The Mac Lane method of construction and classification of extensions of cyclic groups by their automorphism groups

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Abstract

The Mac Lane method has been applied to the construction of the second cohomology group in the extension $C_{12} \times \mathrm{Aut}\, C_{12}$. The method simplifies significantly the difficult problem of construction of nonequivalent extensions and allows to investigate their structure.

1 Introduction

Extensions of groups have many applications in physics. Space groups in crystallography being extensions of a three-dimensional translation group by a point group is a classical example of such an application. Many other physical phenomena and theorems can be described in the formalism of extension of groups, too. From this it follows that it is necessary to search methods of obtaining the extensions and investigate their structures.

In this paper we present the application of the Mac Lane method [1,2] to the construction of extensions of finite cyclic groups by a group of its automorphisms. Physical motivation for the investigation of such extensions arises from group-theoretic description of properties of line polymers whose structure is described by a line group [3]. In this case one-dimensional translations form a cyclic group whereas a group of automorphisms, according to the Weyl recipe [4], describes the inner symmetry of a system.

The Mac Lane method allows us to obtain all nonequivalent extensions expressed by a factor system. Calculations were performed for the cyclic group C_{12} having four automorphisms forming a group D_2.

2 Nonequivalent extensions of groups

A group G is an extension of the "passive" group T by an "active" group Q under a given operator action Δ if it has a normal subgroup $T' \lhd G$ isomorphic with T and if the quotient group G/T' is isomorphic with Q.

The groups G, T, Q form an exact sequence:

$$0 \longrightarrow T \xrightarrow{\kappa} G \xrightarrow{\omega} Q \longrightarrow 1. \tag{1}$$

Denoting the elements of the extension G by $\langle t, q\rangle$, $t \in T$, $q \in Q$, the multiplication rule in the group G has the form:

$$\langle t, q\rangle \langle t', q'\rangle = \langle t + qt' + m(q, q'), qq'\rangle . \tag{2}$$

The factors $m(q, q')$ form the so-called factor system, which fully characterises an extension G.

An extension is described for a given operator action $\Delta : Q \to \operatorname{Aut} T$ of the active group Q on a passive group T:

$$\Delta(q) = \begin{pmatrix} t \\ qt \end{pmatrix} \quad q \in Q,\ t \in T. \tag{3}$$

Two extensions G and G' (for the same groups Q and T) are equivalent if there exists an isomorphism $\chi : G \to G'$ such that the diagram which represents sequences of these extensions is commutative i.e. $\chi \circ \kappa = \kappa'$ and $\omega' \circ \chi = \omega$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \xrightarrow{\kappa} & G & \xrightarrow{\omega} & Q & \longrightarrow & 1 \\ & & \downarrow \operatorname{id} T & & \downarrow \chi & & \downarrow \operatorname{id} Q & & \\ 0 & \longrightarrow & T' & \xrightarrow{\kappa'} & G' & \xrightarrow{\omega'} & Q & \longrightarrow & 1 \end{array} \tag{4}$$

The equivalency of extensions can also be defined basing on a factor system. Namely two extensions are equivalent if their factor systems differ from each other by a twocoboundary δc

$$m' = m + \delta c. \tag{5}$$

3 Second group of cohomology

For given groups Q and T one can obtain many extensions. But not all of them differ. Some of them are equivalent. The number of nonequivalent extensions is given by the second cohomology group $H^2_\Delta(Q, T)$ [5,6]:

$$H^2_\Delta(Q, T) = Z^2_\Delta(Q, T)/B^2_\Delta(Q, T), \tag{6}$$

where $Z^2_\Delta(Q, T)$ is the group of all twococycles, while $B^2_\Delta(Q, T)$ is the group of all twocoboundaries. Groups $Z^2_\Delta(Q, T)$ and $B^2_\Delta(Q, T)$ are subgroups of all twocochains $C^2_\Delta(Q, T)$. The order of this group is great:

$$|C^2_\Delta(Q, T)| = |T|^{(|Q|^2)}. \tag{7}$$

This order increases when the orders of the groups T and Q are increased (combinatorial explosion) and e.g. for such small groups Q and T as $|Q| = 3$, $|T| = 4$ the order of the group of twococycles is equal to 262144.

