Three Aspects of Partitions

by

George E. Andrews[1]

1. Introduction

In this paper we shall discuss three topics in partitions. Section 2 is devoted to partitions with difference conditions and is an elucidation of joint work with J. B. Olsson [16]. In Section 3 we discuss certain partition problems which have their origins in statistical mechanics. We take as the theme for this section Euler's article: Exemplum Memorabile Inductionis Fallacis [23]. The material for this section is closely related to the work in [10]. Section 4 contains a discussion of some of Ramanujan's formulas from both his Notebooks and lost Notebook. More extensive accounts of this topic are found in [8] and [11].

2. Partitions with Difference Conditions

The work in this section is based on [16], joint work with J. B. Olsson. In 1989, Olsson was studying Mullineux's conjecture [29] which briefly may be described as a "conjugation" map for p–regular partitions (i.e. partitions with no part repeated more than $p-1$ times). As Mullineux asserts [29; p. 60]: " ... when p is prime it is conjectured that this bijection (i.e. conjugation) arises in the representation theory of the symmetric group S_n of degree n. Farahat, Müller and Peel [24] have shown how to form a 'good' labelling of the irreducible p–modular representations of S_n (for prime p) by p–regular partitions of n. Now the alternating representation of S_n induces in the usual way a bijection (whose square is the identity) upon these representations and hence induces a similar bijection upon the set of p–regular partititons via the labelling. For low values of n this group theoretic bijection agrees with the one constructed here; the verification of this has been carried out using the tables of decomposition numbers found in Kerber and Peel ([27], $p=3,\ n\le 10,\ n\ne 7$); Robinson ([30], $p=3,\ n=7$) and Wagner ([34], $p=5,7,\le 8$)."

[1]Partially supported by National Science Foundation Grant DMS 8702695-03 and the IBM T. J. Watson Research Center.

Typeset by $\mathcal{AMS}$-TEX

Olsson calculated the number of partitions fixed by Mullineux's map and those fixed by the Farahat–Müller–Peel [24] induced map. If the two maps are the same then obviously the cardinalities of the two sets of fixed points will be identical. This calculation led to the following:

Problem. *Let $\lambda = (a_1, a_2, \ldots, a_k)$ be a partition of n. Consider the two following sets of conditions (p an odd prime).*

$$\begin{cases} \text{(1A)} & p \nmid a_i \text{ and } 2 \nmid a_i \text{ for all } i, \\ \text{(1B)} & a_i \neq a_{i+1} \text{ for all } i. \end{cases}$$

$$\begin{cases} \text{(2A)} & 2 \mid a_i \text{ if and only if } p \mid a_i \text{ for all } i, \\ \text{(2B)} & 0 \leq a_i - a_{i+1} \leq 2p \text{ for all } i \ (a_{k+1} = 0) \\ \text{(2C)} & \text{if } a_i = a_{i+1} \text{ then } a_i \text{ is even,} \\ \text{(2D)} & \text{if } a_i - a_{i+1} = 2p \text{ then } a_i \text{ is odd.} \end{cases}$$

Is the number of partitions of n satisfying (1A)–(1B) equal to the number of partitions satisfying (2A)–(2D)?

The simplest possible case is $p = 3$. In this case conditions (1A)–(1B) describe partitions into distinct parts $\equiv 1$ or 5 (mod 6), and this suggests the following result of Schur [32] specialized to fit this instance.

Theorem 2.1 [32]. *The number or partititons of n into distinct parts congruent to 1 or 5 mod 6 equals the number of partitions of n into parts congruent to 0, 1 or 5 mod 6 with the condition that the difference between parts is at least 6 and greater than 6 between two multiples of 6.*

For example there are 11 partitions of 36 into distinct parts congruent to 1 or 5 (mod 6): 35+1, 31+5, 29+7, 25+11, 23+13, 23+7+5+1, 19+17, 19+11+5+1, 17+13+5+1, 17+11+7+1, 13+11+7+5. The second class of eleven partitions arising from Schur's Theorem when $n = 36$ is: 36, 35+1, 31+5, 30+6, 29+7, 25+11, 24+12, 24+11+1, 23+13, 23+12+1, 19+12+5.

