Publ. I.R.M.A. Strasbourg, 1988, 372/S–20
Actes 20^{e} Séminaire Lotharingien, p. 131–171

A NEW CLASS OF SYMMETRIC FUNCTIONS

BY

I. G. MACDONALD

Contents.

1. Introduction

I will begin by reviewing briefly some aspects of the theory of symmetric functions. This will serve to fix notation and to provide some motivation for the subject of these lectures.

Let $x_1, \ldots, x_n$ be independent indeterminates. The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{Z}[x_1,\ldots,x_n]$ by permuting the x's, and we shall write

$$\Lambda_n = \mathbb{Z}[x_1,\ldots,x_n]^{\mathfrak{S}_n}$$

for the subring of symmetric polynomials in $x_1, \ldots, x_n$. If $f \in \Lambda_n$, we may write

$$f = \sum_{r\geq 0} f^{(r)}$$

where $f^{(r)}$ is the homogeneous component of f of degree r; each $f^{(r)}$ is itself symmetric, and so Λ_n is a *graded* ring :

$$\Lambda_n = \bigoplus_{r\geq 0} \Lambda_n^r,$$

where Λ_n^r is the additive group of symmetric polynomials of degree r in $x_1, \ldots, x_n$. (By convention, 0 is homogeneous of every degree.)

If we now adjoin another indeterminate x_{n+1}, we can form $\Lambda_{n+1} = \mathbb{Z}[x_1, \ldots, x_{n+1}]^{\mathfrak{S}_{n+1}}$, and we have a surjective homomorphism (of graded rings)

$$\Lambda_{n+1} \to \Lambda_n$$

defined by setting $x_{n+1} = 0$. The mapping $\Lambda_{n+1}^r \to \Lambda_n^r$ is surjective for all $r \geq 0$, and bijective if and only if $r \leq n$.

Often it is convenient to pass to the limit. Let

$$\Lambda^r = \varprojlim_n \Lambda_n^r$$

for each $r \geq 0$, and let

$$\Lambda = \bigoplus_{r \geq 0} \Lambda^r.$$

By the definition of inverse (or projective) limits, an element of Λ_n^r is a sequence $(f_n)_{n \geq 0}$ where $f_n \in \Lambda_n^r$ for each n, and f_n is obtained from f_{n+1} by setting $x_{n+1} = 0$. We may therefore regard the f_n as the partial sums of an infinite series f of monomials of degree r in infinitely many indeterminates x_1, x_2, ... For example, if $f_n = x_1 + \cdots + x_n$, then $f = \sum_{i=1}^{\infty} x_i$. Thus the elements of Λ are no longer polynomials, and we call them symmetric *functions*. (Of course, they aren't functions either, but they have to be called something!)

For each n there is a surjective homomorphism $\Lambda \to \Lambda_n$, obtained by setting $x_{n+1} = x_{n+2} = \cdots = 0$. The graded ring Λ is the *ring of symmetric functions*. If R is any commutative ring, we write

$$\Lambda_R = \Lambda \otimes_{\mathbb{Z}} R, \qquad \Lambda_{n,R} = \Lambda_n \otimes_{\mathbb{Z}} R$$

for the ring of symmetric functions (resp. symmetric polynomials in n indeterminates) with coefficients in R.

There are various $\mathbb{Z}$-bases of the ring Λ, some of which we shall review. They all are indexed by *partitions*. A partitions λ is a (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

of non-negative integers, such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and

$$|\lambda| = \sum \lambda_i < \infty,$$

so that from a certain point onwards (if the sequence λ is infinite) all the λ_i are zero. We shall not distinguish between two such sequences which differ only by a string of zeros at the end. Thus $(2,1)$, $(2,1,0)$, $(2,1,0,0,\dots)$ are all to be regarded as the same partition.

The nonzero λ_i are called the *parts* of λ, and the number of parts is the *length* $\ell(\lambda)$ of λ. If λ has m_1 parts equal to 1, m_2 parts equal to 2, and so on, we shall occasionally write $\lambda = (1^{m_1}2^{m_2}\dots)$, although strictly speaking we should write this in reverse order.