4 The Mac Lane method

Mac Lane's theorem helps to construct the second cohomology group and enables us to inspect some features of the structure of extension. The essence of the method is in a theorem that a second cohomology group is isomorphic with a quotient group for another exact sequence involving free groups [1]:

$$\begin{tikzcd}
1 \arrow[r] & R \arrow[r, "i"] \arrow[dd, "\varphi"'] & F \arrow[dr, "M"] \arrow[ddl, "\gamma"'] & & \\
& & & Q \arrow[r] & 1 \\
0 \arrow[r] & T \arrow[r, "\kappa"] & G \arrow[ur, "\omega"] & &
\end{tikzcd} \tag{8}$$

where F, R are free groups with alphabet $\langle X\rangle$ and $\langle Y\rangle$ respectively.

The group F is generated freely from a set $A \subset Q$ of generators of an active group Q. The free group $R \lhd F$ is a quotient group of F and its alphabet is formed using the Nielson–Schreier theorem [5]:

$$Y = \{sx\beta(sx)^{-1} \mid x \in X,\ s \in S\}, \quad S = \{f_q \mid q \in Q\}, \tag{9}$$

where S is the Nielson–Schreier set, f_q — representatives of cosets of F. In diagram (8) φ denotes an operator homomorphisms from R to T and γ denotes a crossed homomorphisms from F to T.

Denoting the set of all operator and crossed homomorphisms by $Z^1_{\Delta\circ M}(F,T)$ we can derive the second cohomology group from an isomorphism:

$$H^2_\Delta(Q,T) \cong \mathrm{Hom}_F(R,T)/Z^1_{\Delta\circ M}(F,T)|_R, \tag{10}$$

where $Z^1_{\Delta\circ M}(F,T)$ is restricted to R.

One can construct a group $\mathrm{Hom}_F\,(R,T)$ from a manifold of all mappings from the alphabet Y of the group R on to group T

$$\mathrm{Hom}_F\,(R,T) = T^Y = \{\varphi \mid Y \to T\}. \tag{11}$$

Then we have to select from this manifold T^Y the submanifold of all operator homomorphisms i.e. such mappings, which intertwine the action $\Xi : F \to \mathrm{Aut}\, R$ and $\Delta : Q \to \mathrm{Aut}\, T$. They form conditions for the operator homomorphism:

$$\varphi(xyx^{-1}) = M(x)\varphi(y) \quad x \in X,\ y \in Y. \tag{12}$$

The group $Z^1_{\Delta\circ M}(F,T)$ of crossed homomorphisms is derived from other rules:

$$\gamma(f_1f_2) = \gamma(f_1) + M(f_1)\gamma(f_2), \quad f_1f_2 \in F^2. \tag{13}$$

5 Extensions of $C_{12} \times \mathrm{Aut}\, C_{12}$

The presented method has been applied to the construction of all nonequivalent extensions of groups $C_{12} \times \mathrm{Aut}\, C_{12}$, where the translation group T is a cyclic group $C_{12} = \{j \mid j = 1, 2, \ldots, 12\}$

Table 1: The group $\mathrm{Aut}\, C_{12}$

$\tau \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12
τ_1	1	2	3	4	5	6	7	8	9	10	11	12
τ_5	5	10	3	8	1	6	11	4	9	2	7	12
τ_7	7	2	9	4	11	6	1	8	3	10	5	12
τ_{11}	11	10	9	8	7	6	5	4	3	2	1	12

Table 2: Operator actions $\Delta : Q \to \mathrm{Aut}\, C_{12}$

$\tau \backslash \Delta$	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}	Δ_{13}	Δ_{14}	Δ_{15}	Δ_{16}
τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1	τ_1
τ_5	τ_1	τ_1	τ_1	τ_1	τ_5	τ_5	τ_5	τ_5	τ_7	τ_7	τ_7	τ_7	τ_{11}	τ_{11}	τ_{11}	τ_{11}
τ_7	τ_1	τ_5	τ_7	τ_{11}	τ_1	τ_5	τ_7	τ_{11}	τ_1	τ_7	τ_5	τ_{11}	τ_1	τ_5	τ_7	τ_{11}
τ_{11}	τ_1	τ_5	τ_7	τ_{11}	τ_5	τ_1	τ_{11}	τ_7	τ_7	τ_1	τ_{11}	τ_5	τ_{11}	τ_7	τ_5	τ_1

and the point group Q is the group of automorphisms $\mathrm{Aut}\, C_{12} = \{\tau_r \mid r = 1, 5, 7, 11\}$. The latter one has four elements τ_r described by relation $\tau_r j = rj \bmod 12$, $j \in C_{12}$. This group (cf also Table 1) is isomorphic to the group D_2. All possible operator actions $\Delta : Q \to \mathrm{Aut}\, C_{12}$ have been listed in Table 2.