In contrast, the eleven partitions arising from conditions (2A)–(2D) in the Problem for $n = 36$, $p = 3$ are: 17+11+7+1, 13+11+7+5, 13+11+6+5+1, 12+12+7+5, 12+11+7+5+1, 12+11+6+6+1, 12+11+6+6+5, 11+7+6+6+5+1, 11+6+6+6+6+1, 7+6+6+6+6+5, 6+6+6+6+6+5+1.

Inspection shows that MacMahon's theory of modular partitions for modulus 6 [28] provides a perfect bijection between these two latter classes of partitions. In MacMahon's representation each part is represented by a row of 6's with the residue mod 6 tacked on at the end. Consequently the eleven MacMahon graphs of the final set of partitions above is:

$$\begin{array}{lll} 6 & 6 & 5 \\ 6 & 5 & \\ 6 & 1 & \\ 1 & & \\ & & \end{array} \qquad \begin{array}{lll} 6 & 6 & 1 \\ 6 & 5 & \\ 6 & 1 & \\ 5 & & \\ & & \end{array} \qquad \begin{array}{ll} 6 & 6 \\ 6 & 6 \\ 6 & 1 \\ 5 & \\ & \end{array} \qquad \begin{array}{lll} 6 & 6 & 1 \\ 6 & 5 & \\ 6 & 6 & \\ 5 & & \\ 1 & & \end{array} \qquad \begin{array}{ll} 6 & 6 \\ 6 & 5 \\ 1 & \\ 5 & \\ 1 & \end{array}$$

$$\begin{array}{ll} 6 & 6 \\ 6 & 5 \\ 6 & \\ 6 & \\ 1 & \\ & \\ & \end{array} \qquad \begin{array}{ll} 6 & 6 \\ 6 & 1 \\ 6 & \\ 6 & \\ 5 & \\ & \\ & \end{array} \qquad \begin{array}{ll} 6 & 5 \\ 6 & 1 \\ 6 & \\ 6 & \\ 5 & \\ 1 & \\ & \end{array} \qquad \begin{array}{ll} 6 & 5 \\ 6 & \\ 6 & \\ 6 & \\ 6 & \\ 1 & \\ & \end{array} \qquad \begin{array}{ll} 6 & 1 \\ 6 & \\ 6 & \\ 6 & \\ 6 & \\ 5 & \\ & \end{array} \qquad \begin{array}{l} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 5 \\ 1 \end{array}$$

Now we form a new set of partitions by reading these graphs by columns instead of rows. The result is Schur's second set of partitions, and a little reflection shows that the above mapping always provides a bijection between Schur's partitions and those of the second kind in the Problem where $p = 3$.

The relationship described above suggests that the Problem may be solved by relating it to some generalization of Schur's Theorem and then applying MacMahon's modular theory. While there arose some difficulties, this program eventually produced the following result.

Theorem 2.2 [16]. *Let $A = \{a_1, a_2, \ldots, a_r\}$ be a set of r distinct positive integers arranged in increasing order, and let N be an integer larger than a_r. Let $P_1(A; N, n)$ denote the number of partitions of n into distinct parts each of which is congruent to some a_i modulo N. Let $P_2(A; N, n)$ denote the number of partititons of n into parts each of which is congruent to 0 or to some a_i modulo N, in addition only parts divisible by N may be reapeated, the smallest part is $< N$, the difference between sucessive parts is at most N and strictly less than N if either part is divisible by N. Then for each $n \geq 0$,*

$$P_1(A; N, n) = P_2(A; N, n).$$

In the above theorem, $A = \{a_1, \ldots, a_r\}$ is an arbitrary set of positive integers arranged in increasing order for which $a_r < N$. The relevant generalization of Schur's theorem requires additionally: (i) $\sum_{s=1}^{k-1} a_s < a_k$ $(k \leq r)$, (ii) $\sum_{s=1}^{r} a_s \leq N$, and (iii) all 2^r subsets of A must have distinct sums. Let A' be the set of $2^r - 1$ positive sums arising from the nonempty subsets of A. Let A'_N denote the set of all positive integers that are congruent to some element of A' modulo N. Let $\rho_N(m)$ denote the least positive residue of m modulo N. For $m \in A'$, let $b(m)$ be the number of terms appearing in the sum of distinct elements of A making up m and let $\nu(m)$ denote the least a_i in this sum.

