Algebraic combinatorics related to the free Lie algebra

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During the past decade numerous fruitful contributions to the theory of the free Lie algebra have been made. Results and methods in this area are characterized by a subtle interplay between algebraic and combinatorial ideas. The quantity and the wealth of the material accumulated in the last years might be accompanied by the certainly undesired side-effect of concealing its own roots and history. We have therefore decided to restrict ourselves to a concise approach to some specially chosen topics. It is essentially self-contained and takes its course starting from a completely elementary source. At the same time the attempt is made to duly provide the reader with appropriate references for the various contributions involved. We take the opportunity to refer to [18] as a useful supplement to this article.

In the first section we describe certain aspects of the theory and a number of known results. In the second section we show that these are different branches of the same key result (Ind) for which we then give a proof, by means of elementary combinatorial reasoning, in the third section. The underlying idea of our approach is to transfer problems on free Lie algebras into the area of group rings of symmetric groups. On the one hand, this provides a powerful tool to solve those problems. On the other hand, the arising questions are a challenging contribution to the classical representation theory of the symmetric group: By passing from the free Lie algebra to group rings, several notions are focused which apparently have not been considered as of central importance before. A most interesting problem in this sense is to analyze the role of the Solomon algebra in the general representation theory of the symmetric group. In our fourth section we introduce this algebra and add some hints in that direction.

1 Free Lie algebras

In the following we write $\mathbb{N}_0$ for the set of all non-negative integers and set $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$, $\underline{n} := \{k \mid k \in \mathbb{N}, 1 \leq k \leq n\}$ for all $n \in \mathbb{N}_0$.

Let R be a commutative unitary ring, $n \in \mathbb{N}$, F be the monoid generated freely by n letters $x_1, \ldots, x_n$. Any element $x_{i_1} \cdots x_{i_m} \in F$ is called a *word* (over $\{x_1, \ldots, x_n\}$), the number m its *length* or *degree*. If the number of all $r \in \underline{m}\rfloor$ such that $x_{i_r} = x_j$ is denoted by k_j ($j \in \underline{n}\rfloor$), then $k_1 + \cdots + k_n = m$, and $(k_1, \ldots, k_n)$ is called the *multidegree* of $x_{i_1} \cdots x_{i_m}$.

Let A_R be the free R-module with basis F. The multiplication in F extends canonically to A_R which thus becomes an associative R-algebra generated freely by $\{x_1, \ldots, x_n\}$. For all $m \in \mathbb{N}_0$ let $A_{R,m}$ be the R-submodule of A_R generated by all words of length m, and for all $(k_1, \ldots, k_n) \in \mathbb{N}_0^n$ let $A_R(k_1, \ldots, k_n)$ be the R-submodule of A_R generated by all words of multidegree $(k_1, \ldots, k_n)$. Then

$$
\begin{aligned}
A_R &= \bigoplus_{m \in \mathbb{N}_0} A_{R,m} \\
A_{R,m} &= \bigoplus_{k_1 + \cdots + k_n = m} A_R(k_1, \ldots, k_n) \qquad (m \in \mathbb{N}_0).
\end{aligned}
\tag{1}
$$

The standard Lie product,

$$a \circ b := ab - ba \quad \text{for all } a, b \in A_R,$$

turns A_R into a Lie algebra over R. By [31], [6, II, §3, Theorem 1], $\{x_1, \ldots, x_n\}$ generates freely a Lie subalgebra of A_R which will be denoted by L_R. A *Lie monomial* in A_R is an element of the $\circ$-closure of $\{x_1, \ldots, x_n\}$. A Lie monomial of the particular form $(\cdots(x_{i_1} \circ x_{i_2}) \circ \cdots) \circ x_{i_\ell}$ is called *left-normed.* For simplicity, we shall use for it the bracket-free notation $x_{i_1} \circ x_{i_2} \circ \cdots \circ x_{i_\ell}$. It is easy to see that the R-module L_R is generated by the set of all left-normed Lie monomials. Surprisingly, no R-basis of L_R consisting of left-normed Lie monomials is known. Set $L_{R,m} := L_R \cap A_{R,m}$ for all $m \in \mathbb{N}_0$, $L_R(k_1, \ldots, k_n) = L_R \cap A_R(k_1, \ldots, k_n)$ for all $(k_1, \ldots, k_n) \in \mathbb{N}_0^n$. The *multidegree* of a Lie monomial $a \neq 0$ in A_R is the unique n-tuple $(k_1, \ldots, k_n)$ such that $a \in L_R(k_1, \ldots, k_n)$. We have

$$
\begin{aligned}
L_R &= \bigoplus_{m \in \mathbb{N}_0} L_{R,m} \\
L_{R,m} &= \bigoplus_{k_1 + \cdots + k_n = m} L_R(k_1, \ldots, k_n) \qquad (m \in \mathbb{N}_0).
\end{aligned}
\tag{2}
$$

By [6, II, §2.5], the following holds:

1.1 Proposition. The mapping

$$
\begin{aligned}
\{x_1 \ldots, x_n\} &\to R \otimes_{\mathbb{Z}} L_{\mathbb{Z}} \\
x_j &\mapsto 1 \otimes x_j
\end{aligned}
$$

extends uniquely to a Lie R-algebra isomorphism ϕ_R of L_R onto $R \underset{\mathbb{Z}}{\otimes} L_{\mathbb{Z}}$. In particular, $L_{R,m}\phi_R = R \underset{\mathbb{Z}}{\otimes} L_{\mathbb{Z},m}$ for all $m \in \mathbb{N}_0$, and $L_R(k_1, \dots, k_n)\phi_R = R \underset{\mathbb{Z}}{\otimes} L_{\mathbb{Z}}(k_1, \dots, k_n)$ for all $(k_1, \dots, k_n) \in \mathbb{N}_0^n$.

As an abelian group, $L_{\mathbb{Z}}(k_1, \dots, k_n)$ is torsion-free and finitely generated (for example, by the set of all Lie monomials in $A_{\mathbb{Z}}$ of multidegree $(k_1, \dots, k_n)$). Therefore, $L_{\mathbb{Z}}(k_1, \dots, k_n)$ is a free $\mathbb{Z}$-module of finite rank. As a consequence, $L_R(k_1, \dots, k_n)$ is a free R-module of the same rank. This rank is known to be the so-called *necklace number* as has been proved by Witt [31]. By 1.1, it is justified to specialize R, and for our purposes it is convenient to put $R := \mathbb{C}$. Subsequently we simply write A (L resp.) for $A_{\mathbb{C}}$($L_{\mathbb{C}}$ resp.), similarly A_m ($A(k_1, \dots, k_n)$ resp.) for the space $A_{\mathbb{C},m}$ ($A_{\mathbb{C}}(k_1, \dots, k_n)$ resp.) of all homogeneous elements of degree m (of multidegree $(k_1, \dots, k_n)$ resp.), etc. Now **W**itt's **D**imension **F**ormula reads as follows:

$$\textbf{(WDF)} \qquad \mathit{dim}\ L(k_1, \dots, k_n) = \frac{1}{m} \sum_{d \mid k_1, \dots, k_n} \mu(d) \frac{\frac{m}{d}!}{\frac{k_1}{d}! \cdots \frac{k_n}{d}!}$$

where $m = k_1 + \cdots + k_n$. The necklace number on the right-hand side of (WDF) is the number of Lyndon words in F of multidegree $(k_1, \dots, k_n)$. In group theoretic terms, it is the number of orbits of length m of the subgroup $\langle (1 \dots m) \rangle$ of the symmetric group S_m with respect to its left action on the set of left cosets of a Young subgroup of S_m of isomorphism type $\mathrm{S}_{k_1} \times \cdots \times \mathrm{S}_{k_n}$.

Putting $(x_{i_1} \cdots x_{i_\ell})^{\circ} := x_{i_1} \circ \cdots \circ x_{i_\ell}$ for all $i_1, \dots, i_\ell \in \underline{n}$, we obtain a vector space epimorphism ${}^{\circ} : A \to L$, sometimes called the *Dynkin mapping.* Grün (see [20, footnote 12]) expressed this mapping by means of the *Weyl action* of S_m on the space A_m: By the rule

$$\sigma x_{i_1} \cdots x_{i_m} := x_{i_{1\sigma}} \cdots x_{i_{m\sigma}} \qquad (i_1, \dots, i_m \in \underline{n}, \sigma \in \mathrm{S}_m),$$

A_m is made into a $\mathbb{C}\mathrm{S}_m$-left module. Obviously, the spaces $A(k_1, \dots, k_n)$ where $k_1 + \cdots + k_n = m$ are $\mathbb{C}\mathrm{S}_m$-submodules of A_m. Let

$$\mathcal{X}_m := \{\pi \mid \pi \in \mathrm{S}_m, 1\pi > 2\pi > \ldots > 1 < \ldots < (m-1)\pi < m\pi\}$$

and

$$\omega_m := \sum_{\pi \in \mathcal{X}_m} (-1)^{1\pi^{-1}-1} \pi \in \mathbb{C}\mathrm{S}_m.$$

Then

$$(3) \qquad a^{\circ} = \omega_m a \text{ for all } a \in A_m.$$

The important criterion by **D**ynkin [8], **S**pecht [25], **W**ever [29] characterizes the Lie elements of A_m by means of the Dynkin mapping:

$$(\mathbf{DSW}) \qquad a \in L_m \Longleftrightarrow a^\circ = ma, \text{ for any } a \in A_m.$$

Putting $\nu_m := \frac{1}{m}\omega_m$, we have, by (3),

$$\nu_m A_m = L_m, \tag{4}$$

as the Dynkin mapping is onto. Hence the essential content of (DSW) is the following:

$$\nu_m^2 = \nu_m. \tag{5}$$

In the special case of $m = n$, (4) implies that

$$\nu_n A(1, \underset{n}{\ldots}, 1) = L(1, \underset{n}{\ldots}, 1). \tag{6}$$

As an operator, every $\phi \in \mathbb{C}\mathrm{S}_n$ such that $\phi A(1, \underset{n}{\ldots}, 1) = L(1, \underset{n}{\ldots}, 1)$ maps the space of all homogeneous elements of degree n of an *arbitrary* free associative algebra onto its subspace of homogeneous Lie elements of degree n. This is due to the fact that A is free over $\{x_1, \ldots, x_n\}$.

