Valuated Matroids
– A new Look at the Greedy Algorithm

by

Andreas W.M. Dress and Walter Wenzel

Abstract: *In this note we study a variant of the greedy algorithm for* **weight functions** *defined on the system of m-subsets of a given set E and characterize completely those classes of weight functions for which this algorithm works. Well known examples come from matroid theory, new ones come from valuation theory.*
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Introduction When matroids were defined in 1935 by H. Whitney, they served the purpose of clarifying on an abstract level the concept of linear (in)dependence. It took more than thirty years before D. Gale observed 1968 in [G], based on earlier work of J.B. Kruskal [K] and R. Rado [R], that they can also be defined by means of their close relationship with *greedy algorithms*. More precisely, he showed that given a finite set E and a family $\mathcal{J}$ of subsets of E with "$J \subseteq I \in \mathcal{J} \Rightarrow J \in \mathcal{J}$" we can find for any map w from E into the set $\mathbb{R}_+$ of nonnegative real numbers a set $I = I_{max} \in \mathcal{J}$ with $f(I) := \sum\limits_{x \in I} f(x) \geq f(J)$ for all $j \in \mathcal{J}$ by using the greedy algorithm (that is by first choosing an element $x_1 \in E$ with $\{x_1\} \in \mathcal{J}$ and $f(x_1) = max(f(y) \mid \{y\} \in \mathcal{J})$, then an element $x_2 \in E \setminus \{x_1\}$ with $\{x_1, x_2\} \in \mathcal{J}$ and $f(x_2) = max(f(y) \mid \{x_1, y\} \in \mathcal{J}, y \neq x_1)$ and so on) if and only if $\mathcal{J}$ is the set of independent subsets of E with respect to some matroid M, defined on E.

Since then, many further interesting relations between the feasibility of greedy algorithms and various combinatorial structures have been discovered (cf. ...) and proved to be rather useful in combinatorial optimization. In this note we want to discuss an apparently new perspective in this context. We start with the observation that in the situation studied by Gale one knows in advance that the resulting set I_{max} is necessarily a basis of the matroid M, that is, it is a maximal subset of $\mathcal{J}$. According to ..., this can be used to speed up the optimization process a little bit as follows: start with an arbitrary basis $B = \{e_1, \ldots, e_m\}$ of M, then choose $x_1 \in E$ such that $B_1 := \{x_1, e_2, \ldots, e_m\}$ is a basis, too, and $f(B_1) \geq f(\{x, e_2, \ldots, e_m\})$ for all bases $\{x, e_2, \ldots, e_m\}$. Then replace e_2 by some $x_2 \in E$ for which $B_2; = \{x_1, x_2, e_3, \ldots, e_m\}$ is a basis and $f(B_2) \geq (\{x_1, xe_3, \ldots, e_m\})$ for all bases $\{x_1, x, e_3, \ldots, e_m\}$ and so on. Then we have $f(\{x_1, \ldots, x_m\}) \geq f(J)$ for all $J \in \mathcal{J}$. One therefore may ask for an arbitrary map v from the set $\binom{E}{m}$ of all m-subsets of E into $\mathbb{R} \cup \{-\infty\}$ for conditions under which the corresponding algorithm of replacing the elements $e_1, \ldots, e_m$ of some m-subset $\{e_1, \ldots, e_m\}$ of E with $v(\{e_1, \ldots, e_m\}) \neq -\infty$ consecutively by some $x_1, \ldots, x_m$ in a locally optimal or *greedy* fashion, the resulting set $\{x_1, \ldots, x_m\}$ satisfies $v(\{x_1, \ldots, x_m\}) \geq v(\{y_1, \ldots, y_m\})$ for all m-subsets $\{y_1, \ldots, y_m\}$ of E.