Let $\mathcal{P}$ denote the set of all partitions, and $\mathcal{P}_n$ the set of all partitions of n (i.e. partitions λ such that $|\lambda| = n$). The *natural* (or dominance) *partial ordering* in $\mathcal{P}$ is defined as follows :

$$\lambda \geq \mu \iff |\lambda| = |\mu| \text{ and } \lambda_1 + \cdots + \lambda_r \geq \mu_1 + \cdots + \mu_r \text{ for all } r \geq 1. \tag{1.1}$$

It is a total order on $\mathcal{P}_n$ for $n \leq 5$, but not for $n \geq 6$.

With each partition λ we associate a *diagram*, consisting of the points $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$. We adopt the convention (as with matrices) that the first coordinate i (the row index) increases as one goes downwards, and the second coordinate j (the column index) increases from left to right. Often it is more convenient to replace the lattice points (i,j) by squares, and then the diagram of λ consists of λ_1 boxes in the top row, λ_2 boxes in the second row, and so on; the whole arrangement of boxes being left-justified.

If we read the diagram of a partition λ by columns, we obtain the *conjugate partition* λ'. Thus λ'_j is the number of boxes in the j-th column of λ, and hence is equal to the number of parts of λ that are $\geq j$. It is not difficult to show that

$$\lambda \geq \mu \iff \mu' \geq \lambda'.$$

Bases of Λ.

1. Monomial symmetric functions. — Let λ be a partition. It defines a monomial $x^\lambda = x_1^{\lambda_1}x_2^{\lambda_2}\dots$ The monomial symmetric function m_λ is the sum of all *distinct* monomials obtainable from x^λ by permutations of the x's. For example, $m_{(2,1)} = \sum x_i^2 x_j$, summed over all (i,j) such that $i \neq j$.

In particular, when $\lambda = (1^r)$ we have

$$m_{(1^r)} = e_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \dots x_{i_r},$$

the r-th *elementary symmetric function*. Their generating function is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod (1 + x_i t), \tag{1.2}$$

where t is another indeterminate, and $e_0 = 1$.

At the other extreme, when $\lambda = (r)$ we have

$$m_{(r)} = p_r = \sum x_i^r,$$

the r-th *power sum.*

It is clear that every $f \in \Lambda$ is uniquely expressible as a finite linear combination of the m_λ, so that $(m_\lambda)_{\lambda\in\mathcal{P}}$ is a $\mathbb{Z}$-*basis* of Λ.

2. — For any partition λ, let

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots$$

The e_λ form another $\mathbb{Z}$-basis of Λ. Equivalently, we have $\mathbb{Z}[e_1, e_2, \dots]$ and the e_r are algebraically independent. Indeed, it is not difficult to show that

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} m_\mu$$

for suitable coefficients $a_{\lambda\mu}$, from which the assertion follows immediately.

3. — For each $r \geq 0$ let

$$h_r = \sum_{|\lambda|=r} m_\lambda,$$

the sum of *all* monomials of total degree r in the x's. The generating function for the h_r is

$$H(t) = \sum_{r\geq 0} h_r t^r = \prod (1 - x_i t)^{-1}, \tag{1.3}$$

as one sees by expanding each factor $(1 - x_i t)^{-1}$ in the product on the right as a geometric series, and then multiplying these series together. From (1.2) and (1.3) it follows that

$$H(t)E(-t) = 1$$

so that

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \tag{1.4}$$

for each $n \geq 1$.

Since the e_r are algebraically independent, we may define a ring homomorphism $\omega : \Lambda \to \Lambda$ by

$$\omega(e_r) = h_r$$

for all $r \geq 1$. The symmetry of the relations (1.4) as between the e's and the h's then shows that $\omega(h_r) = e_r$, i.e., $\omega^2 = 1$. Thus ω is an *automorphism* (of period 2) of Λ, and therefore we have $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$. Equivalently, the products

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots = \omega(e_\lambda)$$

form another $\mathbb{Z}$-basis of Λ.

