singular words of the Fibonacci word, we can easily obtain the following results of the local isomorphism of the Fibonacci word.

Theorem 4 *We have*
1) If we change a finite number of letters of F_∞, then the obtained infinite word F'_∞ is not locally isomorphic to F_∞.
2) Let $u \in A^*$, then $F_\infty \simeq uF_\infty$ if and only if there exists $m > -1$, such that $u \triangleright w_m w_m^*$, where w_m^* is defined as in property 2.12).
3) For any $k \geq 1$, $T^k(F_\infty) \simeq F_\infty$.

Proof. 1) Let $F_\infty = \prod_{j=-1}^{\infty}$ as in theorem 1, because we only change a finite number of letters of F_∞, we can find an integer m and words $u, v \in A^*$, such that

$$F'_\infty = uv\prod_{j=m}^{\infty} w_j,$$

where $|v| = f_{m-1},\ v \neq w_{m-1}$. Therefore by corollary 3, $vw_m \not\prec F_\infty$, that is $F_\infty \not\simeq F'_\infty$.

2) From theorem 1 and property 2.12), for any $k > 0$ and $m \geq 0$,

$$w_{2m}aF_\infty = w_{2m}a\left(\prod_{j=-1}^{2m+2k-1} w_j\right)\left(\prod_{j=2m+2k}^{\infty} w_j\right) = w_{2m}w_{2m+2k+1}\left(\prod_{j=2m+2k}^{\infty} w_j\right),$$

then, by corollary 3, $w_{2m}w_{2m+2k+1} \prec F_\infty$, that is, for any $v \prec w_{2m}aF_\infty$, we can find an integer k, such that $v \prec w_{2m}w_{2m+2k+1}$, so $v \prec F_\infty$. The case of $w_{2m+1}b$ can be proved in the same way. That is, if $u \triangleright w_m w_m^*$ for some m, then $F_\infty \simeq uF_\infty$. If u is not a right factor of any $w_m w_m^*$, then by the discussion similar that of 1), we see that $uF_\infty \not\simeq F_\infty$.

3) The proof follows from the property 1.6).

Example 3. Study of special words of F_∞

Berstel [2] introduced the special words of F_∞ as follows: if $ua, ub \prec F_\infty$, then the word u is called a special word of F_∞. The following theorem is due to Berstel [2] which we shall give another proof by using singular word.

Theorem 5 *A word $w \prec F_\infty$ is a special word if and only if, for some $n \geq 0$, $w \triangleright \overline{F_n}$.*

Proof. It is easily checked that, for any $n \geq 0$, $\overline{F_n}$ is a special word, therefore the theorem is reduced to show that, for any $n \geq 0$, $|\Omega_n| = n+1$.

Now let $u \prec F_\infty$ and let $f_k < |u| \leq f_{k+1}$. By an analogous argument with that for lemma 2, it is readily to see that the word u must be one of the three following forms:

- $u = sw_nt, \ |st| \leq f_{n-1}$;
- $u = sF_nt, \ s, t \neq \varepsilon, \ |st| \leq f_{n-1}, \ s \triangleright F_n, \ t \triangleleft F_n$;
- $u = st, \ s \triangleright F_n, \ t \triangleleft F_n$.

In the first case, by corollary 3, the factor $s \triangleright w_{n-1}$, (resp. t) are determined uniquely. Moreover, since w_n has no overlap, if $s \neq s'$, then $sw_nt \neq s'w_nt'$. Hence there are exactly $|u| - f_k + 1$ different words sw_nt which correspond with $|s| = 0, 1, \ldots, n - f_k$.

In the two later cases, from property 1.1), it is readily to prove that there are exactly f_k different factors of length $|u|$ of F_k^3.

Summarize the discussions above, we get $|\Omega_{|u|}| = f_k + (|u| - f_k + 1) = |u| + 1$.

Example 4. Overlap of the subwords of the Fibonacci word

In this example, we shall determine the factors which have overlap.

Recall that: Let $u \prec F_\infty$, if there exist words x, y and z such that $u = xy = yz$ and $\hat{u}(y) := uz = xyz \prec F_\infty$. Then we shall say that the word u has overlap with the overlap factor y (or overlap length $|y|$), the word $\hat{u}(y)$ is called the overlap of u with the overlap factor y. We denote by $\mathcal{O}(\mathcal{F}_\infty) = \mathcal{O}$ the set of factors having overlap.

Evidently, if $u \in \mathcal{O}$, we have

$$|u| + 1 \leq |\hat{u}(y)| \leq 2|u| - 1, \qquad (*)$$

where y is any overlap factor of u.