We now fix $n \in \mathbb{N}$ and choose a primitive n-th root of unity ε. An element $\phi \in \mathbb{C}\mathrm{S}_n$ is called a *Lie idempotent* if $\phi A(1, \underset{n}{\ldots}, 1) = L(1, \underset{n}{\ldots}, 1)$ and $\phi^2 = \phi$. By (5) and (6),

$$\nu_n = \frac{1}{n} \sum_{\pi \in \mathcal{X}_n} (-1)^{1\pi^{-1}-1} \pi$$

is a Lie idempotent. Using (5), it is easy to check, for an arbitrary element $\phi \in \mathbb{C}\mathrm{S}_n$, that

$$\phi \text{ is a Lie idempotent if and only if } \phi\nu_n = \nu_n, \ \nu_n\phi = \phi. \tag{7}$$

This is a first example of the general phenomenon that significant notions in the theory of free Lie algebras may be characterized by means of formally simple equations in the group ring $\mathbb{C}\mathrm{S}_n$. Applying (7), we have:

(8)

The Lie idempotents in $\mathbb{C}\mathrm{S}_n$ are the idempotent generators of the right ideal $\nu_n\mathbb{C}\mathrm{S}_n$.

A second important example of a Lie idempotent was given by Klyachko in 1974. For every $\sigma \in \mathrm{S}_n$ set

$$ind\ \sigma := \sum \{ j \mid j \in \underline{n-1}\!\!\lrcorner\, , j\sigma > (j+1)\sigma \}$$

(called the *(major) index* of σ, [19, sect. III, VI, 104.]). Then

$$\lambda_n := \frac{1}{n} \sum_{\sigma \in \mathrm{S}_n} \varepsilon^{ind\ \sigma} \sigma$$

is a Lie idempotent ([16], [2,3.4.3]). [1])

Starting from any Lie idempotent, an analysis of [3, 1^{st} Theorem 2.3] yields a simple method of constructing a family of related Lie idempotents:

1.2 Proposition. Let K be a subfield of $\mathbb{C}$, $\phi = \sum\limits_{\sigma\in \mathrm{S}_n} c_\sigma \sigma \in K\mathrm{S}_n$ be a Lie idempotent, B be a $\mathbb{Q}$-basis of K such that $1 \in B$. For all $b \in B$ let $\phi_b \in \mathbb{Q}\mathrm{S}_n$ such that $\phi = \sum\limits_{b\in B} b\phi_b$. (Almost all ϕ_b are 0.) Then

$$\phi_1 + \sum_{b\in B\setminus\{1\}} d_b\phi_b$$

is a Lie idempotent, for every choice of the coefficients $d_b \in \mathbb{C}$.

Proof. As $\nu_n \in \mathbb{Q}\mathrm{S}_n$, (7) implies that $\phi_1\nu_n = \nu_n$, $\phi_b\nu_n = 0$ for all $b \in B\setminus\{1\}$, and $\nu_n\phi_b = \phi_b$ for all $b \in B$. But these equations imply our claim, again by (7). $\square$

Of course, new Lie idempotents are obtained by means of (7) only if not all the coefficients c_σ of ϕ are rational. The case of $\phi := \lambda_n$, $K := \mathbb{Q}(\varepsilon)$, $B := \{1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots\}$ has been considered in [3, (2.20)].

In 1986, Reutenauer discovered a further Lie idempotent; it has rational coefficients: For every $\sigma \in \mathrm{S}_n$ we define the *defect set* of σ by

$$\mathrm{D}(\sigma) := \{j \mid j \in \underline{n-1}\!\lrcorner\,, j\sigma > (j+1)\sigma\},$$

and the *defect* of σ by

$$d(\sigma) := |\mathrm{D}(\sigma)|.$$

[1]For any variable t we have the identity

$$\sum_{\sigma\in \mathrm{S}_n} t^{ind\ \sigma}\sigma = \prod_{j=1}^{n}(id + t\tau_j + t^2\tau_j^2 + \cdots + t^{j-1}\tau_j^{j-1}) \tag{9}$$

where $\tau_j = (j \ldots 1)$. This yields a product representation of λ_n if we put $t := \varepsilon$. Furthermore, if Φ is a representation of the group ring of S_n over the field $\mathbb{C}(t)$, then (9) implies that

$$\sum_{\sigma\in \mathrm{S}_n} t^{ind\ \sigma}\Phi(\sigma) = \prod_{j=1}^{n}(\Phi(id) + t\Phi(\tau_j) + t^2\Phi(\tau_j)^2 + \cdots + t^{j-1}\Phi(\tau_j)^{j-1}).$$

In the special case of the 1-dimensional representations this reduces to the following identities:

$$\begin{aligned} \textstyle\sum_{\sigma\in \mathrm{S}_n} t^{ind\ \sigma} &= \textstyle\prod_{j=1}^{n}(1 + t + t^2 + \cdots + t^{j-1}) \qquad ([27, 4.5.9]),\\ \textstyle\sum_{\sigma\in \mathrm{S}_n} sgn(\sigma)t^{ind\ \sigma} &= \textstyle\prod_{j=1}^{n}(1 + (-1)^{j+1}t + t^2 + (-1)^{j+1}t^3 + \cdots + (-1)^{j+1}t^{j-1}) \text{ resp.} \end{aligned}$$

Then

$$\rho_n := \frac{1}{n} \sum_{\sigma \in \mathrm{S}_n} \frac{(-1)^{d(\sigma)}}{\binom{n-1}{d(\sigma)}} \sigma$$

is a Lie idempotent ([21, (1.4)]).

We have an easy characterization of the elements $\pi \in \mathcal{X}_n$ which occur in ν_n, in terms of defect sets: Let $\pi \in \mathrm{S}_n$ and $r := d(\pi)$. Then the following three statements are equivalent:

(10)
$$\begin{cases} \pi \in \mathcal{X}_n \\ \mathrm{D}(\pi) = \underline{r} \\ \pi = (j_1 \ldots 1) \cdots (j_r \ldots 1) \text{ for some } j_1, \ldots, j_r \in \underline{n} \text{ such that } j_1 > j_2 > \ldots > j_r > 1. \end{cases}$$

The equation $\pi = (j_1 \ldots 1) \cdots (j_r \ldots 1)$ where $j_1 > j_2 > \ldots > j_r > 1$ implies that $j_\ell = \ell\pi$ for all $\ell \in \underline{r}$, hence $\mathrm{D}(\pi)\pi = \{j_1, \ldots, j_r\}$. In particular, we have:

(11) For every $C \subseteq \underline{n} \setminus \{1\}$ there exists a unique $\pi \in \mathcal{X}_n$ such that $\mathrm{D}(\pi)\pi = C$. The mapping $\pi \mapsto \mathrm{D}(\pi)\pi$ is a bijection of $\mathcal{X}_n$ onto the power set of $\underline{n} \setminus \{1\}$.

The following three simple properties of the elements $\pi \in \mathcal{X}_n$ will be useful at a later stage:

$$\mathrm{D}(\pi^{-1}) = \mathrm{D}(\pi)\pi - 1 \tag{12}$$

(In particular, the inverses of any two distinct elements of $\mathcal{X}_n$ have distinct defect sets.) For $k, \ell \in \underline{n}$ we have

$$\text{If } k < \ell \text{ and } k\pi > \ell\pi, \text{ then } k \in \mathrm{D}(\pi). \tag{13}$$

$$\text{If } k < \ell \text{ and } k\pi < \ell\pi, \text{ then } \ell \notin \mathrm{D}(\pi). \tag{14}$$

Moreover, by (10),

$$\nu_n = \frac{1}{n} \sum_{\pi \in \mathcal{X}_n} (-1)^{d(\pi)} \pi = \frac{1}{n} (id - (n \ldots 1))(id - (n-1 \ldots 1)) \cdots (id - (21)). \tag{15}$$

This last description of ν_n as a product has first been noted by Magnus [20].

