Δ-matroids and Pfaffian Forms

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In analogy to the fact that representable matroids may be defined in terms of Grassmann-Plücker maps we want to describe representability of Δ-matroids by skew-symmetric matrices in terms of Pfaffian forms. In the sequel we assume $n \in \mathbb{N}, S := \{1, \ldots, n\}$, and K denotes some commutative field.

Definition 1 (cf. [B1, §6] or [B2, §1]):
Assume $\mathcal{F} \subseteq \mathcal{P}(S)$. The pair $(S, \mathcal{F})$ is a **Δ-matroid**, if $\mathcal{F}$ satisfies

(SEA) For $F_1, F_2 \in \mathcal{F}$ and $e \in F_1 \Delta F_2$ there exists some $f \in F_1 \Delta F_2$ with $F_1 \Delta \{e, f\} \in \mathcal{F}$.

For $\mathcal{F} \subseteq \mathcal{P}(S)$ and $T \subseteq S$ we put $\mathcal{F}\Delta T := \{F \Delta T \mid F \in \mathcal{F}\}$.

Definition 2 (cf. [B2, §4]):
A Δ-matroid $(S, \mathcal{F})$ is representable over K by a skew-symmetric matrix $A = (a_{ij})_{i,j \in S}$, if for some $T \subseteq S$ we have

$$\mathcal{F}\Delta T = \mathcal{F}(A) := \{S' \subseteq S \mid A' := (a_{ij})_{i,j \in S'} \text{ is nonsingular}\},$$

where $(a_{ij})_{i,j \in \emptyset}$ is considered to be nonsingular. A is then called a **presentation** of $(S, \mathcal{F})$.

Theorem 3: A Δ-matroid $(S, \mathcal{F})$ is representable over K by a skew-symmetric matrix if and only if there exists some map $P : \mathcal{P}(S) \to K$ and some $T \in \mathcal{F}$ such that

(P0) For $I \subseteq S$ we have $P(I) \neq 0$ if and only if $I \Delta T \in \mathcal{F}$.
(P1) If $I \subseteq S$ and $\#I \equiv 1 \mod 2$, then $P(I) = 0$.
(P2) If $I_1, I_2 \subseteq S$ and $I_1 \Delta I_2 = \{i_1, \ldots, i_k\}$ with $i_j < i_{j+1}$ for $1 \leq j \leq k-1$, then

$$\sum_{j=1}^{k} (-1)^j \cdot P(I_1 \Delta \{i_j\}) \cdot P(I_2 \Delta \{i_j\}) = 0.$$

If $(S, \mathcal{F})$ is representable by $A = (a_{ij})_{i,j \in S}$, then P may be choosen to be the corresponding Pfaffian form, which may be defined by $P(I) := 0$ for $\#I \equiv 1 \mod 2$, $P(\emptyset) := 1$, $P(\{i, j\}) := a_{ij}$ for $1 \leq i < j \leq n$, and

$$P(\{i_1, \ldots, i_{2m}\}) := \sum_{j=2}^{2m} (-1)^j \cdot P(\{i_1, i_j\}) \cdot P(\{i_2, \ldots, i_{2m}\} \setminus \{i_j\})$$

$$\text{for } 2 \leq m \leq \frac{n}{2} \text{ and } i_j < i_{j+1} \text{ for } 1 \leq j \leq 2m-1.$$

Vice versa, if P satisfies (P0), (P1), (P2), then $(S, \mathcal{F})$ is representable by the matrix $A = (a_{ij})_{i,j \in S}$ given by $a_{ii} := 0$ for $1 \leq i \leq n$, $a_{ij} = -a_{ji} = P(\{i,j\})$ for $1 \leq i < j \leq n$.

Corollary: A Δ-matroid $(S, \mathcal{F})$, representable by some skew-symmetric matrix, satisfies the following strong exchange property:

$$\begin{aligned}&\text{For } F_1, F_2 \in \mathcal{F} \text{ and } e \in F_1 \Delta F_2 \text{ there exists some}\\ &f \in (F_1 \Delta F_2) \setminus \{e\} \text{ with } F_1 \Delta \{e,f\} \in \mathcal{F} \text{ and } F_2 \Delta \{e,f\} \in \mathcal{F}.\end{aligned}$$

Furthermore, Theorem 3 suggests

Definition 4: Assume $F \subseteq \mathcal{P}(S)$.

i) $(S, \mathcal{F})$ is an **orientable Δ-matroid**, if there exists some $T \in \mathcal{F}$ and some map $P : \mathcal{P}(S) \to \{0, 1, -1\}$ satisfying

(OP0) For $I \subseteq S$ we have $P(I) \in \{1, -1\}$ if and only if $I \Delta T \in \mathcal{F}$.

(OP1) For $I \subseteq S$ with $\# I \equiv 1 \bmod 2$ we have $P(I) = 0$.

(OP2) If $I_1, I_2 \subseteq S$, $I_1 \Delta I_2 = \{i_1, \ldots, i_k\}$ with $i_j < i_{j+1}$ for $1 \leq j \leq k-1$, and if for some $w \in \{1, -1\}$ we have $\kappa_j := w \cdot (-1)^j \cdot P(I_1 \Delta \{i_j\}) \cdot P(I_2 \Delta \{i_j\}) \geq 0$ for $1 \leq j \leq k$, then $\kappa_j = 0$ for every j.

ii) Assume $P : \mathcal{P}(S) \to \mathbb{R}^+ \dot{\cup} \{0\}$ is some map. $(S, \mathcal{F}, P)$ is a **valuated Δ-matroid**, if for some $T \in \mathcal{F}$ we have

(VP0) For $I \subseteq S$ we have $P(I) > 0$ if and only if $I \Delta T \in \mathcal{F}$.

(VP1) For $I \subseteq S$ with $\# I \equiv 1 \bmod 2$ we have $P(I) = 0$.

(VP2) If $I_1, I_2 \subseteq S$ with $\# I_1 \equiv \# I_2 \equiv 0 \bmod 2$, then for every $i \in I_1 \Delta I_2$ there exists some $j \in (I_1 \Delta I_2) \setminus \{i\}$ with

$$P(I_1) \cdot P(I_2) \leq P(I_1 \Delta \{i,j\}) \cdot P(I_2 \Delta \{i,j\}).$$

References

[B1] A. Bouchet: Greedy algorithm and symmetric matroids, Mathematical Programming 38 (1987), 147-159.

[B2] A. Bouchet: Representability of Δ-matroids, Colloquia Mathematica Societatis János Bolyai 52. Combinatorics, Eger (Hungary), 1987, 167-182.

[DW] A.W.M. Dress and W. Wenzel: Grassmann-Plücker relations and matroids with coefficients, to appear in Advances in Mathematics.

[K] G. Kowalewski: Determinantentheorie, Chelsea Publishing Company, New York, 1948.

[S] J.R. Stembridge: Nonintersecting Paths, Pfaffians, and Plane Partitions, Advances in Mathematics 83 (1990), 96-131.

[W] W. Wenzel: Pfaffian Forms and Δ-Matroids, in preparation.