Lemma 6 *Let $f_n < |u| \leq f_{n+1}$, and let $u \neq w_{n+1}$, then $u \in \mathcal{O}$ if and only if $w_n \not\prec u$.*

Proof. Let $w_n \prec u$ and write $u = sw_nt$. If $u \in \mathcal{O}$, notice that $w_n \notin \mathcal{O}$, thus overlap of u must be of the form sw_nvw_nt. By corollary 4,

$$|sw_nvw_nt| \geq |s| + |t| + 2f_n + f_{n-1} = |u| + f_{n+1} \geq 2|u|,$$

which is in contradiction with the inequality $(*)$.

Now suppose that $w_n \not\prec u$, then discuss as in theorem 5, we have either

- $u = sF_nt$, where $s, t \neq \varepsilon$, $|s| + |t| \leq f_{n-1}$, $s \triangleright F_n$, $t \triangleleft F_n$;

- or $u \prec F_n^2$.

In the first case, if $|t| = f_n - 1$, then $u = w_{n+1} \notin \mathcal{O}$. Now consider $|t| < f_n - 1$. Since $|s| + |t| \le f_{n-1}$, $s \triangleright F_n$, $t \triangleleft F_n$, we can write $F_n = txs$. Since $|t| < f_n - 1$, by theorem 3.4),

$$(C_{|t|}(F_n))^3 = (xst)^3 = xstxstxst \prec F_\infty,$$

that is, $u = sF_nt = stxst$ has overlap with factor st.

In the second case, notice that $u \prec F_n^2$ and $|u| > f_n$, so if we write $u = st$, with $|t| = f_n$, then $t = C_k(F_n)$ for some k, and $s \triangleright t$, thus $u = sxs$. On the other hand, since $u = sC_k(F_n) \prec F_n^2$, so $sxsxs = s(C_k(F_n))^2 \prec F_n^3 \prec F_\infty$, that is $u = sxs$ has overlap with overlap factor s.

Lemma 7 *If $u \in \mathcal{O}$, then the overlap of u is unique.*

Proof. Let $f_n < |u| \le f_{n+1}$, and let w be the largest singular word contained in u. By corollary 6, w is one of w_{n-1}, w_n and w_{n+1}. Since $u \in \mathcal{O}$, w must be w_{n-1} from lemma 6, so we can write $u = sw_{n-1}t$. Now suppose that there are two different overlaps of u, then w_{n-1} will appear three times in one of these two overlaps. Since $w_{n-1} \notin \mathcal{O}$, this overlap must be of the form $sw_{n-1}v_1w_{n-1}v_2w_{n-1}t$, then by an analogous argument with lemma 6, we shall get a contradiction of $(*)$.

From lemma 7 and the proof of the lemma 6, we obtain immediately

Corollary 6 *Let $f_n < |u| \le f_{n+1}$, and let $u \in \mathcal{O}$, then $u = vv'v$, where $|v|$ is the overlap length.*

Summarize the results above, we have

Theorem 6 *Let $f_n < |u| \le f_{n+1}$ and let $u \ne w_{n+1}$, $u \prec F_\infty$, then $u \in \mathcal{O}$ if and only if $w_n \not\prec u$. If $u \in \mathcal{O}$, then the overlap of u is unique and $u = vv'v$, where v is the factor of overlap and $|v| = |u| - f_n$.*

In particular, $C_k(F_n) \in \mathcal{O}$ if and only if $0 \le k \le f_n - 2$.

Note that:
i) $f_{n+1} < 2f_n < f_{n+2} < 3f_n < f_{n+3}$;
ii) for any k, $w_{n+1} \not\prec (C_k(F_n))^2$;
iii) for any k, $w_{n+2} \not\prec (C_k(F_n))^3$.
We get immediately from theorem 6

Corollary 7 *For any k, $(C_k(F_n))^2 \in \mathcal{O}$, $(C_k(F_n))^3 \in \mathcal{O}$.*

Remark 3 *If $w^2 \not\prec F_\infty$ and w has no overlap, then the adjacent words of w will be positively separate. Moreover we can prove that for these words, there is a decomposition similar to the singular words.*

Let $w = abab$, by theorem 3.3) and theorem 6, $w^2 \prec F_\infty$ and $w \notin \mathcal{O}$, so w is separated positively. the following decomposition illustrates the remark above:

$$aba(abab)\underline{aaba}(abab)\underline{a}(abab)\underline{aaba}(abab)\underline{aaba}(abab)\underline{a}(abab)\underline{aaba}(abab)\underline{a}(abab)\cdots$$

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