The coefficients of any Lie idempotent have a remarkable property which was discovered by Wever [30, Satz 4] in the case of the particular Lie idempotent ν_n (see also [13], [5]). The following general version of this result is due to Garsia [10, Proposition 5.1]:

1.3 Theorem. Let $\phi = \sum\limits_{\sigma \in \mathrm{S}_n} c_\sigma \sigma \in \mathbb{C}\mathrm{S}_n$ be a Lie idempotent. Then for any conjugacy class C of S_n

$$\sum_{\sigma \in C} c_\sigma = \begin{cases} \frac{\mu(d)}{n} & \text{if } d \mid n \text{ and } (1 \ldots n)^{\frac{n}{d}} \in C \\ 0 & \text{otherwise} \end{cases} .$$

By (8), $\phi \mathbb{C}\mathrm{S}_n = \nu_n \mathbb{C}\mathrm{S}_n$ for all Lie idempotents $\phi \in \mathbb{C}\mathrm{S}_n$. Hence the general statement 1.3 is implied by Wever's special result by means of the following proposition due to Frobenius [9, §1], [7, §9, exerc. 16]:

$$\text{(16)} \qquad \begin{array}{l} \text{If } \phi = \sum\limits_{\sigma \in \mathrm{S}_n} c_\sigma \sigma,\ \psi = \sum\limits_{\sigma \in \mathrm{S}_n} d_\sigma \sigma \text{ are idempotent elements of } \mathbb{C}\mathrm{S}_n \text{, then} \\ \phi \mathbb{C}\mathrm{S}_n \underset{\mathbb{C}\mathrm{S}_n}{\cong} \psi \mathbb{C}\mathrm{S}_n \text{ if and only if } \sum\limits_{\sigma \in C} c_\sigma = \sum\limits_{\sigma \in C} d_\sigma \text{ for all conjugacy classes } C \text{ of } \mathrm{S}_n. \end{array}$$

We now turn to another aspect of the theory which concerns representations of general linear groups. The vector space A_n may be identified with the n-fold tensor product $V \otimes \underset{n}{\ldots} \otimes V$ where $V = A_1$. Therefore, in a natural way, A_n is a $GL(V)$-(right) module. In his doctoral thesis of 1901 [22] and in a famous paper of 1927 [23], Schur described the decomposition of A_n into irreducible $GL(V)$-modules in terms of irreducible representations of S_n: If p is a partition of n ($p \vdash n$) and U^p is an irreducible $\mathbb{C}\mathrm{S}_n$-module corresponding to a Young diagram of shape p, then $U^p \underset{\mathbb{C}\mathrm{S}_n}{\otimes} A_n$ is either 0 or is an irreducible $GL(V)$-module. A_n is a direct sum of modules of this type, and the multiplicity of $U^p \underset{\mathbb{C}\mathrm{S}_n}{\otimes} A_n$ in A_n is the number of standard Young tableaux of shape p, denoted by st^p. That is,

$$A_n \underset{GL(V)}{\cong} \bigoplus_{p \vdash n} st^p (U^p \underset{\mathbb{C}\mathrm{S}_n}{\otimes} A_n). \tag{17}$$

Obviously, L_n is a $GL(V)$-submodule of A_n. More than fifty years ago, the question was raised as to how the $GL(V)$-module structure of L_n could be described [28]. In the meantime, various contributions to this problem have been achieved, but a satisfactory answer in the spirit of Schur's result (17) was discovered only recently. Let us first recall a module isomorphism of preliminary character proved by Klyachko in 1974. We write C_n for the eigenspace of the cycle $(1 \ldots n)$ in A_n with respect to the eigenvalue ε.

1.4 Proposition ([16, Proposition 1]). $L_n \underset{GL(V)}{\cong} C_n$.

The desired decomposition of L_n into $GL(V)$-irreducible constituents was finally obtained in 1987: For every Young tableau T put

$$maj\ T := \sum \{ j \mid j \in \underline{n-1}\!\!\lrcorner\ ,\ j+1 \text{ is in a lower row of } T \text{ than } j \}$$

(called the *major index* of T). For $p \vdash n$ and $i \in \underline{n}$, let st_i^p be the number of all standard Young tableaux T of shape p such that $maj\ T \equiv i\ mod\ n$. The main result on the $GL(V)$-module structure of L_n is the following:

1.5 Theorem (see [10, 8.]). $L_n \underset{GL(V)}{\cong} \bigoplus_{p \vdash n} st_1^p (U^p \underset{\mathbb{C}\mathrm{S}_n}{\otimes} A_n)$.

2 On (Ind), a key result

A proof of 1.5 is obtained by means of two non-trivial results which are interesting in their own right (2.1 and (Ind)).

For every $j \in \underline{n} \cup \{0\}$ let M_j be a 1-dimensional $\langle(1 \ldots n)\rangle$-module over $\mathbb{C}$ such that the character of $(1 \ldots n)$ is ε^j. The first result to be mentioned here is the following:

2.1 Theorem (Kraskiewicz, Weyman [17], (see also Springer [26, 4.5]))

$$M_j^{\mathrm{S}_n} \underset{\mathbb{C}\mathrm{S}_n}{\cong} \bigoplus_{p \vdash n} st_j^p U^p \qquad \text{for every } j \in \underline{n} \cup \{0\}.$$

As for the second result, we remark first that the natural action of S_n on $\{x_1, \ldots, x_n\}$ gives rise to a $\mathbb{C}\mathrm{S}_n$-right module structure on the spaces $A(1, \underset{n}{\ldots}, 1)$ and $L(1, \underset{n}{\ldots}, 1)$. We observe

$$(18) \qquad A(1, \underset{n}{\ldots}, 1) \text{ is a regular } \mathbb{C}\mathrm{S}_n\text{-right module,}$$

and

$$(19) \qquad L_n \underset{GL(V)}{\cong} L(1, \underset{n}{\ldots}, 1) \underset{\mathbb{C}\mathrm{S}_n}{\otimes} A_n.$$

The following statement proves to be a key result for the whole context as the discussion in this section will show. An equivalent form of it is already contained in Wever's paper [30, Satz 5] and was rediscovered in 1974 by Klyachko [16, Corollary 1]:

$$(\textbf{Ind}) \qquad L(1, \underset{n}{\ldots}, 1) \underset{\mathbb{C}\mathrm{S}_n}{\cong} M_1^{\mathrm{S}_n}.$$

The induced $\mathbb{C}\mathrm{S}_n$-module $M_1^{\mathrm{S}_n}$ is obviously isomorphic to the right ideal of $\mathbb{C}\mathrm{S}_n$ generated by the following idempotent element:

$$\zeta_n := \frac{1}{n} \sum_{i=0}^{n-1} \varepsilon^{-i} (1 \ldots n)^i.$$

Hence $M_1^{\mathrm{S}_n} \underset{\mathbb{C}\mathrm{S}_n}{\otimes} A_n$ is $GL(V)$- isomorphic to the eigenspace C_n.

(a) Now 1.4 is a consequence of the isomorphisms

$$L_n \underset{GL(V)}{\cong} L(1, \underset{n}{\ldots}, 1) \underset{\mathbb{C}\mathrm{S}_n}{\otimes} A_n \underset{GL(V)}{\cong} M_1^{\mathrm{S}_n} \underset{\mathbb{C}\mathrm{S}_n}{\otimes} A_n$$

which follow from (19) and (Ind).

(b) Applying 2.1 (where $j = 1$), we obtain 1.5.

(c) Theorem 1.3, too, follows easily from (Ind): Let $\phi \in \mathbb{C}\mathrm{S}_n$ be any Lie idempotent. By (18), $A(1, \underset{n}{\ldots}, 1) \underset{\mathbb{C}\mathrm{S}_n}{\cong} \mathbb{C}\mathrm{S}_n$, hence

$$\phi\mathbb{C}\mathrm{S}_n \underset{\mathbb{C}\mathrm{S}_n}{\cong} \phi A(1, \underset{n}{\ldots}, 1) = L(1, \underset{n}{\ldots}, 1) \underset{\mathbb{C}\mathrm{S}_n}{\cong} M_1^{\mathrm{S}_n} \underset{\mathbb{C}\mathrm{S}_n}{\cong} \zeta_n\mathbb{C}\mathrm{S}_n.$$

Therefore, by (16), the property stated in the formula of 1.3 for the coefficients of ϕ follows once it is verified for the coefficients of ζ_n. But for ζ_n it is an easy consequence of well-known properties of roots of unity.

(d) Finally, we sketch a short proof of (WDF) exploiting (Ind) (see [4,2.] for more details). Let $k_1, \ldots, k_n \in \mathbb{N}_0$ and $m = k_1 + \cdots + k_n$. Let Y be a Young subgroup of type $\mathrm{S}_{k_1} \times \cdots \times \mathrm{S}_{k_n}$ of S_m, χ be the character of the $\mathbb{C}\mathrm{S}_m$-right module $L(1, \underset{m}{\ldots}, 1)$, and ψ be a faithful irreducible character of $\langle(1 \ldots m)\rangle$. Now (Ind) implies that $(\chi, 1_Y^{\mathrm{S}_m})_{\mathrm{S}_m} = (\psi^{\mathrm{S}_m}, 1_Y^{\mathrm{S}_m})_{\mathrm{S}_m}$. But

$$(\chi, 1_Y^{\mathrm{S}_m})_{\mathrm{S}_m} = (\chi\mid_Y, 1_Y)_Y = dim\ C_{L(1,\underset{m}{\ldots},1)}(Y)$$

which is equal to the dimension of $L(k_1, \ldots, k_m)$, and

$$(\psi^{\mathrm{S}_m}, 1_Y^{\mathrm{S}_m})_{\mathrm{S}_m} = (\psi, 1_Y^{\mathrm{S}_m}\mid_{<(1\ldots m)>})_{<(1\ldots m)>}$$

which is the number of orbits of length m of the subgroup $\langle(1 \ldots m)\rangle$ of S_m with respect to its left action on the set of left cosets of Y.