In [2], the main result may be restated as follows:

Theorem 2.3. *Let $E(A'_N;n)$ denote the number of partitions of n into parts taken from $A'_N : n = c_1 + c_2 + \cdots + c_s$, $c_i \geq c_{i+1}$,*

$$c_i - c_{i+1} \geq Nb(\rho_N(c_{i+1})) + \nu(\rho_N(c_{i+1})) - \rho_N(c_{i+1}).$$

Then $E(A';n) = P_1(A;N,n)$.

The proof of Theorem 2.3 was successfully altered to yield Theorem 2.2. In addition when conditions (i)–(iii) listed above apply to A and N in Theorem 2.2, then MacMahon's modular partitions may be utilized to show the equivalence of the two results.

Finally it should be mentioned that C. Bessenrodt [20] has proved a generalization of Theorem 2.2 using purely combinatorial methods. Also K. Alladi and B. Gordon [1] have a nice study of related continued fractions when A is the two element set $\{a_1, a_2\}$.

3. Euler's "Exemplum Memorabile Inductionis Fallacis"

In [12], [13] and [14] a model generalizing the hard hexagon model was solved using several q–analogs of trinomial coefficients. The trinomial coefficients $\binom{m}{j}_2$ may be defined by

$$\sum_{j=-m}^{m} \binom{m}{j}_2 x^j = (1 + x + x^{-1})^m. \tag{3.1}$$

In this way for given m, the largest coefficient is $\binom{m}{0}_2$. These numbers fit a modified Pascal triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & 1 & 1 & & & & \\
 & & & 1 & 2 & 3 & 2 & 1 & & & \\
 & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & \\
 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 & \\
1 & : & : & : & : & : & : & : & : & : & 1
\end{array}$$

Euler discovered a sufficiently mysterious aspect of the central column of this array that he wrote a short note entitled, "Exemplum Memorabile Inductionis Fallacis" (A Remarke Example of Misleading Induction).

Euler first computed $\binom{m}{0}_2$ for $0 \leq m \leq 9$:

$$1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, \ldots$$

He then tripled each entry in a row shifted one to the right:

$$\begin{array}{l}
1,1,3,7,19,51,141,393,1107,\quad 3139,\ldots \\
\quad 3,3,9,21,57,153,423,1179,3321,\ldots,
\end{array}$$

and starting with the first two–entry column, he subtracted the first row from the second:

$$2, 0, 2, 2, 6, 12, 30, 72, 182, \ldots,$$

each of which may be factored into two consecutive integers:

$$1 \cdot 2,\ 0 \cdot 1,\ 1 \cdot 2,\ 1 \cdot 2,\ 2 \cdot 3,\ 3 \cdot 4,\ 5 \cdot 6,\ 8 \cdot 9,\ 13 \cdot 14, \ldots.$$

The first factors make up the Fibonacci sequence F_n defined by $F_{-1} = 1$, $F_0 = 0$, $F_n = F_{n-1} + F_{n-2}$ for $n > 0$.

Surprisingly, however, this marvelous rule

$$3\binom{m+1}{0}_2 - \binom{m+2}{0}_2 = F_m(F_m + 1), \quad -1 \le m \le 7, \tag{3.2}$$

is *false* for $m > 7$. In order to understand (3.2) we define

$$E_m(a, b) = \sum_{\lambda=-\infty}^{\infty} \left(\binom{m}{10\lambda + a}_2 - \binom{m}{10\lambda + b}_2 \right). \tag{3.3}$$

As part of Theorem 3.1, we show that

$$2E_{m+1}(0, 1) = F_m(F_m + 1), \tag{3.4}$$

from which (3.2) follows by inspection.