This question has a surprisingly simple answer if, as suggested by D. Gale's result, one modifies it as follows: Given some $v : \binom{E}{m} \to \mathbb{R} \cup \{-\infty\}$ with $v(\binom{E}{m}) \neq \{-\infty\}$, find necessary and sufficient conditions for v such that for all maps $\varphi : E \to \mathbb{R}$ the maximal value of $v_\varphi : \binom{E}{m} \to \mathbb{R} \cup \{-\infty\}$ given by

$$v_\varphi(\{e_1, \ldots, e_m\}) := v(\{e_1, \ldots, e_m\}) + \varphi(e_1) + \ldots + \varphi(e_m)$$

can be found by the greedy algorithm, explained above. As we will show, this is possible for all $\varphi : E \to \mathbb{R}$ if and only if the following variant of the matroid exchange property holds:

(V1) *For all* $B_1, B_2 \in \binom{E}{m}$ *and* $e \in B_1 \setminus B_2$ *there exists some* $f \in B_2 \setminus B_1$ *with*

$$v(B_1) + v(B_2) \leq v((B_1 \setminus \{e\}) \cup \{f\}) + v((B_2 \setminus \{f\}) \cup \{e\}).$$

Interesting examples of maps $v : \binom{E}{m} \to \mathbb{R} \cup \{-\infty\}$, satisfying this condition, come – surprisingly enough – from p-adic analysis (or, more generally, from valuation theory – actually, it was in this context, where the above condition occured first, (see [DW1]):

If E is a finite subset of $\mathbb{Q}^m$ which spans $\mathbb{Q}^m$, then the Grassmann-Plücker relations imply that for a given prime number p the map $v_p : \binom{E}{m} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$v_p(\{e_1, \ldots, e_m\}) := \begin{cases} -\infty & \text{if } det(e_1, \ldots, e_m) = 0 \\ n & \text{if } det(e_1, \ldots, e_m) = p^{-n} \cdot \frac{a}{b} \\ & \text{with } n \in \mathbb{Z}, a, b \in \mathbb{Z} \setminus p \cdot \mathbb{Z} \end{cases}$$

satisfies our condition. Hence, as an application of our result, one could compute a basis $e_1, \ldots, e_m$ of $\mathbb{Q}^m$, contained in E, for which the p-part of $det(e_1, \ldots, e_m)$ is as small as possible, by the above greedy algorithm.

It may also be interesting to review later work on the greedy algorithm from this perspective, in particular the work of B. Korte and L. Lovasz on greedoids (cf. [...]).

In the sequel we assume that E is some finite set and $m \in \mathbb{N}$ satisfies $m \leq \#E$.

For a map $v : \binom{E}{m} \to \mathbb{R} \cup \{-\infty\}$ we put

$$v(e_1, \ldots, e_m) := \begin{cases} v(\{e_1, \ldots, e_m\}) & \text{if } \{e_1, \ldots, e_m\} \in \binom{E}{m} \\ -\infty & \text{otherwise .} \end{cases}$$

Definition 1: A valuated matroid of rank m with values in $\mathbb{R}$, is a pair (E, v), consisting of a finite set E with $\#E \geq m$ together with a map $v : \binom{E}{m} \to \mathbb{R} \cup \{-\infty\}$ satisfying (V1) and

(V0) there exists some $B \in \binom{E}{m}$ with $v(B) \neq -\infty$.

An m-set $B \in \binom{E}{m}$ is called a **basis** of the valuated matroid (E, v), if $v(B) \neq -\infty$.

Remarks: i) By (V1) it is clear that the bases of a valuated matroid are also the bases of a combinatorial geometry (or matroid in the classical sense).

Vice versa, if M is a combinatorial geometry of rank m, defined on E, then any map v from $\binom{E}{m}$ into $\mathbb{R}\cup\{-\infty\}$ which satisfies (V0) and (V1) is called a valuation of M, if for all $B \in \binom{E}{m}$ one has $v(B) \neq -\infty$ if and only if B is a basis of M.

ii) A valuation $v : \binom{E}{m} \to \mathbb{R}\cup\{-\infty\}$ is called **trivial**, if there exists some $\alpha \in \mathbb{R}$ with $v(B) \in \{\alpha, -\infty\}$ for all $B \in \binom{E}{m}$.