3 A self-contained approach

We now present an elementary combinatorial approach to the theory by giving self-contained proofs of (5) and (Ind). For every $D \subseteq \underline{n-1}$ we call

$$\mathrm{S}_n(D) := \{\sigma \mid \sigma \in \mathrm{S}_n, \mathrm{D}(\sigma) = D\}$$

the *defect class* of D in S_n and put

$$\delta_D := \sum_{\sigma \in \mathrm{S}_n(D)} \sigma \in \mathbb{C}\mathrm{S}_n.$$

Any defect class contains exactly one of the inverses of the elements of $\mathcal{X}_n$ (cf. (12)).

The following basic lemma by F. Bergeron, N. Bergeron, and Garsia [3,(1.11)] reveals a surprising connection between the concept of the Lie multiplication and that of the defect of permutations:

3.1 Lemma. $\delta_D \nu_n = (-1)^{|D|} \nu_n$ for all $D \subseteq \underline{n-1}$.

A direct simple proof of 3.1 would be of interest, as has been remarked already in [3]. We propose to proceed as follows: For every $\sigma \in \mathrm{S}_n$ we put $\mathrm{D}_0(\sigma) := \mathrm{D}(\sigma) \cup \{0\}$,

$$\begin{aligned} P_\sigma &:= \mathrm{D}(\sigma) \backslash (1 + \mathrm{D}_0(\sigma)), \\ T_\sigma &:= (1 + \mathrm{D}_0(\sigma)) \backslash \mathrm{D}(\sigma) \qquad (= (1 + \mathrm{D}_0(\sigma)) \backslash \mathrm{D}_0(\sigma)), \end{aligned}$$

and call the elements of P_σ the *peaks*, the elements of T_σ the *troughs* of σ. Let $j \in \underline{n}$. Then

$$j \in P_\sigma \text{ if and only if } j \neq 1, j \neq n, \text{ and } (j-1)\sigma,\ (j+1)\sigma < j\sigma$$

and

$$\begin{aligned} j \in T_\sigma \text{ if and only if } & j = 1 \text{ and } 2\sigma > 1\sigma, \text{ or } j = n \text{ and } (n-1)\sigma > n\sigma, \\ & \text{or } 1 < j < n \text{ and } (j-1)\sigma,\ (j+1)\sigma > j\sigma. \end{aligned}$$

By (10), we have

$$\sigma \in \mathcal{X}_n \Longleftrightarrow P_\sigma = \emptyset \Longleftrightarrow T_\sigma = \{1\sigma^{-1}\}. \tag{20}$$ [2])

[2]As an easy consequence, we mention a further characterization: $\sigma \in \mathcal{X}_n$ if and only if $\underline{k}\sigma^{-1}$ is an interval, for every $k \in \underline{n}$.

3.2 Proposition. Let $\sigma \in \mathrm{S}_n$, $D \subseteq \underline{n-1}\rfloor$, $L := (1 + (D\backslash \mathrm{D}(\sigma))) \cup (\mathrm{D}(\sigma)\backslash D)$. (Then $T_\sigma \cap L = \emptyset$.)
a) There is an element $\pi \in \mathcal{X}_n$ such that $\mathrm{D}(\sigma\pi^{-1}) = D$ if and only if $P_\sigma \subseteq (D\backslash(1+D)) \cup ((1+D)\backslash D)$.
b) Suppose that there is an element $\psi \in \mathcal{X}_n$ such that $\mathrm{D}(\sigma\psi^{-1}) = D$. Let $\pi \in \mathcal{X}_n$. Then $\mathrm{D}(\sigma\pi^{-1}) = D$ if and only if $L\sigma \subseteq \mathrm{D}(\pi)\pi \subseteq (T_\sigma \cup L)\sigma$.

Proof. For all $A, B \subseteq \mathbb{Z}$ it is straightforward to verify that

(21) $((1+(A\backslash B)) \cup (B\backslash A))\backslash((1+(A\cup B))\backslash(B\cap A)) = (B\backslash(1+B))\backslash((A\backslash(1+A)) \cup ((1+A)\backslash A))$,

(22) $((1+(A\cup B))\backslash(A\cap B))\backslash((1+(A\backslash B)) \cup (B\backslash A)) = (1+B)\backslash B$.

Set $A := D \cup \{0\}$, $B := \mathrm{D}_0(\sigma)$, $R := (1+(A\cup B))\backslash(A\cap B)$. Obviously, $L = (1+(A\backslash B)) \cup (B\backslash A)$. Now (21) and (22) easily imply that $L\backslash R = P_\sigma\backslash((D\backslash(1+D)) \cup ((1+D)\backslash D))$, $R\backslash L = T_\sigma$. Hence

$$R = T_\sigma \cup L \Longleftrightarrow L \subseteq R \Longleftrightarrow P_\sigma \subseteq (D\backslash(1+D)) \cup ((1+D)\backslash D). \tag{23}$$

The main step of our proof is to show the following, for all $\pi \in \mathcal{X}_n$:

$$\mathrm{D}(\sigma\pi^{-1}) = D \Longleftrightarrow L \subseteq \mathrm{D}(\pi)\pi\sigma^{-1} \subseteq R. \tag{24}$$

Suppose first that $\mathrm{D}(\sigma\pi^{-1}) = D$. For every $i \in L$, one of the following two statements holds:

$$\left\langle \begin{array}{llllll} i \neq 1, & i\sigma\pi^{-1} & < & (i-1)\sigma\pi^{-1}, & \text{and} & (i-1)\sigma < i\sigma \\ i \neq n, & i\sigma\pi^{-1} & < & (i+1)\sigma\pi^{-1}, & \text{and} & (i+1)\sigma < i\sigma \end{array} \right. .$$

By (13), $i\sigma\pi^{-1} \in \mathrm{D}(\pi)$. Hence $L \subseteq \mathrm{D}(\pi)\pi\sigma^{-1}$. Furthermore, for every $i \in \underline{n}\rfloor\backslash R$, one of the following two statements holds:

$$\left\langle \begin{array}{llllll} i \neq 1, & i\sigma\pi^{-1} & > & (i-1)\sigma\pi^{-1}, & \text{and} & (i-1)\sigma < i\sigma \\ i \neq n, & i\sigma\pi^{-1} & > & (i+1)\sigma\pi^{-1}, & \text{and} & (i+1)\sigma < i\sigma \end{array} \right. .$$

By (14), $i\sigma\pi^{-1} \notin \mathrm{D}(\pi)$. Hence $\mathrm{D}(\pi)\pi\sigma^{-1} \subseteq R$.

Conversely, suppose that $L \subseteq \mathrm{D}(\pi)\pi\sigma^{-1} \subseteq R$. We show, for all $i \in \underline{n-1}\rfloor$, that

$$i \in \mathrm{D}(\sigma\pi^{-1}) \Longleftrightarrow i \in D. \tag{25}$$

Suppose first that $i \in \mathrm{D}(\sigma)$. Then $(i+1)\sigma < i\sigma$. By hypothesis, $\mathrm{D}(\sigma)\backslash D \subseteq \mathrm{D}(\pi)\pi\sigma^{-1} \subseteq \underline{n}\rfloor\backslash(\mathrm{D}(\sigma) \cap D)$, and therefore

$$i \in D \Longleftrightarrow i \notin \mathrm{D}(\pi)\pi\sigma^{-1} \Longleftrightarrow i\sigma\pi^{-1} \notin \mathrm{D}(\pi) \Longleftrightarrow i\sigma\pi^{-1} > (i+1)\sigma\pi^{-1},$$

by (13) and (14). Similarly, if $i \notin D(\sigma)$, then $i\sigma < (i+1)\sigma$. By hypothesis, $1+(D\backslash \mathrm{D}(\sigma)) \subseteq \mathrm{D}(\pi)\pi\sigma^{-1} \subseteq 1+(D\cup \mathrm{D}_0(\sigma))$, and therefore

$$i \in D \Longleftrightarrow i+1 \in \mathrm{D}(\pi)\pi\sigma^{-1} \Longleftrightarrow (i+1)\sigma\pi^{-1} \in \mathrm{D}(\pi) \Longleftrightarrow i\sigma\pi^{-1} > (i+1)\sigma\pi^{-1},$$

by (14) and (13). Thus in both cases (25) holds. The proof of (24) is complete.

As $L_\sigma \subseteq \underline{n}\backslash\{1\}$, the statements (24) and (11) imply that

$$(\exists \pi \in \mathcal{X}_n \quad \mathrm{D}(\sigma\pi^{-1}) = D) \Longleftrightarrow L \subseteq R. \tag{26}$$

By (23), this implies a). Under the hypothesis of b), (26) implies that $L \subseteq R$, hence $R = T_\sigma \cup L$, by (23). By means of (24), we obtain b). □

3.3 Corollary. Let $\sigma \in \mathrm{S}_n,\ D \subseteq \underline{n-1}\,,\ \mathcal{X}(\sigma, D) := \{\pi | \pi \in \mathcal{X}_n,\ \mathrm{D}(\sigma\pi^{-1}) = D\}$.
a) If $\sigma \in \mathcal{X}_n$, then $\mathcal{X}(\sigma, D)$ contains exactly one element π, and we have $(-1)^{d(\pi)} = (-1)^{|D|+d(\sigma)}$.
b) If $\sigma \notin \mathcal{X}_n$, then $\sum\limits_{\pi\in\mathcal{X}(\sigma,D)} (-1)^{d(\pi)} = 0$.