4. — The generating function for the power-sums $p_r = \sum x_i^r$ is

$$\begin{aligned} P(t) &= \sum_{r \geq 1} p_r t^{r-1} \\ &= \sum_i \sum_{r \geq 1} x_i^r t^{r-1} \\ &= \sum_i \frac{x_i}{1 - x_i t} \end{aligned}$$

and therefore

$$P(t) = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)}. \tag{1.5}$$

Hence we have $H'(t) = H(t)P(t)$, so that

$$n h_n = \sum_{r=1}^{n} p_r h_{n-r}$$

for all $n \geq 1$. These relations enable us to express the h's in terms of the p's, and vice versa, and show that

$$\begin{aligned} h_n &\in \mathbb{Q}[p_1, \ldots, p_n], \\ p_n &\in \mathbb{Z}[h_1, \ldots, h_n] \end{aligned}$$

so that

$$\mathbb{Q}[p_1, \ldots, p_n] = \mathbb{Q}[h_1, \ldots, h_n]$$

for all $n \geq 1$. Letting $n \to \infty$, we see that

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[h_1, h_2, \ldots] = \mathbb{Q}[p_1, p_2, \ldots].$$

For each partition λ let

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

(with the understanding that $p_0 = 1$). The p_λ form a $\mathbb{Q}$-basis of $\Lambda_{\mathbb{Q}}$, but do *not* form a $\mathbb{Z}$-basis of Λ.

Analogously to (1.5) we have

$$P(-t) = \frac{d}{dt}\log E(t). \tag{1.6}$$

Since the involution ω interchanges $E(t)$ and $H(t)$, it follows from (1.5) and (1.6) that it interchanges $P(t)$ and $P(-t)$, so that

$$\omega(p_r) = (-1)^{r-1}p_r \tag{1.7}$$

for all $r \geq 1$.

Finally, we may compute h_n as a polynomial in the power sums, as follows : from (1.5) we have

$$\begin{aligned} H(t) &= \exp\Bigl(\sum_{r\geq 1}\frac{p_r t^r}{r}\Bigr) \\ &= \prod_{r\geq 1}\exp\frac{p_r t^r}{r} \\ &= \prod_{r\geq 1}\sum_{m_r\geq 0}\frac{1}{m_r!}\Bigl(\frac{p_r t^r}{r}\Bigr)^{m_r}. \end{aligned}$$

Let us pick out the coefficient of p_λ in the product. If $\lambda = (1^{m_1}2^{m_2}\dots)$, it is z_λ^{-1}, where

$$z_\lambda = \prod_{r\geq 1}(r^{m_r}.m_r!) \tag{1.8}$$

and therefore

$$h_n = \sum_{|\lambda|=n} z_\lambda^{-1}p_\lambda.$$

This numerical function z_λ (which will occur frequently in the sequel) has the following interpretation. Let $|\lambda| = n$, and let $w \in \mathfrak{S}_n$ be a permutation of cycle-type λ. Then z_λ is the order of the centralizer of w in $\mathfrak{S}_n$.

5. Schur functions. — Let λ be a partition of length $\leq n$, and form the determinant

$$D_\lambda = \det\Bigl(x_i^{\lambda_j+n-j}\Bigr)_{1\leq i,j\leq n}.$$

This vanishes whenever any two of the x's are equal, hence is divisible by the Vandermonde determinant

$$D_0 = \prod_{i<j}(x_i - x_j).$$

The quotient

$$s_\lambda(x_1,\ldots,x_n) = D_\lambda/D_0$$

is a homogeneous symmetric polynomial of degree $|\lambda|$ in $x_1, \ldots, x_n$. Moreover we have

$$s_\lambda(x_1,\ldots,x_n,0) = s_\lambda(x_1,\ldots,x_n)$$

and hence for each partition λ a well-defined element $s_\lambda \in \Lambda$, homogeneous of degree $|\lambda|$. These are the *Schur functions.*

Let us define a scalar product $\langle\ ,\ \rangle$ on Λ as follows :

$$\text{(1.9)} \qquad \langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu} z_\lambda$$

where $\delta_{\lambda\mu} = 1$ if $\lambda = \mu$, and $\delta_{\lambda\mu} = 0$ otherwise. Then one can show (see e.g. [M_1], ch. I) that the Schur functions s_λ have the following properties, which characterize them uniquely :

(A) $\quad s_\lambda = m_\lambda + \sum\limits_{\lambda<\mu} K_{\lambda\mu} m_\mu$

for suitable coefficients $K_{\lambda\mu}$;

(B) $\quad \langle s_\lambda, s_\mu\rangle = 0 \quad \text{if } \lambda \neq \mu.$