Theorem 3.1.

$$2E_m(1, 2) = 2E_{m-1}(0, 3) = 2E_{m+1}(0, 1) = F_m(F_m + 1), \tag{3.5}$$

$$E_m(1, 4) = E_{m+1}(2, 3) = F_{m+1}F_m, \tag{3.6}$$

$$2E_m(3, 4) = 2E_{m-1}(2, 5) = 2E_{m+1}(4, 5) = F_m(F_m - 1), \tag{3.7}$$

$$2E_m(1, 3) = F_{2m} + F_m, \tag{3.8}$$

$$2E_m(2, 4) = F_{2m} - F_m, \tag{3.9}$$

$$2E_m(1, 5) = F_{2m+1} - F_{m-1}, \tag{3.10}$$

$$2E_m(0, 4) = F_{2m+1} + F_{m-1}, \tag{3.11}$$

$$2E_m(0, 2) = F_{2m-1} + F_{m+1}, \tag{3.12}$$

$$2E_m(3, 5) = F_{2m-1} - F_{m+1}, \tag{3.13}$$

$$E_m(0, 5) = F_{2m-1} + F_m F_{m-1}. \tag{3.14}$$

Sketch of Proof. We note that

$$E_m(a, b) = -E_m(b, a), \quad E_m(10r + a, 10s + b) = E_m(a, b), \tag{3.15}$$

$$E_m(10 - a, b) = E_m(a, b), \tag{3.16}$$

$$E_m(10 - a, b) = E_m(a, b) = E_m(a, 10 - b), \tag{3.16}$$

and that

$$E_m(a,b) = E_{m-1}(a,b) + E_{m-1}(a-1,b-1). \tag{3.17}$$

Equations (3.15)–(3.17) totally define $E_m(a,b)$ together with appropriate initial values. The rest follows by induction. □

As a corollary of Theorem 3.1 it is easy to show that

$$\begin{aligned} 3\binom{m+1}{0}_2 - \binom{m+2}{0}_2 &= 2\binom{m+1}{0}_2 - 2\binom{m+1}{1}_2 \\ &= 2E_{m+1}(0,1) \qquad (\text{for } m \le 7) \\ &= F_m(F_{m+1}+1) \qquad (\text{by } (3.5)). \end{aligned}$$

The natural question that arises is: Are there q–analogs of at least portions of Theorem 2.1 and if so, what are the implications for the Rogers–Ramanujan type identities?

In [10], q–analogs of (3.8)–(3.11) were found. For example, we recall Schur's polynomials $G_1(q) = G_2(q) = 1$, $G_n(q) = G_{n-1}(q) + q^{n-2}G_{n-2}(q)$ for $n > 2$. Schur [31] showed that

$$G_{n+1}(q) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda+1)/2} \begin{bmatrix} n \\ \lfloor \frac{n-5\lambda}{2} \rfloor \end{bmatrix}_q . \tag{3.18}$$

where $\lfloor x \rfloor$ is the greatest integer $\le x$ and

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{cases} \frac{(1-q^A)(1-q^{A-1})\cdots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\cdots(1-q)} & 0 \le B \le A, \\ 0 & \text{otherwise.} \end{cases}$$

The q–analog of trinomial coefficients appropriate for our discussion here is

$$\binom{m;B;q}{A}_2 = \sum_{j\ge 0} q^{j(j+B)} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m-j \\ j+A \end{bmatrix} . \tag{3.19}$$