We express nonequivalent extensions for the operator action Δ_7 (Table 2) in terms of a factor system defined by

$$m(q_1, q_2) = \varphi(\varrho(q_1, q_2)), \quad (q_1, q_2) \in Q^2. \tag{14}$$

The function $\varrho(q_1, q_2)$ in (14) is described by a product

$$f_{q_1} f_{q_2} = \varrho(q_1, q_2) f_{q_1 q_2}, \tag{15}$$

where f_q is a representative of the coset in the decomposition

$$F = \bigcup_{q \in Q} R f_q, \tag{16}$$

and R is the kernel of the epimorphism $M : F \to Q$. The group D_2 has an alphabet $X = \{x_1, x_2\}$ and the Schreier set consists of four elements $S = \{e_F, x_1, x_2, x_1 x_2\}$. This set determines the factor system $\varrho : Q \times Q \to R$ by (15). For our case the factor system $\varrho(q_1, q_2)$ has been presented in Table 3.

The alphabet Y of the subgroup R can be identify with the set of all non-trivial elements of the second and third columns of Table 3. Thus we have

$$Y = \{y_1 = x_1^2, y_2 = x_2^2, y_3 = x_2 x_1 x_2^{-1} x_1^{-1}, y_4 = x_1 x_2 x_1 x_2^{-1}, y_5 = x_1 x_2^2 x_1^{-1}. \tag{17}$$

The factor system $\varrho : Q \times Q \to R$, expressed in terms of the alphabet Y is presented in Table 4. Having the alphabets X and Y we can construct conditions for operator homomorphisms (12)

Table 3: The factor system $\varrho : Q \times Q \to R$ in the alphabet X

	e_F	x_1	x_2	x_1x_2
e_F	e_F	e_F	e_F	e_F
x_1	e_F	x_1^2	e_F	x_1^2
x_2	e_F	$x_2x_1x_2^{-1}x_1^{-1}$	x_2^2	$x_2x_1x_2x_1^{-1}$
x_1x_2	e_F	$x_1x_2x_1x_2^{-1}$	$x_1x_2^2x_1^{-1}$	$x_1x_2x_1x_2$

Table 4: The factor system $\varrho : Q \times Q \to R$ in the alphabet Y

	E	u_x	u_y	u_z
E	e_F	e_F	e_F	e_F
u_x	e_F	y_1	e_F	y_1
u_y	e_F	y_3	y_2	y_3y_5
u_z	e_F	y_4	y_5	y_4y_2

(see Table 5). Both, operator and crossed homomorphisms, forming groups $\mathrm{Hom}_F\,(R, C_{12})$ and $Z^1_{\Delta\circ M}$, are collected in Table 6 and 7, respectively.

According to formula (6) we construct cosets determined by $Z^1_{\Delta\circ M}(F,T)$ in $\mathrm{Hom}_F\,(R,T)$ (see Table 8) forming the second cohomology group $H^2_{\Delta_7}(Q,T)$. This group has eight elements. It yields that we have eight nonequivalent extensions $C_{12} \times \mathrm{Aut}\, C_{12}$ (under operator action Δ_7). Such extensions are listed in Table 9 in the form of factor systems $m : Q \times Q \to C_{12}$ for the coset representatives chosen as in Table 8.

6 Conclusions

The second cohomology group for an extension $C_{12} \times \mathrm{Aut}\, C_{12}$ is of the order 8 (under the operator action Δ_7 described in Table 2). Each element of this group $m \in H^2(D_2, C_{12})$ forms the factor system (Table 9) determined by the Seitz formula (2) and gives an nonequivalent

Table 5: Conditions for operator homomorphisms $\varphi : R \to T$

	x_1	x_2
y_1	$\varphi(y_1) = 5\varphi(y_1)$	$\varphi(y_3) + \varphi(y_4) = 7\varphi(y_1)$
y_2	$\varphi(y_5) = 5\varphi(y_2)$	$\varphi(y_2) = 7\varphi(y_2)$
y_3	$\varphi(y_4) - \varphi(y_1) = 5\varphi(y_3)$	$\varphi(y_2) - \varphi(y_5) - \varphi(y_3) = 7\varphi(y_3)$
y_4	$\varphi(y_1) + \varphi(y_3) = 5\varphi(y_4)$	$\varphi(y_1) - \varphi(y_2) + \varphi(y_3) + \varphi(y_5) = 7\varphi(y_4)$
y_5	$\varphi(y_2) = 5\varphi(y_5)$	$\varphi(y_5) = 7\varphi(y_5)$