Proof. a) If $\sigma \in \mathcal{X}_n$, then (20) implies that $P_\sigma = \emptyset$ and $T_\sigma = \{1\sigma^{-1}\}$. By 3.2a), $\mathcal{X}(\sigma, D) \neq \emptyset$. If $\pi \in \mathcal{X}(\sigma, D)$, then $L\sigma \subseteq \mathrm{D}(\pi)\pi \subseteq \{1\} \cup L\sigma$ by 3.2b), hence $\mathrm{D}(\pi)\pi = L\sigma$. By (11), $\mathcal{X}(\sigma, D) = \{\pi\}$. Furthermore, $\mathrm{D}(\sigma) = \underline{d(\sigma)}$, and therefore $(1+(D\backslash\mathrm{D}(\sigma)))\cap(\mathrm{D}(\sigma)\backslash D) = \emptyset$. Hence

$$d(\pi) = |L| = |D\backslash\mathrm{D}(\sigma)| + |\mathrm{D}(\sigma)\backslash D| \equiv |D| + |\mathrm{D}(\sigma)| \qquad \mathit{mod}\, 2.$$

b) If $\sigma \notin \mathcal{X}_n$, then $|T_\sigma| \geq 2$ by (20). By 3.2b) and (11), there is a 1-1 correspondence between $\mathcal{X}(\sigma, D)$ and the power set of $T_\sigma\backslash\{1\sigma^{-1}\}$. Hence
$\sum\limits_{\pi\in\mathcal{X}(\sigma,D)} (-1)^{|\mathrm{D}(\pi)|} = (-1)^{|L|} \cdot \sum\limits_{S\subseteq T_\sigma\backslash\{1\sigma^{-1}\}} (-1)^{|S|} = 0.$ □

Proof of 3.1. For all $D \subseteq \underline{n-1}$ we have, by 3.3,

$$\delta_D\omega_n = \sum_{\sigma\in\mathrm{S}_n(D)} \sum_{\pi\in\mathcal{X}_n} (-1)^{d(\pi)}\sigma\pi = \sum_{\rho\in\mathrm{S}_n} \sum_{\substack{\pi\in\mathcal{X}_n \\ \mathrm{D}(\rho\pi^{-1})=D}} (-1)^{d(\pi)}\rho = \sum_{\rho\in\mathcal{X}_n} (-1)^{|D|+d(\rho)}\rho = (-1)^{|D|}\omega_n.$$

□

As a first application of 3.1, we obtain a simple **proof of (5)**: For every $\pi \in \mathcal{X}_n$ we have $d(\pi) = 1\pi^{-1} - 1$, and therefore

$$\omega_n = \sum_{d=0}^{n-1} (-1)^d \delta_{\underline{d}} \tag{27}$$

(cf. [3, Theorem 1.1]). Hence $\nu_n^2 = \frac{1}{n}\sum\limits_{d=0}^{n-1} (-1)^d\delta_{\underline{d}}\nu_n = \frac{1}{n}\sum\limits_{d=0}^{n-1} (-1)^{2d}\nu_n = \nu_n$. □

A further immediate consequence of 3.1 is the following:

$$(28) \qquad \sum_{D \subseteq \underline{n-1}\rfloor} t^{\Sigma D} \delta_d \nu_n = \prod_{j=1}^{n-1} (1-t^j)\nu_n \qquad (t \text{ a variable}),$$

where $\sum D := \sum_{i \in D} i$ ([3, Theorem 2.1], [4, (9)]). Putting $t := \varepsilon$ we obtain

$$(29) \qquad \lambda_n \nu_n = \nu_n .$$

This equation leads to a short **proof of (Ind)** and, simultaneously, of the fact that λ_n is a Lie idempotent:

By a direct calculation one has the equation $\lambda_n \zeta_n = \lambda_n$ ([16, Lemma 2, 1)], [2,3.4.3]). Hence, by (29), $\nu_n \mathbb{C}\mathrm{S}_n = \lambda_n \zeta_n \nu_n \mathbb{C}\mathrm{S}_n \subseteq \lambda_n \zeta_n \mathbb{C}\mathrm{S}_n$. Now $dim\ \lambda_n \zeta_n \mathbb{C}\mathrm{S}_n \leq dim\ \zeta_n \mathbb{C}\mathrm{S}_n = dim\ M_1^{\mathrm{S}_n} = (n-1)!$, and $dim\ \nu_n \mathbb{C}\mathrm{S}_n = dim\ \nu_n A(1, \underset{n}{\ldots}, 1) = dim\ L(1, \underset{n}{\ldots}, 1)$ by (18) and (6). It is well known that the Lie monomials $x_1 \circ x_{2\sigma} \circ \cdots \circ x_{n\sigma}$ ($\sigma \in Stab_{\mathrm{S}_n}(1)$) form a basis of $L(1, \underset{n}{\ldots}, 1)$ (cf., e.g., [2, 4.8.1]). Hence $dim\ \nu_n \mathbb{C}\mathrm{S}_n = |Stab_{\mathrm{S}_n}(1)| = (n-1)!$.[3]) We conclude that

$$(30) \qquad \nu_n \mathbb{C}\mathrm{S}_n = \lambda_n \zeta_n \mathbb{C}\mathrm{S}_n = \lambda_n \mathbb{C}\mathrm{S}_n ,$$

and the left multiplication by λ_n induces a $\mathbb{C}\mathrm{S}_n$-right module isomorphism of $\zeta_n \mathbb{C}\mathrm{S}_n$ onto $\nu_n \mathbb{C}\mathrm{S}_n$. This yields (Ind). □

As ν_n is an idempotent, (30) implies that

$$(31) \qquad \nu_n \lambda_n = \lambda_n .$$

Now (29) and (31) show that λ_n is a Lie idempotent, by (7). □

We conclude this section by a further application of 3.2:

3.4 Corollary. Let $D \subseteq \underline{n-1}\rfloor$, $0 \leq k < n$. For all $S \subseteq \underline{n-1}\rfloor$ set $b_S := \binom{|S \backslash (1+\mathrm{S}_0)|}{k - |(1+(D\backslash S)) \cup (S \backslash D)|}$ where $\mathrm{S}_0 := S \cup \{0\}$. Then

$$\delta_D \delta_{\underline{k}\rfloor} = \sum_S b_S \delta_S$$

where the sum ranges over all $S \subseteq \underline{n-1}\rfloor$ such that $S \backslash (1+\mathrm{S}_0) \subseteq ((1+D)\backslash D) \cup (D \backslash (1+D))$.

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[3]Alternatively, we may use the fact that ν_n is an idempotent to derive this formula for $dim\ \nu_n \mathbb{C}\mathrm{S}_n$ from a result due to Frobenius [9]: The coefficient of id in ν_n is $\frac{1}{n}$. Hence, if χ is the character of the $\mathbb{C}\mathrm{S}_n$-module $\nu_n \mathbb{C}\mathrm{S}_n$, then $\sum_{\sigma \in \mathrm{S}_n} \frac{1}{n} = \chi(id) = dim\ \nu_n \mathbb{C}\mathrm{S}_n$.

Proof. For every $\sigma \in \mathrm{S}_n$ put $\mathcal{X}^k(\sigma, D) := \{\pi | \pi \in \mathcal{X}_n,\ \mathrm{D}(\sigma\pi^{-1}) = D \text{ and } d(\pi) = k\}$. We show:

(32)

If $\sigma \in \mathrm{S}_n$ such that $S \backslash (1 + \mathrm{S}_0) \subseteq ((1 + D) \backslash D) \cup (D \backslash (1 + D))$ (where $S := \mathrm{D}(\sigma)$), then $|\mathcal{X}^k(\sigma, D)| = b_S$.