Note that

$$\binom{m;B;1}{A}_2 = \binom{m}{A}_2 . \tag{3.20}$$

Schur [31] deduced the first Rogers-Ramanujan identity as a limiting case of (3.18). For q–analogs of (3.11) and (3.10) respectively, we discover that

$$\begin{aligned} &\frac{1}{2}(G_{2m+1}(q^{1/2}) + G_{2m+1}(-q^{1/2})) \\ &\quad = \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2-2\lambda} \binom{m;10\lambda;q}{10\lambda}_2 - \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+22\lambda+4} \binom{m;10\lambda+4;q}{10\lambda+4}_2 , \end{aligned} \tag{3.21}$$

$$(3.22)\quad \frac{q^{-1/2}}{2}(G_{2m+1}(q^{1/2}) - G_{2m+1}(-q^{1/2}))$$
$$= \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+8\lambda}\binom{m;10\lambda+1;q}{10\lambda+1}_2 - \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+32\lambda+8}\binom{m;10\lambda+5;q}{10\lambda+5}_2.$$

In contrast with Schur's identity (3.18), we find [9] that

$$(3.23)\qquad G_{n+1}(q) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda+1)/2}\binom{n;\lfloor\frac{5\lambda+1}{2}\rfloor;q}{\lfloor\frac{5\lambda+1}{2}\rfloor}_2.$$

From (3.23) one can again deduce the first Rogers-Ramanujan identity as a limiting case; however, the simple replacement of the Gaussian polynomial in (3.18) by a q-trinomial in (3.23) is at the very least quite surprising.

Furthermore the limiting cases of (3.21) and (3.22) do not lead to the first Rogers-Ramanujan identity, but rather to the Rogers-Ramanujan series split into even and odd parts. Namely

$$1+\sum_{j=1}^{\infty}\frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \frac{1}{\prod_{n=1}^{\infty}(1-q^{2n})}$$
$$\left\{\sum_{\lambda=-\infty}^{\infty}(q^{60\lambda^2-4\lambda} - q^{60\lambda^2+44\lambda+8}) + q\sum_{\lambda=-\infty}^{\infty}(q^{60\lambda^2+16\lambda} - q^{60\lambda^2+64\lambda+16})\right\}.$$

The main riddle concerning all this is precisely the combinatorics. In [3], [4; Ch. 9], [15], [21], we see clearly the partition-theoretic significance of (3.18). However the q-trinomial version (3.23) is still a complete conbinatorial mystery.

4. Ramanujan

The work in the last several years stemming from Ramanujan's discoveries has truly been amazing. Bruce Berndt at the University of Illinois has been the chief architect of much of the work. He is bringing out edited versions [17], [18], [19] of Ramanujan's famous Notebooks. In addition, the book Ramanujan Revisited [7], edited by Berndt and others, describes recent research on a number of topics related to Ramanujan's work, and the Lost Notebook has been published in photostatic reproduction by Springer–Narosa in 1987. The Mock Theta Conjectures arising from the Lost Notebook were described as follows by Ian Stewart in Nature [33]:

> One of the most unusual people in the annals of mathematical research is Srinivasa Ramanujan, a self-taught Indian mathematician whose premature death left a rich legacy of unproved theorems. Ramanujan was preeminent in an unfashionable field — the manipulation of formulas. He tended to state his results without proofs — indeed on many occasions it is unclear whether he possessed proofs in the accepted sense —

yet he had an uncanny knack of penetrating to the heart of the matter. Over the years, many of Ramanujan's claims have been established in full rigour, although seldom easily. The most recent example, the 'mock theta conjectures', is especially striking, because the results in question were stated in Ramanujan's final correspondence with his collaborator Godfrey H. Hardy. The conjectures have recently been proved by Dean Hickerson [26]
... The proof involves delicate manipulations of infinite series of a kind that would have delighted Ramanujan. The astonishing complexity of the proof underlines, yet again, the depth of Ramanujan's genius. It is very hard to see how anyone could have been led to such results without getting bogged down in the fine detail. Ramanujan was the formula man *par exellence*, operating in a period when formulas were out of fashion. Today's renewed emphasis on combinatorics, inspired in a part by the digital nature of computers, has provoked a renewed interest in formula manipulations. The half–forgotten ideas of Srinivasa Ramanujan are breathing new life into number theory and combinatorics.

In what follows we provide a sketch of recent work arising from Ramanujan's Notebooks.