Table 6: The group $\mathrm{Hom}_F(R, C_{12})$ of operator homomorphisms (for operation action Δ_7)

	y_1	y_2	y_3	y_4	y_5		y_1	y_2	y_3	y_4	y_5
φ_1	3	2	5	4	10	φ_{25}	9	2	5	10	10
φ_2	3	2	11	10	10	φ_{26}	9	2	11	4	10
φ_3	3	4	1	8	8	φ_{27}	9	4	1	2	8
φ_4	3	4	7	2	8	φ_{28}	9	4	7	8	8
φ_5	3	6	3	6	6	φ_{29}	9	6	3	12	6
φ_6	3	6	9	12	6	φ_{30}	9	6	9	6	6
φ_7	3	8	5	4	4	φ_{31}	9	8	5	10	4
φ_8	3	8	11	10	4	φ_{32}	9	8	11	4	4
φ_9	3	10	1	8	2	φ_{33}	9	10	1	2	2
φ_{10}	3	10	7	2	2	φ_{34}	9	10	7	8	2
φ_{11}	3	12	3	6	12	φ_{35}	9	12	3	12	12
φ_{12}	3	12	9	12	12	φ_{36}	9	12	9	6	12
φ_{13}	6	2	2	4	10	φ_{37}	12	2	2	10	10
φ_{14}	6	2	8	10	10	φ_{38}	12	2	8	4	10
φ_{15}	6	4	4	2	8	φ_{39}	12	4	4	8	8
φ_{16}	6	4	10	8	8	φ_{40}	12	4	10	2	8
φ_{17}	6	6	6	12	6	φ_{41}	12	6	6	6	6
φ_{18}	6	6	12	6	6	φ_{42}	12	6	12	12	6
φ_{19}	6	8	2	4	4	φ_{43}	12	8	2	10	4
φ_{20}	6	8	8	10	4	φ_{44}	12	8	8	4	4
φ_{21}	6	10	4	2	2	φ_{45}	12	10	4	8	2
φ_{22}	6	10	10	8	2	φ_{46}	12	10	10	2	2
φ_{23}	6	12	6	12	12	φ_{47}	12	12	6	6	12
φ_{24}	6	12	12	6	12	φ_{48}	12	12	12	12	12

Table 7: The group $Z^1_{\Delta \circ M}$ of crossed homomorphisms (for operation action Δ_7)

	y_1	y_2	y_3	y_4	y_5
γ_1	6	8	2	4	4
γ_2	6	4	10	8	8
γ_3	6	12	6	12	12
γ_4	12	8	8	4	4
γ_5	12	4	4	8	8
γ_6	12	12	12	12	12

Table 8: Coset representatives $\mathrm{Hom}_F\,(R,T)/Z^1_{\Delta\circ M}(F,T)$ (for operation action Δ_7)

	y_1	y_2	y_3	y_4	y_5
r_0	12	12	12	12	12
r_1	9	6	9	6	6
r_2	3	6	9	12	6
r_3	3	12	3	6	12
r_4	9	12	3	12	12
r_5	6	6	12	6	6
r_6	6	6	6	12	6
r_7	6	12	12	6	12

Table 9: Factor systems $m : Q \times Q \to C_{12}$ (for operation action Δ_7)

$$r_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 9 & 6 & 3 \\ 0 & 9 & 6 & 3 \\ 0 & 6 & 6 & 0 \end{pmatrix}$$

$$r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 9 & 6 & 3 \\ 0 & 0 & 6 & 6 \end{pmatrix} \quad r_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & 0 & 3 \\ 0 & 6 & 0 & 6 \end{pmatrix}$$

$$r_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 9 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 6 \\ 0 & 0 & 6 & 6 \\ 0 & 6 & 6 & 0 \end{pmatrix}$$

$$r_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 6 \\ 0 & 6 & 6 & 0 \\ 0 & 0 & 6 & 6 \end{pmatrix} \quad r_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 6 \end{pmatrix}$$

extension. However, the factor system depends on a choice of coset representatives listed in table 8. This choice corresponds to the gauge transformation [2], which is connected with equivalent extensions.

References

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