Gauge symmetries in the Heisenberg model of a magnet

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Abstract

A relation between geometric and gauge symmetries for the Heisenberg model of magnet is pointed out. The space of quantum states of the magnet exhibits the structure of a bundle, with the base consisting of nodes of the crystal, and the typical fiber spanned on the set of the single-node spin projections. The geometric symmetry group acts on the base, the gauge group — on the typical fiber, and all combined operations form the wreath product. In particular, all global gauge transformations yield the direct product group — a subgroup of the wreath product.

1 Introduction

The Heisenberg model of a magnet exhibits a rich combinatorial structure, associated with a group enumeration of various classes of magnetic configurations [1, 2]. These classes are orbits of certain groups of the magnet. The most obvious symmetry is the geometric one, consisting of such operations that preserve the distances between magnetic nodes. Here we aim to point out some other operations, called gauge symmetries [3], which involve transformations of internal spaces of each magnetic node. They are well described in terms of fiber structure of space of all quantum states of the magnet [4, 5]. We also discuss here the relation of gauge symmetries to wreath products.

2 The Heisenberg model and combinatorics

Kinematics of the Heisenberg model of a magnetic crystal is specified by the space of all its quantum states. Let X be the set of all nodes of the crystal, and let each node carries a spin s. Let Y be the set of all projections of the single-node spin. Let

$$|X| = n, \quad |Y| = m, \tag{1}$$

so that n is the number of the crystal nodes, and $m = 2s + 1$ defines the value of the spin s. Then the set

$$Y^X =: \{f : X \to Y\} \tag{2}$$

of all mappings

$$f = |i_1, \ldots, i_n\rangle\,, \quad i_j \in Y, \;\; j \in X, \tag{3}$$

is the set of all magnetic configurations. Eq. (3) displays the Dirac ket notation for a magnetic configuration f. The linear closure

$$L = \mathrm{lc}_{\mathbb{C}} Y^X \tag{4}$$

of the set Y^X over the field $\mathbb{C}$ of complex numbers is a linear space. This space is equipped with the unitary structure by demand that the set Y^X is an orthonormal basis in it. L is the space of all quantum states of the magnet.

Combinatorial enumeration of orbits is given in terms of various group actions. Let S_n and S_m be the symmetric group on the set X and Y, respectively. Then the defining actions $U : S_n \times X \to X$ and $V : S_m \times Y \to Y$ are naturally lifted to $P : S_n \times Y^X \to Y^X$ and $Q : S_m \times Y^X \to Y^X$. Subductions of actions P and/or Q to various subgroups of S_n and/or S_m yield orbits, associated with appropriate symmetries.

3 Geometric symmetries

Let $H \subset S_n$ be the geometric symmetry group of the crystal (it can be a point group or a finite extension of translation group, resulting from the periodic Born–von Kármán boundary conditions). Similarly, let $G \subset S_m$ be a subgroup of the single-node group S_m, e.g. the two-element group involving the time reversal. In such a case, the direct product $H \times G$ describes the total space-time symmetry of the magnet. This symmetry is realized in terms of the subduced action $(P \downarrow H) \times (Q \downarrow G)$.

The geometric, and — more generally — space-time symmetry of the magnet is therefore related to the direct product group $H \times G$. A more general combinatorial construction is the wreath product $G \wr H$. There is thus a temptation to ask whether the action of the wreath product $G \wr H$ on the set Y^X of all magnetic configurations has also a physical meaning? We propose an answer in Sec. 6.

4 Fiber structure of the space L of quantum states of the magnet

Let

$$W = \mathrm{lc}_{\mathbb{C}} Y \tag{5}$$

be the linear unitary space of quantum states of single-node spin s, with Y as its orthonormal basis. We can think of X as the base of a bundle E with the typical fiber W. To be more specific, let $\phi_j : W \to W_j$, $j \in X$, be a copying isomorphism, which produces a faithful copy of the space W, centered at the node j of the crystal X. Then the collection of all such copies,

$$E = \bigcup_{j \in X} W_j \tag{6}$$

is the fiber bundle with the base X and the typical fiber W. The canonical bundle projection $p : E \to X$ is given by

$$p(e) = j \quad \text{for} \quad e \in W_j \subset E. \tag{7}$$

Every mapping $\psi : X \to E$ with the property

$$p \circ \psi = \mathrm{id}_X, \tag{8}$$

where id_X is the identity mapping on the base X, is called a section of the bundle E. We observe that if

$$\psi(j) = i_j, \ \ j \in X, \tag{9}$$

then $\psi \in Y^X$. Thus some sections of the bundle E can be identified as magnetic configurations.