In [22], D. Bressoud gave a very simple proof of the Rogers–Ramanujan identities. We may for purposes of example slightly rephrase his proof. Namely, he noted that

$$\alpha_n = \begin{cases} 1 & \text{if} \quad n=0 \\ (-1)^n \left(zq^{\binom{n}{2}} + z^{-n} q^{\binom{n+1}{2}} \right) & \text{if} \quad n>0, \end{cases} \tag{4.1}$$

and

$$\beta_n = \frac{(z)_n \, (q/z)_n}{(q)_{2n}} \tag{4.2}$$

form a Bailey pair [6; p. 26], i.e.

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q)_{n-j} \, (aq)_{n+j}} \tag{4.3}$$

where

$$(A)_n = (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{4.4}$$

A weak iterated version of Bailey's Lemma asserts that if α_n and β_n form a Bailey pair, then

$$\frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} q^{kn^2} a^{kn} \alpha_n = \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{\alpha^{n_1+\cdots+n_a} q^{n_1^2+\cdots+n_k^2} \beta_{n_1}}{(q)_{n_k-n_{k-1}} (q)_{n_{k-1}-n_{k-2}} \cdots (q)_{n_2-n_1}}. \tag{4.5}$$

Bressoud's proof [22] can be viewed as setting $z = 1$ in (4.1) and (4.2) and inserting the resulting pair in (4.5) with $k = 2$.
If instead, we take (4.1) and (4.2) as they are and insert them into (4.5) with $k = 1$ and $\alpha = 1$, we find

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(z)_n\,(q/z)_n\,q^{n^2}}{(q)_{2n}} &= \frac{1 + \sum\limits_{n=1}^{\infty} (-1)^n \left(z^n q^{\binom{n}{2}} + z^{-n} q^{\binom{n+1}{2}}\right) q^{n^2}}{(q)_\infty} \\
&= \frac{\sum\limits_{n=-\infty}^{\infty} (-z)^n q^{n(3n-1)/2}}{(q)_\infty} \\
&= \frac{(q^3;q^3)_\infty (zq;q^3)_\infty (z^{-1}q^2;q^3)_\infty}{(q)_\infty},
\end{aligned} \tag{4.6}$$

a result from the Lost Notebook.

If we differentiate each entry in (4.6) and then set $z = 1$, we deduce

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(1-q)(1-q^2)\cdots(1-q^{n-1})q^{n^2}}{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n})} &= \frac{\sum\limits_{n=-\infty}^{\infty} (-1)^n n q^{n(3n-1)/2}}{(q)_\infty} \\
&= \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}} \right) \\
&= \sum_{n=1}^{\infty} q^n \left(\sum_{d/n} \left(\frac{d}{3}\right) \right),
\end{aligned} \tag{4.7}$$

where $\left(\frac{d}{3}\right)$ is the Legendre symbol.

Thus just beneath the surface of (4.6) is a q-series (namely the left–hand side of (4.7)) with multiplicative coefficients all non–negative, all indeed $O(n^2)$.

This suggests that the underlying combinatorics of (4.7) is well worth a look, and we shall not be disappointed. Following the program outlined in [5], we consider

$$f(t,q) = \sum_{n=1}^{\infty} \frac{t^{2n} q^{n^2} (-tq)_{n-1}}{(t)_{n+1}\,(t^2q;q^2)_n}. \tag{4.8}$$

The function $f(t,q)$ was constructed to both satisfy a first order nonhomogeneous q-difference equation

$$f(t,q) = \frac{t^2 q}{(1-t)(1-tq)(1-t^2q)} + \frac{t^2 q(1+tq)}{(1-t)(1-t^2q)} f(tq,q), \tag{4.9}$$

and to reduce in the case $t = -1$ to the left–hand side of (4.7):

$$f(-1,q) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{q^{n^2} (q)_{n-1}}{(q^{n+1};q)_n}.$$