By 3.2a), the hypothesis of (32) implies that there exists an element $\pi \in \mathcal{X}_n$ such that $\mathrm{D}(\sigma\pi^{-1}) = D$. By 3.2b) there is then a 1-1 correspondence between $\mathcal{X}^k(\sigma, D)$ and the set of all subsets of order k of $(L \cup T_\sigma) \backslash \{1\sigma^{-1}\}$ containing L. Therefore

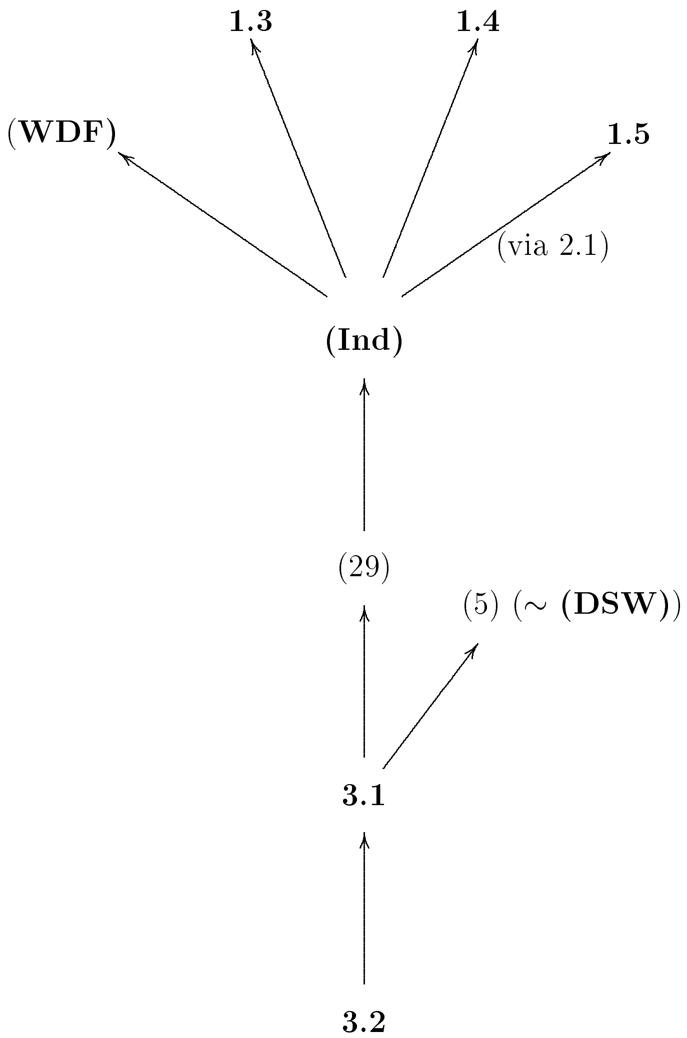
$$|\mathcal{X}^k(\sigma, D)| = \binom{|T_\sigma| - 1}{k - |L|} = b_S,$$

as $|S \backslash (1 + \mathrm{S}_0)| = |(1 + \mathrm{S}_0) \backslash S| - 1$. This shows (32). We conclude that

$$\delta_D \delta_{\underline{k}\rfloor} = \sum_{\rho \in \mathrm{S}_n(D)} \sum_{\substack{\pi \in \mathcal{X}_n \\ d(\pi) = k}} \rho\pi = \sum_{\sigma \in \mathrm{S}_n} |\mathcal{X}^k(\sigma, D)| \sigma = \sum b_{\mathrm{D}(\sigma)} \sigma,$$

where the last sum ranges over all $\sigma \in \mathrm{S}_n$ such that $P_\sigma \subseteq ((1 + D) \backslash D) \cup (D \backslash (1 + D))$.□

We summarize the logical structure of the principal parts of the preceding sections by means of the following diagram:



4. Some remarks about Solomon's descent algebra

Let Δ_n be the subspace of $\mathbb{C}S_n$ generated by all elements δ_S ($S \subseteq \underline{n-1}$). A particular aspect of 3.4 is that for every $D \subseteq \underline{n-1}$, $0 \leq k < n$, the product $\delta_D \delta_{\underline{k}}$ is contained in Δ_n. This is a special case of the following result due to Solomon [24]:

4.1 Theorem. Δ_n is multiplicatively closed.

Hence Δ_n is a subalgebra of $\mathbb{C}S_n$, called the *Solomon algebra* (with respect to $\underline{n}$). It should be noted that all Lie idempotents mentioned before (ν_n, λ_n, ρ_n) are elements of Δ_n. The equations in (5), 3.1, (29), (31), 3.4 may be viewed as details of the multiplicative structure of Δ_n. Garsia and Reutenauer proved a remarkable characterization of the Solomon algebra: By means of certain Lie terms, they defined a set of subspaces of the free associative algebra which are normalized by an element $\gamma \in \mathbb{C}S_n$ if and only if $\gamma \in \Delta_n$ ([12, Theorem 4.5]).

In the following we give two rather different but equally simple proofs of 4.1. In the first one we introduce a graph structure on the set of points S_n. The second one will consist in showing 4.3.

We define a lexicographic ordering on S_n by putting $\pi \underset{lex}{<} \rho$ if $\pi \neq \rho$ and $i\pi < i\rho$ for the smallest $i \in \underline{n\rfloor}$ such that $i\pi \neq i\rho$ $(\pi, \rho \in \mathrm{S}_n)$.

An element $\sigma^* \in \mathrm{S}_n$ is called a *neighbour* of $\sigma \in \mathrm{S}_n$ if there is a number $k \in \underline{n-1\rfloor}$ such that

$$\sigma^* = \sigma(k, k+1) \text{ and } |k\sigma^{-1} - (k+1)\sigma^{-1}| \neq 1.$$

The relation on S_n defined in this manner is obviously symmetric and hence yields a non-oriented graph structure on the set of vertices S_n. We denote by $[\sigma]$ the component of σ. Then we have $\mathrm{D}(\rho) = \mathrm{D}(\sigma)$ for every $\rho \in [\sigma]$. This observation is the trivial part of the following result:

4.2 Proposition. Let $D \subseteq \underline{n-1\rfloor}$, $\sigma \in \mathrm{S}_n(D)$. Then $[\sigma] = \mathrm{S}_n(D)$.

Proof. For all $\ell \in \mathbb{N}_0$ we put $M_\ell := \{\mu | \mu \in \mathrm{S}_\ell,\ (i+1)\mu = i\mu - 1 \text{ for all } i \in \mathrm{D}(\mu)\}$. The following statement is easily seen:

(33) Let $\lambda \in \mathrm{S}_n$, $k := n\lambda$. Then $\lambda \in M_n$ if and only if $(k+j)\lambda = n - j$ for all $j \in \underline{n-k\rfloor} \cup \{0\}$

and $\lambda|_{\underline{k-1\rfloor}} \in M_{k-1}$.

As a consequence, we show

(34) For every $T \subseteq \underline{n-1\rfloor}$ there exists a unique element $\mu_n^T \in M_n \cap \mathrm{S}_n(T)$,

in other words, M_n is a set of representatives for the defect classes in S_n. In order to prove (34) by induction on n, we put $k := max((\underline{n-1\rfloor} \cup \{0\}) \backslash T) + 1$. Then we may assume that $M_{k-1} \cap \mathrm{S}_{k-1}(T \cap \underline{k-1\rfloor})$ contains a unique element μ. By (33), $M_n \cap \mathrm{S}_n(T)$ contains the permutation

$$\lambda := \begin{pmatrix} 1 & \dots & k-1 & k & k+1 & \dots & n-1 & n \\ 1\mu & \dots & (k-1)\mu & n & n-1 & \dots & k+1 & k \end{pmatrix}$$

as its only element.

Our next step is to prove

(35) If $\rho \in \mathrm{S}_n \backslash M_n$, then there exists a neighbour ρ^* of ρ such that $\rho^* \underset{lex}{<} \rho$.

We have to show that there is a number $k \in \underline{n-1}\rfloor$ such that $k\rho^{-1} - (k+1)\rho^{-1} > 1$ as then the element $\rho^* := \rho(k, k+1)$ has the required properties. By hypothesis we have $i\rho - (i+1)\rho \geq 2$ for some $i \in \underline{n-1}\rfloor$. Now it suffices to put $k := min\{j|(i+1)\rho \leq j \leq i\rho,\ j\rho^{-1} \geq i\} - 1$.

By (34), $M_n \cap [\rho] \subseteq M_n \cap \mathrm{S}_n(\mathrm{D}(\rho)) = \{\mu_n^{\mathrm{D}(\rho)}\}$ for all $\rho \in \mathrm{S}_n$. The assumption that $\mu_n^{\mathrm{D}(\rho)} \notin [\rho]$ leads, by (35), to the contradiction that $[\rho]$ is infinite. Hence $\mu_n^{\mathrm{D}(\rho)} \in [\rho]$ for all $\rho \in \mathrm{S}_n$. In particular, $\mu_n^D \in [\rho] \cap [\sigma]$ for all $\rho \in \mathrm{S}_n(D)$. □

Proof of 4.1. For any $\sigma, \sigma^* \in \mathrm{S}_n$ which are neighbours of each other we set

$$\begin{aligned} N^1_{\sigma,\sigma^*} &:= \{(\xi,\rho)|\xi,\rho \in \mathrm{S}_n,\quad \xi\rho = \sigma,\quad (\sigma^*\sigma^{-1})\xi \quad \text{is a neighbour of } \xi\}, \\ N^2_{\sigma,\sigma^*} &:= \{(\xi,\rho)|\xi,\rho \in \mathrm{S}_n,\quad \xi\rho = \sigma,\quad \rho(\sigma^{-1}\sigma^*) \quad \text{is a neighbour of } \rho\}. \end{aligned}$$

Let $k \in \underline{n-1}\rfloor$ such that $\sigma^{-1}\sigma^* = (k, k+1)$. If $\xi, \rho \in \mathrm{S}_n$ such that $\xi\rho = \sigma$, then $\xi(k\rho^{-1}, (k+1)\rho^{-1}) = \xi\rho(k, k+1)\rho^{-1} = \sigma(k, k+1)\rho^{-1} = \sigma^*\sigma^{-1}\xi$. Hence $\sigma^*\sigma^{-1}\xi$ is a neighbour of ξ if and only if $|k\rho^{-1} - (k+1)\rho^{-1}| = 1$, i.e., if and only if $\rho\sigma^{-1}\sigma^*$ is not a neighbour of ρ. We write Π_σ for the set of all pairs $(\xi, \rho) \in \mathrm{S}_n \times \mathrm{S}_n$ such that $\xi\rho = \sigma$. Then it follows that

