The Tutte Group of Projective Spaces

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The Tutte group of a matroid offers an algebraic approach to matroid theory. In particular, this group reflects some important geometric properties of matroids in algebraic terms.

In the sequel let M denote a matroid defined on some possibly infinite set E of finite rank n with $\mathcal{H} = \mathcal{H}_M$ as its set of hyperplanes and $\mathcal{L} = \mathcal{L}_M$ as its set of hyperlines.

Definition of the extended Tutte group:
Let $\mathbb{F}_M^{\mathcal{H}}$ denote the free abelian group generated by the symbols ε and $X_{H,a}$ for $H \in \mathcal{H},\ a \in E \setminus H$, and let $\mathbb{K}_M^{\mathcal{H}}$ denote the subgroup of $\mathbb{F}_M^{\mathcal{H}}$ generated by ε^2 and all elements of the form

$$\varepsilon \cdot X_{H_1,a_2} \cdot X_{H_1,a_3}^{-1} \cdot X_{H_2,a_3} \cdot X_{H_2,a_1}^{-1} \cdot X_{H_3,a_1} \cdot X_{H_3,a_2}^{-1}$$

for $H_1, H_2, H_3 \in \mathcal{H},\ L := H_1 \cap H_2 \cap H_3 = H_i \cap H_j \in \mathcal{L}$ for $i \neq j$ and $a_i \in H_i \setminus L$ for $1 \leq i \leq 3$.
Then the **(extended) Tutte group** $\mathbb{T}_M^{\mathcal{H}}$ of M is defined by

$$\mathbb{T}_M^{\mathcal{H}} := \mathbb{F}_M^{\mathcal{H}} / \mathbb{K}_M^{\mathcal{H}}.$$

Let ε_M denote the image of ε and $H(a)$ the image of $X_{H,a}$ in $\mathbb{T}_M^{\mathcal{H}}$, respectively.

Definition of the inner Tutte group:
Define the homomorphism $\Phi : \mathbb{T}_M^{\mathcal{H}} \to \mathbb{Z}^{\mathcal{H}} \times \mathbb{Z}^E$ by

$$\begin{array}{ll} \Phi(\varepsilon_M) & := 0, \\ \Phi(H(a)) & := (\delta_H, \delta_a) \text{ for } H \in \mathcal{H},\ a \in E \setminus H. \end{array}$$

$\mathbb{T}_M^{(0)} := ker\Phi$ is the **inner Tutte group** of M.

Proposition 1:
Assume M is representable over the skewfield K; that is, there exists a family of maps $f_H : E \to K$ ($H \in \mathcal{H}$) with

(H1) $f_H^{-1}(\{0\}) = H$ for all $H \in \mathcal{H}$;

(H2) if $H_1, H_2, H_3 \in \mathcal{H}$ are pairwise distinct and $L := H_1 \cap H_2 \cap H_3 \in \mathcal{L}$, then there exist $c_1, c_2, c_3 \in K^\star$ such that for all $e \in E$ we have

$$f_{H_1}(e) \cdot c_1 + f_{H_2}(e) \cdot c_2 + f_{H_3}(e) \cdot c_3 = 0.$$

Put $K_C := K^\star/[K^\star, K^\star]$.
Then the homomorphism $\psi : \mathbb{F}_M^{\mathcal{H}} \to K^\star$ given by

$$\begin{aligned}&\psi(\varepsilon) := -1,\\ &\psi(X_{H,a}) := f_H(a) \text{ for } H \in \mathcal{H},\ a \in E \setminus H\end{aligned}$$

induces a homomorphism $\varphi = \overline{\psi} : \mathbb{T}_M^{\mathcal{H}} \to K_C$.

Proposition 2:
Assume $H_1, H_2, H_3 \in \mathcal{H}$ with $H_1 \cap H_2 \cap H_3 \in \mathcal{L}$ and $H_3 \neq H_i$ for $i \in \{1, 2\}$. Then for $a, b \in H_3 \setminus (H_1 \cup H_2)$ we have

$$H_1(a) \cdot H_2(a)^{-1} = H_1(b) \cdot H_2(b)^{-1}.$$

Definition of cross ratios in Tutte groups:
If H_1, H_2, H_3 are as in Proposition 2, put

$$\left| \begin{matrix} H_1 & & H_2 \\ & H_3 & \end{matrix} \right| := H_1(a) \cdot H_2(a)^{-1} \text{ for } a \in H_3 \setminus (H_1 \cup H_2).$$

Assume $H_1, H_2, H_3, H_4 \in \mathcal{H}$ with $H_1 \cap H_2 \cap H_3 \cap H_4 \in \mathcal{L}$ and $H_1, H_2 \neq H_3, H_4$, i.e. $\{H_1, H_2\} \cap \{H_3, H_4\} = \emptyset$. The **cross ratio** $\left[\begin{matrix} H_1 & H_2 \\ H_3 & H_4 \end{matrix} \right] \in \mathbb{T}_M^{(0)}$ is defined by

$$\left[\begin{matrix} H_1 & H_2 \\ H_3 & H_4 \end{matrix} \right] := \left| \begin{matrix} H_1 & & H_2 \\ & H_3 & \end{matrix} \right| \cdot \left| \begin{matrix} H_2 & & H_1 \\ & H_4 & \end{matrix} \right|.$$