If the magic of Ramanujan's mathematics is operating here, then $f(t,q)$ should be an interesting generating function of polynomials (in q), and $f(q,q^2)$, $\lim_{t\to 1^-} f(t,q)(1-t)$ should also exhibit interesting structure. In this regard, we find

$$f(t,q)=\sum_{n=2}^{\infty} t^n p_n(q)$$

with

$$p_n(q)=\sum_{\lambda=-\infty}^{\infty} q^{(2\lambda+1)(3\lambda+1)} \begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor - 3\lambda - 1 \end{bmatrix},$$

$$\lim_{t\to 1^-}(1-t)f(t,q)=\sum_{n=1}^{\infty}\frac{(1+q)^2(1+q^2)^2\cdots(1+q^{n-1})^2(1+q^n)q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^{2n})}$$

$$=q\prod_{n=1}^{\infty}\frac{(1-q^{12n})(1+q^{12n-1})(1+q^{12n-11})}{(1-q^n)},$$

and finally

$$\frac{1}{1-q}+(1+q)f(q,q^2)=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}(-q;q^2)_n}{(q)_{2n+1}(-q^2;q^2)_n}=\prod_{n=1}^{\infty}\frac{(1-q^{6n})(1+q^{6n-1})(1+q^{6n-5})}{(1-q^{2n})}.$$

It should be added that the above discoveries all resulted from a consideration of four seemingly benign identities of Ramanujan [11]. The simplest of which is

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{\binom{n+2}{2}}(1-q^n)}{(1+q^n)^2}=\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\right)^2\sum_{n=1}^{\infty}\frac{nq^{\binom{n+1}{2}}}{1-q^n}.$$

The proof of this result relies in an essential way on Bressoud's Bailey pair (4.1), (4.2) differentiated with respect to z and with z then set equal to 1 together with a general q-hypergeometric identity of Bailey [25, p. 42, eq. (2.10.10)].

5. Conclusion

These lectures being expository lack the details necessary for a full understanding of the underlying proofs. In this regard, Section 2 is an exposition of [16]; Section 3 of [10], and Section 4 of [5], [8] and [11]. Related background material may be found in [2] and [4] for Section 2, [9] for Section 3 and [6; Ch.9] for Section 4.

References

1. ALLADI K. and GORDON B., *Partition identities and a continued fraction of Ramanujan*, J. Combin. Theory Ser. A **63**, 275–300.
2. ANDREWS G.E., *A general theorem on partitions with difference conditions*, Amer. J. Math. **91** (1969), 18–24.