Every combinatorial geometry M of rank m, defined on E, has a trivial valuation $v : \binom{E}{m} \to \mathbb{R}\cup\{-\infty\}$, namely

$$v(B) := \begin{cases} 0 & \text{if } B \text{ is a base of } M \\ -\infty & \text{otherwise .} \end{cases}$$

This is nothing but a reformulation of the strong exchange property for bases of M.

Definition 2: Assume $v : \binom{E}{m} \to \mathbb{R}\cup\{-\infty\}$ is some map with $v(B) \neq -\infty$ for at least one $B \in \binom{E}{m}$. The **greedy algorithm** runs as follows:

Step 0 : Choose some $e_1, \ldots, e_m \in E$ with $v(\{e_1, \ldots, e_m\}) \neq -\infty$.

Step $k(1 \leq k \leq m)$: Assume that $x_1, \ldots, x_{k-1} \in E$ are already determined and choose some $x_k \in E$ such that

$$v(\{x_1, \ldots, x_k, e_{k+1}, \ldots, e_m\}) \geq v(\{x_1, \ldots, x_{k-1}, x, e_{k+1}, \ldots, e_m\})$$

for all $x \in E$.

We say that the greedy algorithm **works** for v if for all starting sequences $e_1, \ldots, e_m \in E$ with $v(\{e_1, \ldots, e_m\}) \neq -\infty$ and all permitted choices of the $x_1, \ldots, x_m$ one has $v(\{x_1, \ldots, x_m\}) \geq v(B)$ for all $B \in \binom{E}{m}$ in which case v is called **admissible**.

For $e_1, \ldots, e_{m-1} \in E$ put

$$\begin{aligned} &M_v(e_1, \ldots, e_{m-1}) \\ &:= \{x \in E \mid v(e_1, \ldots, e_{m-1}, x) \geq v(e_1, \ldots, e_{m-1}, y) \text{ for all } y \in E\}. \end{aligned}$$

Obviously, v is admissible if and only if for all $e_1, \ldots, e_m \in E$ with $v(e_1, \ldots, e_m) \neq -\infty$ and all $x_1, \ldots, x_m \in E$ with

$$x_i \in M_v(x_1, \ldots, x_{i-1}, e_{i+1}, \ldots, e_m) \text{ for } 1 \leq i \leq m$$

we have $v(x_1, \ldots, x_m) \geq v(y_1, \ldots, y_m)$ for all $y_1, \ldots, y_m \in E$.

Definition 3: Two maps $v, w : \binom{E}{m} \to \mathbb{R}\cup\{-\infty\}$ are called **projectively equivalent**, if there exists some $\alpha \in \mathbb{R}$ and some map $\varphi : E \to \mathbb{R}$ such that

$$w(e-1, \ldots, e_m) = \alpha + \sum_{i=1}^{m} \varphi(e_i) + v(e_1, \ldots, e_m) \text{ for all } e_1, \ldots, e_m \in E.$$

If this is the case, we write $w := v(\alpha, \varphi)$.

Remark: If $v : \binom{E}{m} \to \mathbb{R} \cup \{-\infty\}$ is a valuation of some combinatorial geometry M, then it is clear that $v(\alpha, \varphi)$ is also a valuation of M for all $\alpha \in \mathbb{R}$ and all maps $\varphi : E \to \mathbb{R}$.

Now we can show

Theorem: Assume E is a finite set with $\#E \geq m$ and $v : \binom{E}{m} \to \mathbb{R} \cup \{-\infty\}$ is some map satisfying (V0). Then (E, v) is a valuated matroid, if and only if the greedy algorithm works for v_φ for all $\alpha \in \mathbb{R}$ and all maps $\varphi : E \to \mathbb{R}$.

Proof: At first we assume that v is a valuation of some combinatorial geometry M, defined of E. By the last remark it is enough to show that v is admissible.