(36) $$\Pi_\sigma \text{ is the disjoint union of } N^1_{\sigma,\sigma^*} \text{ and } N^2_{\sigma,\sigma^*}.$$

It is straightforward to verify the following, for any $\xi, \rho \in \mathrm{S}_n$:

(37) $$(\xi,\rho) \in N^1_{\sigma,\sigma^*} \Longrightarrow (\sigma^*\sigma^{-1}\xi, \rho) \in N^1_{\sigma^*,\sigma},$$

(38) $$(\xi,\rho) \in N^2_{\sigma,\sigma^*} \Longrightarrow (\xi, \rho\sigma^{-1}\sigma^*) \in N^2_{\sigma^*,\sigma}.$$

By symmetry, we conclude that, in particular, $|N^j_{\sigma,\sigma^*}| = |N^j_{\sigma^*,\sigma}|$ $(j = 1, 2)$. We observe that the mapping

$$(\xi,\rho) \mapsto \begin{cases} (\sigma^*\sigma^{-1}\xi, \rho) & \text{if } (\xi,\rho) \in N^1_{\sigma,\sigma^*} \\ (\xi, \rho\sigma^{-1}\sigma^*) & \text{if } (\xi,\rho) \in N^2_{\sigma,\sigma^*} \end{cases}$$

is a bijection of Π_σ onto Π_{σ^*}. In (37), we have $\mathrm{D}(\xi) = \mathrm{D}(\sigma^*\sigma^{-1}\xi)$, and in (38), similarly, $\mathrm{D}(\rho) = \mathrm{D}(\rho\sigma^{-1}\sigma^*)$. As a consequence, we obtain

(39) $$|\Pi_\sigma \cap (\mathrm{S}_n(D) \times \mathrm{S}_n(D'))| = |\Pi_{\sigma^*} \cap (\mathrm{S}_n(D) \times \mathrm{S}_n(D'))| \text{ for all } D, D' \subseteq \underline{n-1}\rfloor.$$

Up to this point our hypothesis was that σ, σ^* were neighbours of each other. But 4.2 shows now that (39) holds, in fact, for any $\sigma, \sigma^* \in \mathrm{S}_n$ such that $\mathrm{D}(\sigma) = \mathrm{D}(\sigma^*)$. Hence

$$\delta_D \cdot \delta_{D'} = \sum_{\sigma \in \mathrm{S}_n} |\Pi_\sigma \cap (\mathrm{S}_n(D) \times \mathrm{S}_n(D'))|\sigma = \sum_{T \subseteq \underline{n-1}\rfloor} |\Pi_{\sigma_T} \cap (\mathrm{S}_n(D) \times \mathrm{S}_n(D'))|\delta_T$$

where, for $T \subseteq \underline{n-1}\rfloor$, the element σ_T is an arbitrary representative of $\mathrm{S}_n(T)$. $\square$

It is obvious that the coefficient of the basis element δ_T in the representation of $\delta_D \cdot \delta_{D'}$ must necessarily be the one given in the last formula of the proof *once it is known that* $\delta_D \cdot \delta_{D'}$ *is contained in* Δ_n. Therefore, the essential statement which yields the claim is not that formula but the foregoing assertion (39) and its subsequent comment.

We now describe another basis of Δ_n and prove a formula for the product of any two of its elements as a linear combination of basis elements whose coefficients, by contrast, turn out to be combinatorially interesting and not obvious at all (4.3). This formula, in a more general context, is essentially due to Solomon ([24, Theorem 1]). A simple application of Coleman's lemma [15, 4.3.7] suffices to gather from Solomon's result the special version which is of interest here. It has been stated explicitly by Garsia and Reutenauer in [12, Proposition 1.1] who refer to [11]. We give an independent elementary proof.

If $q_1, \ldots, q_\ell \in \mathbb{N}$ such that $q_1 + \cdots + q_\ell = n$, the ℓ-tuple $q = (q_1, \ldots, q_\ell)$ is called a *decomposition* of n, and we write $q \models n$. We define the *length* of q by $|q| := \ell$. Then $D(q) := \{q_1, q_1 + q_2, \ldots, q_1 + \cdots + q_{\ell-1}\}$ is a subset of $\underline{n-1}\rfloor$. The mapping $q \mapsto D(q)$ is a bijection from the set of all decompositions of n onto the power set of $\underline{n-1}\rfloor$. We put

$$\Xi^q := \sum_{T \subseteq D(q)} \delta_T.$$

Then $\{\Xi^q | q \models n\}$ is a basis of Δ_n. For $r = (r_1, \ldots, r_k)$, $q = (q_1, \ldots, q_\ell) \models n$ let $\mathcal{M}_{r,q}$ be the set of all $(k \times \ell)$-matrices $M = (m_{ij})$ over $\mathbb{N}_0$ such that

$$\begin{aligned} \sum_{j \in \underline{\ell}\rfloor} m_{ij} &= r_i \quad \text{for all } i \in \underline{k}\rfloor \,, \\ \sum_{i \in \underline{k}\rfloor} m_{ij} &= q_j \quad \text{for all } j \in \underline{\ell}\rfloor \,. \end{aligned}$$

In short, summing up the rows (columns resp.) of M gives r (q resp.). Furthermore, let $w(M)$ be the sequence of all non-zero entries of M written according to the natural order of its rows. More formally, if t is the number of all non-zero entries of M and $s \in \underline{t}\rfloor$, we set $w_s := m_{ij}$ if $m_{ij} \neq 0$ and if there exist exactly $s-1$ entries $m_{x,y} \neq 0$ such that (x, y) is lexicographically smaller than (i, j). Then $w(M) = (w_1, \ldots, w_t)$.

4.3 Proposition. $\Xi^r \cdot \Xi^q = \sum\limits_{M \in \mathcal{M}_{r,q}} \Xi^{w(M)}$ for all $r, q \models n$.

The sum on the right is equal to $\sum\limits_{s \models n} c_{r,q,s} \Xi^s$ where $c_{r,q,s}$ is the number of all $M \in \mathcal{M}_{r,q}$ such that $w(M) = s$. Obviously, 4.1 is an immediate consequence of 4.3. We will derive 4.3 from our next lemma for which we need some more preparations. For every $q = (q_1, \ldots, q_\ell) \models n$, the *standard partition* relative to q is defined to be the ℓ-tuple $P^q := (P_1^q, \ldots, P_\ell^q)$ where

$$P_j^q := (q_1 + \cdots + q_{j-1}) + \underline{q_j}\rfloor \qquad (j \in \underline{\ell}\rfloor\,).$$

The stabilizer of P^q in S_n is a Young subgroup Y^q of S_n of type $\mathrm{S}_{q_1} \times \cdots \times \mathrm{S}_{q_\ell}$. Every coset $Y^q\sigma$ ($\sigma \in \mathrm{S}_n$) contains a lexicographically smallest element. The set $\mathcal{S}^q$ of these elements is called the *Solomon system* of Y^q in S_n. We have the following obvious characterization:

$$\sigma \in \mathcal{S}^q \text{ if and only if } \sigma|_{P_j^q} \text{ is increasing, for all } j \in \underline{\ell} \,. \tag{40}$$

This implies that

$$\Xi^q = \sum_{\sigma \in \mathcal{S}^q} \sigma \qquad \text{for all } q \models n.$$

For every $r, q \models n$ and $M = (m_{ij}) \in \mathcal{M}_{r,q}$ we put (following the main idea of Coleman's lemma [15, 4.3.7])

$$\mathcal{S}^r(M) := \{\rho | \rho \in \mathcal{S}^r,\ |P_i^r \cap P_j^q \rho^{-1}| = m_{ij} \text{ for all } i \in \underline{k}\,, j \in \underline{\ell}\,\}$$

where $k = |r|,\ \ell = |q|$. We have the following remark:

$$P_i^r \cap P_j^q \rho^{-1} \text{ is an interval, for all } i \in \underline{k}\,,\ j \in \underline{\ell}\,, \rho \in \mathcal{S}^r. \tag{41}$$

For, if $x, y \in P_i^r \cap P_j^q \rho^{-1}$ and $z \in \mathbb{N}$ such that $x < z < y$, then $z \in P_i^r$ and, by (40), $x\rho < z\rho < y\rho$. Now $x\rho, y\rho \in P_j^q$, hence $z\rho \in P_j^q$, i.e., $z \in P_j^q \rho^{-1}$. [4])

4.4 Lemma.[5]) Let $r, q \models n$ and $M \in \mathcal{M}_{r,q}$. Then the product mapping $(\rho, \sigma) \mapsto \rho\sigma$ ($\rho, \sigma \in \mathrm{S}_n$) induces a bijection of $\mathcal{S}^r(M) \times \mathcal{S}^q$ onto $\mathcal{S}^{w(M)}$.