By applying the more elementary part of Tutte's homotopy theory one proves

Proposition 3:

i) $\mathbb{T}_M^{(0)}$ is the subgroup of $\mathbb{T}_M^{\mathcal{H}}$ generated by ε_M and all cross ratios $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$ for which H_1, H_2, H_3, H_4 are pairwise distinct hyperplanes in M with $H_1 \cap H_2 \cap H_3 \cap H_4 \in \mathcal{L}$.

ii) $\mathbb{T}_M^{(0)}$ is generated by all cross ratios $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$ as in i) if and only if M is not regular.

Definition:
Put

$$\mathcal{H}^{(4)} := \{(H_1, H_2, H_3, H_4) \in \mathcal{H}^4 \mid \begin{array}{l} H_1 \cap H_2 \cap H_3 \cap H_4 = H_i \cap H_j \in \mathcal{L} \\ \text{for } i \in \{1,2\},\ j \in \{3,4\}\ \}. \end{array}$$

i) $(H_1, H_2, H_3, H_4), (H_1', H_2', H_3', H_4') \in \mathcal{H}^{(4)}$ are called **projective neighbours of each other**, if $\bigcap_{i=1}^4 H_i \neq \bigcap_{i=1}^4 H_i'$ and there exists some $H \in \mathcal{H}$ with

   (I) $\bigcap_{i=1}^4 H_i \subsetneq H,\ \bigcap_{i=1}^4 H_i' \subsetneq H$ and

   (II) for $1 \le i \le 4$ we have $H_i = H_i'$ or $H_i \cap H_i' = H_i \cap H = H_i' \cap H \in \mathcal{L}$.

   If this is the case, we write $(H_1, H_2, H_3, H_4) \wedge (H_1', H_2', H_3', H_4')$.

ii) $G = (H_1, H_2, H_3, H_4)$ and $G' = (H_1', H_2', H_3', H_4') \in \mathcal{H}^{(4)}$ are called **projectively equivalent**, if there exist $k \ge 0$ and $G_0, G_1, \ldots, G_k \in \mathcal{H}^{(4)}$ such that
$$G_0 = G,\ G_k = G',\ G_{i-1} \wedge G_i \text{ for } 1 \le i \le k.$$
   In this case we write $(H_1, H_2, H_3, H_4) \overset{pr}{\sim} (H_1', H_2', H_3', H_4')$.

Proposition 4:
Assume $(H_1, H_2, H_3, H_4) \overset{pr}{\sim} (H_1', H_2', H_3', H_4')$. Then we have

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} = \begin{bmatrix} H_1' & H_2' \\ H_3' & H_4' \end{bmatrix}.$$

From now on we assume that M is a projective space of dimension $m \ge 2$.

Proposition 5:
Assume $H_1, H_2 \in \mathcal{H}$ are distinct, put

$$\mathcal{H}_{1,2} := \{H \in \mathcal{H} \mid H_1 \cap H_2 \subseteq H\}$$

and let $\mathcal{G}_{(H_1,H_2)}$ denote the group of projectivities $p : \mathcal{H}_{1,2} \to \mathcal{H}_{1,2}$ with $p(H_i) = H_i$ for $i \in \{1,2\}$. Then $\eta : \mathcal{G}_{(H_1,H_2)} \twoheadrightarrow \mathbb{T}_M^{(0)}$ defined by

$$\eta(p) := \begin{bmatrix} H_1 & H_2 \\ H_3 & p(H_3) \end{bmatrix} \text{ for } H_3 \in \mathcal{H}_{1,2} \setminus \{H_1, H_2\}$$

is a well defined group epimorphism; that is, η does not depend on H_3.

Proposition 6:
If $H_1, H_2, H_3 \in \mathcal{H}$ are pairwise distinct with $L := H_1 \cap H_2 \cap H_3 \in \mathcal{L}$, then

$$\mathbb{T}_M^{(0)} = \left\{ \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \mid L \subseteq H_4 \in \mathcal{H} \setminus \{H_1, H_2\} \right\}.$$

By applying Proposition 1 and Proposition 6 one proves

Proposition 7:
If M is the projective space over the skewfield K of dimension $m \geq 2$, then $\mathbb{T}_M^{(0)} \cong K_C$.
If, in particular, K is a field, then $\mathbb{T}_M^{(0)} \cong K^\star$.

In general it follows from [G], Theorem 2, that in case M is a finite non-desarguesian projective plane its inner Tutte group $\mathbb{T}_M^{(0)}$ is necessarily trivial.

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