Assume $e_1, \ldots, e_m \in E$ with $v(\{e_1, \ldots, e_m\}) \neq 0$ and $x_1, \ldots, x_m \in E$ such that $x_i \in M_v(x_1, \ldots, x_{i-1}, e_{i+1}, \ldots, e_m)$ for $1 \leq i \leq m$. Put $B_0 := \{x_1, \ldots, x_m\}$. We must prove

$$v(B) \leq v(B_0) \text{ for all } B \in \tbinom{E}{m}. \tag{1}$$

At first we show

$$v((B_0 \setminus x_j) \cup x) \leq v(B_0) \text{ for } 1 \leq j \leq m \text{ and all } x \in E. \tag{2}$$

By our assumption (2) is clear for $j = m$.

To prove (2) for $1 \leq j \leq m-1$ we may assume by induction that

$$v((B_0 \setminus \{x_j, x_m\}) \cup \{x, e_m\}) \leq v((B_0 \setminus x_m) \cup e_m) \text{ for all } x \in E, \tag{2a}$$

because $v' : \binom{E \setminus e_m}{m-1} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$v'(A) := v(A \cup e_m) \ (A \in \tbinom{E \setminus e_m}{m})$$

is obviously a valuation of the contraction $M' := M/\{e_m\}$ of rank $m-1$.

Now assume $v((B_0 \setminus x_j) \cup x) \neq -\infty$ for some fixed j with $1 \leq j \leq m-1$ and some fixed $x \in E$. Consider the bases $B_1 := (B_0 \setminus x_m) \cup e_m$ and $B_2 := (B_0 \setminus x_j) \cup x$ of M. (2) clearly holds in case $x_j = x$. Otherwise we have $x_j \in B_1 \setminus B_2$. Thus there exists $e \in B_2 \setminus B_1 \subseteq \{x_m, x\}$ with

$$v(B_1) + v(B_2) \leq v((B_1 \setminus x_j) \cup e) + v((B_2 \setminus e) \cup x_j).$$

But (2a) yields $v((B_1 \setminus x_j) \cup e) \leq v(B_1)$ and thus

$$v(B_2) \leq ((b_2 \setminus e) \cup x_j). \tag{2b}$$

Furthermore, we have $\{x_1, \ldots, x_{m-1}\} \subseteq (B_2 \setminus e) \cup x_j$, and thus our choice of x_m implies

$$v((B_2 \setminus e) \cup x_j) \leq v(B). \tag{2c}$$

(2) follows now from (2b) and (2c).

Now we derive (1) from (2) by induction on $n := \#(\{y_1,\ldots,y_m\} \setminus \{x_1,\ldots,x_m\})$. The cases $n=0$ and $v(y_1,\ldots,y_m)=0$ are trivial, while for $n=1$ we are done by (2).

Now assume $2 \le n \le m$, say, $\#\{y_1,\ldots,y_n,x_1,\ldots,x_m\} = n+m$ and $y_k = x_k$ for $n+1 \le k \le m$. Then by (V1), (2) and our induction hypothesis there exists i with $1 \le i \le n$ and

$$\begin{aligned}
&v(\{y_1,\ldots,y_n,x_{n+1},\ldots,x_m\}) + v(B_0)\\
\le& v(\{x_i,y_2,\ldots,y_n,x_{n+1},\ldots,x_m\}) + v((B_0\setminus x_i)\cup y_1)\\
\le& 2\cdot v(B_0).
\end{aligned}$$

Since $v(B_0) \ge v(\{e_1,\ldots,e_m\}) > -\infty$, this means

$$v(y_1,\ldots,y_n,x_{n+1},\ldots,x_m) \le v(B_0).$$

Now assume that, vice versa, v_φ is admissible for all $\alpha \in \mathbb{R}$ and all maps $\varphi : E \to \mathbb{R}$. We have to show that v satisfies (V1). Otherwise assume $e_1,\ldots,e_m,f_1,\ldots,f_m \in E$ are such that for $B_1 := \{e_1,\ldots,e_m\}$ and $B_2 := \{f_1,\ldots,f_m\}$ we have

$$v(B_1)+v(B_2) > v((B_2\setminus f_i)\cup e_1) + v((B_1\setminus e_1)\cup f_i) \tag{3}$$

for all i with $1 \le i \le m$. Then we must have $e_1 \notin B_2$ and $v(B_2) \neq -\infty$.