Clearly, the action $V : S_m \times Y \to Y$ can be extended to the action $V' : U(m) \times W \to W$ of the unitary group $U(m)$ on the typical fiber W. The unitary group $U(m)$ is the maximal quantum symmetry group for the single spin s.

5 Gauge symmetries

Now we are in a position to consider a subgroup $\Gamma \subset U(m)$ as the gauge group. Some prominent examples from physics are $\Gamma = U(1)$ for quantum electrodynamics, $\Gamma = SU(2)$ for isospin symmetry, and $\Gamma = SU(3)$ for quarks.

We consider the action of the gauge group Γ on the space L of the Heisenberg magnet. The simplest case is when each $g \in \Gamma$ acts in the same way on each fiber W_j, i.e. when

$$(g\psi)(j) = \phi_j(g\phi_j^{-1}(\psi(j))), \ \ j \in X. \tag{10}$$

Eq. (10) defines a global gauge transformation in the space L, imposed by the element $g \in \Gamma$. Clearly, combination of global gauge transformations with geometric symmetries H yields the action of the direct product group $\Gamma \times H$ in the space L.

We proceed to introduce local gauge transformations. To this aim, we consider a mapping $c : X \to \Gamma$, called a cochain, such that each element $c(j) \in \Gamma$, $j \in X$, of the gauge group Γ acts on its own fiber W_j. Under such an action, each section $\psi \in L$ transforms to a new section

$$c\psi = \psi' \in L, \tag{11}$$

given by

$$(c\psi)(j) = \phi_j(c(j)\phi_j^{-1}(\psi(j))), \ \ j \in X. \tag{12}$$

Eq. (12) defines the local gauge transformation in the space L. It is worth to notice that $c\psi$ is not a composition of mappings, but only a short notation for a new section $\psi' \in L$, the image of ψ under the local gauge transformation c.

The group of all local gauge transformations in the space L, imposed by the gauge group Γ coincides with the set

$$C^1(X, \Gamma) = \Gamma^X = \{c : X \to \Gamma\} \tag{13}$$

of all cochains on the set X, valued in the group Γ, with the group multiplication imposed by Γ.

6 Wreath product

Assume that the crystal X is an orbit of the geometric symmetry group $H \subset S_n$. Then the group (13) of all local gauge transformations imposed by $\Gamma \subset U(m)$ can be identified as the base group of the wreath product

$$\Gamma \wr H = \Gamma^X \Box H \tag{14}$$

of the gauge symmetry group Γ by the permutation group H on the set X of nodes of the crystal. We have thus arrived at a clear physical interpretation of the wreath product (14). In particular, operations $(c, 1_H) \in \Gamma \wr H$ of the base group Γ^X of the wreath product are local gauge transformations. Operations of the form $(1_\Gamma, h) \in \Gamma \wr H$ coincide with purely geometric transformations, whereas general operations $(c, h) \in \Gamma \wr H$ are combinations of geometric and gauge symmetries.

7 Conclusions

We have shown a relation between the wreath product structure and gauge symmetries of the Heisenberg model of a magnet. This relation is readily expressed in terms of fiber bundle structure of the space L of all quantum states of the magnet. The geometric symmetries, described by a subgroup H of the symmetric group S_n, are related to the base X of the bundle E, whereas the gauge group Γ is the subgroup of the unitary group $U(m)$. The former describes the spatial distribution of nodes of the crystal, whereas the latter emerges from the internal symmetry of the typical fiber W, spanned on the set Y of single-node spin projections. All combinations of geometric and gauge symmetries are given by the wreath product $\Gamma \wr H$.

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