3. ANDREWS G.E., *Sieves in the theory of partitions*, Amer. J. Math. **94** (1972), 1214–1230.
4. ANDREWS G.E., *The Theory of Partitions*, Encycl. Math. and Its Application, Vol. 2, G.-C. Rota, ed., Addison-Wesley, Reading, 1976, (Reissued: Cambridge University Press, London and New York, 1985).
5. ANDREWS G.E., *Combinatorics and Ramanujan's "lost" notebook*, London Math. Soc. Lecture Series, Vol. 103, Cambridge University Press, London, 1985, pp. 1–23.
6. ANDREWS G.E., *q-Series*, C.B.M.S Series in Math., Vol. 66, Amer. Math. Soc., Providence, 1986.
7. ANDREWS G.E., *Ramanujan Revisited*, Academic Press, New York (1988).
8. ANDREWS G.E., *Mock theta functions*, Proc. Symp. in Pure Math. **49** (1989), 283–298.
9. ANDREWS G.E., *q-Trinomial coefficients and the Rogers-Ramanujan identities*, Analytic Number Theory (Berndt et al., eds.), Birkhäuser, Boston, 1990, pp. 1–11.
10. ANDREWS G.E., *Euler's "Exemplum Memorabile Inductionis Fallacis" and q-trinomial coefficients*, J. Amer. Math. Soc. **3** (1990), 653–669.
11. ANDREWS G.E., *Bailey chains and generalized Lambert series: I. four identities of Ramanujan*, Illinois J. Math. **36**, 251–274.
12. ANDREWS G.E. and BAXTER R.J., *Lattice gas generalizations of the hard hexagon model: I. star triangle relations and local densities*, J. Stat. Phys. **44** (1986), 259–271.
13. ANDREWS G.E. and BAXTER R.J., *Lattice gas generalizations of the hard hexagon model: II. the local densities as elliptic functions*, J. Stat. Phys. **44** (1986), 713–728.
14. ANDREWS G.E. and BAXTER R.J., *Lattice gas generalizations of the hard hexagon model: III q-trinomial coefficients*, J. Stat. Phys. **47** (1987), 297–230.
15. ANDREWS G.E., BAXTER R.J., BRESSOUD D.M., BURGE W.H., FORRESTER P.J. and VIENNOT G., *Partitions with prescribed hook differences*, Europ. J. Math. **8** (1987), 341–350.
16. ANDREWS G.E. and OLSSON J.B., *Partition identities with an application to group representation theory*, J. Reine und angew. Math. **413** (1991), 198–212.
17. BERNDT B.C., *Ramanujan's Notebooks, Part I*, Springer Verlag, New York and Berlin, 1985.
18. BERNDT B.C., *Ramanujan's Notebooks, Part II*, Springer Verlag, New York and Berlin, 1990.
19. BERNDT B.C., *Ramanujan's Notebooks, Part III*, Springer Verlag, New York and Berlin, 1991.
20. BESSENRODT C., *A combinatorial proof of a refinement of the Andrews-Olsson partition identity*, Europ. J. Combin. **12** (1991), 271–276.
21. BRESSOUD D.M., *Extension of the partition sieve*, J. Number Th. **12** (1980), 87–100.
22. BRESSOUD D.M., *An easy proof of the Rogers-Ramanujan identities*, J. Number Th. **16** (1983), 235–241.
23. EULER L., *Observations analyticae*, Novi Commentarii Academiae Scientarum Petropolitanae **11** (1765), 124–143, also in *Opera Omnia, Series 1* Vol. 15, Teubner, pp. 50–69.
24. FARAHAT H.K., MÜLLER W. and PEEL M.H, *The modular characters of the symmetric groups*, J. Algebra **40** (1976), 354–363.
25. GASPER G. and RAHMAN M., *Basic Hypergeometric Series*, Encycl. Math. and Its Applications, Vol. 35, G.-C. Rota, ed., Cambridge University Press, Cambridge, 1990.
26. HICKERSON D.R., *A proof of the mock theta conjectures*, Invent. math **94** (1989), 639–660.
27. KERBER A. and PEEL M.H., *On the decomposition numbers of symmetric and alternating groups*, Mitt. Math. Sem. Giessen **91** (1971), 45–81.
28. MACMAHON P.A., *The theory of modular partitions*, Proc. Cambridge Phil. Soc. **21** (1923), 197–204, (Reprinted in the Coll. Papers of P.A. MacMahon, Vol. 1, MIT Press, Cambridge, 1978).
29. MULLINEUX G., *Bijections of p-regular partitions and p-modular irreducibles of the symmetric groups*, J. London Math. Soc. (2) **20** (1979), 60–66.
30. ROBINSON G. de B., *Representation Theory of the Symmetric Group*, Edinburgh University Press, 1961.
31. SCHUR I., *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1917), 302–321, (Reprinted in I. Schur, Gesammelte Abhandlungen, Vol. 2, Springer, Berlin, 1973, pp. 117–136).

32. SCHUR I., *Zur additiven Zahlentheorie*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1926), 499–495, (Reprinted in I. Schur, Gesammelte Abhandlungen, Vol. 3 Springer, Berlin, 1973, pp. 43–50).
33. STEWART I., *Mock theta conjectures*, Nature **339** (1989), 341.
34. WAGNER A., *The faithful representations of the least degree of S_n and A_n over a field of odd characteristic*, Math. Z. **154** (1976), 103–114.

IBM
The Thomas J. Watson Research Center
Yorktown Heights, NY 10598 USA
and
The Pennsylvania State University
University Park, PA, PA 16802 USA