Since E is finite, we can choose some $\gamma \in \mathbb{R}$ such that for all $\gamma_0, \delta_0 \in v(\binom{E}{m}) \cap \mathbb{R}$ we have

$$\gamma_0 - \delta_0 < \gamma. \tag{4}$$

Now define $\varphi : E \to \mathbb{R}$ by

$$\begin{aligned}
&\varphi(e_1) := 0,\\
&\varphi(f_i) := \begin{cases} \gamma \text{ if } v(\{f_i,e_2,\ldots,e_m\}) = -\infty\\ v(B_1) - v(\{f_i,e_2,\ldots,e_m\}) \text{ otherwise,}\end{cases}\\
&\varphi(x) := -2m\cdot\gamma \text{ for } x \in E\setminus\{e_1,f_1,\ldots,f_m\}.
\end{aligned}$$

and put $w := v(0,\varphi)$.

By the definition of φ we have

$$v(\{e_1,\ldots,e_m\}) \ge \varphi(x) + v(\{x,e_2,\ldots,e_m\})$$

and therefore

$$v_\varphi(\{e_1,\ldots,e_m\}) \ge v_\varphi(\{x,e_2,\ldots,e_m\}) \text{ for all } x \in E$$

and thus $e_1 \in M_w(e_2,\ldots,e_m)$.

On the other hand, we show next that for all pairwise distinct $x_1, \ldots, x_m \in E$ with $X := \{x_1, \ldots, x_m\} \neq B_2$ we have

$$w(B_2) > w(B); \tag{5}$$

so no base B with $e_1 \in B$ can have maximal v_φ-value in contradiction to our assumption that the greedy algorithm works for v_φ.

Indeed, if $X \nsubseteq B_2 \cup e_1$, then $\varphi(x_0) = -2m \cdot \gamma$ for at least one $x_0 \in X$ and $\varphi(x) \leq \gamma$ for all $x \in X \setminus x_0$. This means in view of $-\gamma \leq \varphi(f_i)$ for $1 \leq i \leq m$ also

$$\begin{aligned} w(X) = \sum_{j=1}^{m} \varphi(x_j) + v(X) &\leq -(m+1) \cdot \gamma + v(x) \\ &< -m \cdot \gamma + v(B_2) \leq w(B_2). \end{aligned}$$

Otherwise, $X = (B_2 \setminus f_i) \cup e_1$ for some i with $1 \leq i \leq m$. If $v(f_i, e_2, \ldots, e_m) \neq -\infty$, then (3) implies

$$w(X) = \sum_{\substack{j=1 \\ j\neq 1}}^{m} \varphi(f_j) + v(X) < \sum_{\substack{j=1 \\ j\neq 1}}^{m} \varphi(f_j) + v(B_1) + v(b_2) - v(f_i, e_2, \ldots, e_m)$$

and therefore

$$w(X) < \sum_{\substack{j=1 \\ j\neq i}}^{m} \varphi(f_j) + v(B_2) + \varphi(f_i) = w(|B_2).$$

Finally, if $v(f_i, e_2, \ldots, e_m) = -\infty$, then

$$w(X) = \sum_{\substack{j=1 \\ j\neq 1}}^{m} \varphi(f_j) + v(X) < \sum_{\substack{j=1 \\ j\neq i}}^{m} \varphi(f_j) + \gamma + v(B_2) = w(B_2).$$

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Universität Bielefeld
Fakultät für Mathematik
Postfach 8640
D-4800 Bielefeld 1
FRG