Proof. Let $M = (m_{ij}),\ k := |r|,\ \ell := |q|$. For all $(i,j) \in \underline{k} \times \underline{\ell}$ we put

$$R_{ij} := \left(\sum_{(x,y) \in \underline{i-1} \times \underline{\ell}} m_{x,y} + \sum_{y \in \underline{j-1}} m_{i,y} \right) + \underline{m_{ij}}.$$

Up to empty sets (which arise if $m_{ij} = 0$), the sequence $(R_{11}, R_{12}, \ldots, R_{k\ell})$ is the standard partition of $\underline{n}$ relative to $w(M)$. As $|P_i^r| = \sum_j m_{ij} = \sum_j |R_{ij}|$, we have

$$P_i^r = \bigcup_j R_{ij} \quad \text{for all } i \in \underline{k}\,. \tag{42}$$

Next we prove

$$P_i^r \cap P_j^q \rho^{-1} = R_{ij} \text{ for all } i \in \underline{k}\,, j \in \underline{\ell}\,, \rho \in \mathcal{S}^r(M). \tag{43}$$

[4]The proof shows that (41) holds for arbitrary intervals I, J instead of $P_i^r,\ P_j^q$ whenever $\rho|_I$ is increasing or decreasing.

[5]D. B.

We have $P_i^r = \bigcup_j (P_i^r \cap P_j^q \rho^{-1})$ and, by definition of $\mathcal{S}^r(M)$, $|P_i^r \cap P_j^q \rho^{-1}| = m_{ij} = |R_{ij}|$. Hence it suffices to show that for any $j_1, j_2 \in \underline{\ell}$ such that $j_1 < j_2$ every element $x_1 \in P_i^r \cap P_{j_1}^q \rho^{-1}$ is smaller than every element $x_2 \in P_i^r \cap P_{j_2}^q \rho^{-1}$. But for such elements x_1, x_2 we have $x_1\rho \in P_{j_1}^q$, $x_2\rho \in P_{j_2}^q$, hence $x_1\rho < x_2\rho$ as P^q is a standard partition of $\underline{n}$. This implies, by (40), that $x_1 < x_2$.

In particular, (43) implies that

$$P_j^q = \bigcup_i R_{ij}\rho \text{ for all } j \in \underline{\ell}, \rho \in \mathcal{S}^r(M). \tag{44}$$

Now we show that

$$\rho\sigma \in \mathcal{S}^{w(M)} \text{ for all } \rho \in \mathcal{S}^r(M),\ \sigma \in \mathcal{S}^q. \tag{45}$$

By (43) and (40), $\rho\sigma|_{R_{ij}}$ is the composition of two increasing functions, hence is increasing. Therefore, $\rho\sigma$ induces an increasing function on every part of the standard partition of $\underline{n}$ relative to $w(M)$. Now (45) follows from (40).

Let $\tau \in \mathcal{S}^{w(M)}$. If $\rho \in \mathcal{S}^r(M)$, $\sigma \in \mathcal{S}^q$ such that $\rho\sigma = \tau$, then, by (44), we have

$$P_j^q \sigma = \left(\bigcup_i R_{ij} \right) \tau \text{ for all } j \in \underline{\ell}. \tag{46}$$

Hence σ is the uniquely determined permutation of $\underline{n}$ which maps P_j^q increasingly onto $\left(\bigcup_i R_{ij} \right) \tau$, for all $j \in \underline{\ell}$. The uniqueness of σ implies that of ρ as $\rho = \tau\sigma^{-1}$. On the other hand, to prove existence, we note first that

$$|P_j^q| = \sum_i m_{ij} = |\bigcup_i R_{ij}| = |(\bigcup_i R_{ij})\tau| \text{ for all } j \in \underline{\ell}.$$

Hence there exists an element $\sigma \in \mathcal{S}^q$ with the property (46). Let $\rho := \tau\sigma^{-1}$. We claim

$$\rho \in \mathcal{S}^r. \tag{47}$$

Let $i \in \underline{k}$ and $x_1, x_2 \in P_i^r$ such that $x_1 < x_2$. By (42), there exist $j_1, j_2 \in \underline{\ell}$ such that $j_1 \leq j_2$ and $x_1 \in R_{i,j_1}$, $x_2 \in R_{i,j_2}$. By definition of σ we have (46), hence $x_h\tau \in R_{i,j_h}\tau \subseteq P_{j_h}^q \sigma$ $(h = 1, 2)$. If $j_1 = j_2$, then $x_1\tau < x_2\tau$ as $\tau \in \mathcal{S}^{w(M)}$, hence

$$x_1\rho = x_1\tau\sigma^{-1} < x_2\tau\sigma^{-1} = x_2\rho, \tag{48}$$

since $\sigma \in \mathcal{S}^q$. Finally, if $j_1 < j_2$, then again (48) holds, because $x_h\rho = x_h\tau\sigma^{-1} \in P^q_{j_h}$ ($h = 1, 2$). This proves (47).

Furthermore, (46) and (42) imply that $P^r_i \cap P^q_j\rho^{-1} = P^r_i \cap P^q_j\sigma\tau^{-1} = \bigcup_g R_{ig} \cap \bigcup_h R_{h,j} = R_{ij}$. This and (47) show that $\rho \in \mathcal{S}^r(M)$. □

Proof of 4.3. Using 4.4, we obtain that for all $r, q \models n$,

$$\begin{aligned} \Xi^r \cdot \Xi^q &= \sum_{\rho\in\mathcal{S}^r} \rho \cdot \sum_{\sigma\in\mathcal{S}^q} \sigma = \left(\sum_{M\in\mathcal{M}_{r,q}} \sum_{\rho\in\mathcal{S}^r(M)} \rho \right) \cdot \sum_{\sigma\in\mathcal{S}^q} \sigma \\ &= \sum_{M\in\mathcal{M}_{r,q}} \left(\sum_{\rho\in\mathcal{S}^r(M)} \rho \cdot \sum_{\sigma\in\mathcal{S}^q} \sigma \right) = \sum_{M\in\mathcal{M}_{r,q}} \sum_{\tau\in\mathcal{S}^{w(M)}} \tau = \sum_{M\in\mathcal{M}_{r,q}} \Xi^{w(M)}. \end{aligned}$$ □

There is a strong connection between the Solomon algebra and the character theory of S_n. For every $q \models n$ we write ξ^q for the Young character with respect to Y^q, that is, $\xi^q = (1_{Y^q})^{\mathrm{S}_n}$. If $q, r \models n$ such that $\xi^q = \xi^r$ (or, equivalently, Y^q is conjugate to Y^r in S_n), we write $q \approx r$.

It is well known that the same rule as in 4.3 holds for the (tensor) product of two Young characters ξ^r, ξ^q if the Ξ's are replaced by ξ's ([14, 2.9.16]). Hence

$$\Xi^q \mapsto \xi^q \quad (q \models n)$$

extends linearly to an algebra epimorphism c of Δ_n onto the character ring $Cl(\mathrm{S}_n)$ of S_n over $\mathbb{C}$. Solomon [24] showed that its kernel is the Jacobson radical of Δ_n:

$$ker\ c = J(\Delta_n). \tag{49}$$

This may be seen as follows: By the semisimplicity of $Cl(\mathrm{S}_n)$ we know that $J(\Delta_n) \subseteq ker\ c$. On the other hand, $ker\ c$ is the linear span of the elements $\Xi^q - \Xi^r$ where $q \approx r$. Exploiting the multiplication rule 4.3, the nilpotency of $ker\ c$ is readily seen (see the proof of Theorem 3.4 in [1]). Hence $ker\ c$ is a nilpotent ideal, and (49) follows.

Moreover, Atkinson has shown that the nilpotency index of $J(\Delta_n)$ is $n - 1$ ([1, 3.5]).

If $\xi \in Cl(\mathrm{S}_n)$ and C is a conjugacy class of S_n, we write $\xi(C)$ for the unique value of ξ on C. The conjugacy classes of S_n may be indexed by the partitions of n. More precisely, we define C^p (where $p \vdash n$) to be the set of all elements of S_n which have a cycle decomposition of type p. We note:

(50) The linear mappings c^p (where $p \vdash n$) such that

$$c^p(\Xi^q) = \xi^q(C^p) \qquad \text{for all } q \models n$$

are a full set of irreducible representations for Δ_n.

(For a different description see [1,3.].) To prove (50) it suffices to define $d^p(\xi) := \xi(C^p)$ for all $\xi \in Cl(\mathrm{S}_n)$, $p \vdash n$, and to observe that the mappings d^p are a full set of irreducible representations of $Cl(\mathrm{S}_n)$. Since c^p is the composition of c and d^p, this yields (50), in view of (49).

Furthermore, it should be mentioned here (without proof) that $c(\delta_D)$ is the character of a certain skew representation of S_n, for every $D \subseteq \underline{n-1}\lrcorner$.

We conclude this exposition by some remarks about the ideal of Δ_n generated by all Lie idempotents in Δ_n. Using (7) and 3.1, this is easily seen to be $\nu_n\Delta_n$. Moreover, the ideal $\nu_n\Delta_n \cap J(\Delta_n)$ has codimension 1 in $\nu_n\Delta_n$, and the coset $\nu_n + (\nu_n\Delta_n \cap J(\Delta_n))$ is the set of all Lie idempotents in Δ_n. We state, without giving details here:

4.5 Proposition. The dimension of $\nu_n\Delta_n$ is the number of all decompositions of n which are Lyndon words over the alphabet $\mathbb{N}$.

(A proof of this result and related topics will be given in a forthcoming paper.)

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