6. Zonal symmetric functions. — These are certain symmetric functions Z_λ, at present more familiar to statisticians than combinatorialists, which (when restricted to a finite number of variables $x_1, \ldots, x_n$) arise naturally in connection with Fourier analysis on the homogeneous space G/K, where $G = \mathrm{GL}_n(\mathbb{R})$ and $K = O(n)$, the orthogonal group (so that G/K may be identified, via $X \mapsto XX^t$, with the space of positive definite real symmetric $n \times n$ matrices). I shall not give a direct definition here, but will only remark that the Z_λ (suitably normalized) are characterized by the following two properties :

(A) $\quad Z_\lambda = m_\lambda + \text{lower terms}$

where by "lower terms" is meant a linear combination of the m_μ such that $\mu < \lambda$,

(B) $\quad \langle Z_\lambda, Z_\mu\rangle_2 = 0 \quad \text{if } \lambda \neq \mu,$

where the scalar product $\langle\ ,\ \rangle_2$ on Λ is defined by

$$\text{(1.10)} \qquad \langle p_\lambda, p_\mu\rangle_2 = \delta_{\lambda\mu}.2^{\ell(\lambda)} z(\lambda),$$

$\ell(\lambda)$ being the length of the partition λ.

7. Jack's symmetric functions [S]. — These are a common generalization of the Schur functions and the zonal symmetric functions, and again I shall not give a direct construction of them at this stage. Let $\alpha \in \mathbb{R}$, $\alpha > 0$, and define a scalar product $\langle\ ,\ \rangle_\alpha$ on $\Lambda_\mathbb{R}$ by

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} . \alpha^{\ell(\lambda)} z_\lambda. \tag{1.11}$$

Then Jack's functions $P_\lambda = P_\lambda(x;\alpha)$ are characterized by the two properties

(A) $P_\lambda = m_\lambda + \text{lower terms}$,
(B) $\langle P_\lambda, P_\mu \rangle_\alpha = 0$ if $\lambda \neq \mu$.

The symmetric functions P_λ depend rationally on α, i.e. they lie in Λ_F where F is the field $\mathbb{Q}(\alpha)$, and we may if we prefer regard the parameter α as an indeterminate rather than a real number. Clearly when $\alpha = 1$ they reduce to the Schur functions s_λ, and when $\alpha = 2$ to the zonal functions Z_λ. They also tend to definite limits as $\alpha \to 0$ and as $\alpha \to \infty$ (even though the scalar product (1.11) collapses), and in fact

$$\begin{aligned} P_\lambda(x;\alpha) &\to e_{\lambda'} \quad \text{as } \alpha \to 0, \\ P_\lambda(x;\alpha) &\to m_\lambda \quad \text{as } \alpha \to \infty. \end{aligned}$$

Also when $\alpha = \frac{1}{2}$ they occur in nature, as zonal spherical functions on the homogeneous space G/K, where now $G = \mathrm{GL}_n(\mathbb{H})$ and $K = U(n, \mathbb{H})$, the quaternionic unitary group of $n \times n$ matrices.

8. Hall-Littlewood symmetric functions [M_1, ch. III]. — These symmetric functions arose originally in connection with the combinatorial and enumerative lattice properties of finite abelian p-groups (where p is a prime number). Let t be an indeterminate, let $F = \mathbb{Q}(t)$ and define a scalar product $\langle\ ,\ \rangle_{(t)}$ on Λ_F by

$$\langle p_\lambda, p_\mu \rangle_{(t)} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i})^{-1}. \tag{1.13}$$

Then the Hall-Littlewood symmetric functions $P_\lambda(x;t)$ are characterized by the two properties

(A) $P_\lambda = m_\lambda + \text{lower terms}$,
(B) $\langle P_\lambda, P_\mu \rangle_{(t)} = 0$ if $\lambda \neq \mu$.

When $t = 0$, the P_λ reduce to the Schur functions s_λ, and when $t = 1$ to the monomial symmetric functions m_λ.

Remarks.

1. — Let n be a positive integer, and arrange the partitions of n in lexicographical order, so that (1^n) comes first and (n) comes last. For example,

$$(1^4),\ (2,1^2),\ (2^2),\ (3,1),\ (4),$$

if $n=4$. This is a *total* ordering L_n of the set $\mathcal{P}_n$ of partitions of n, and correspondingly defines a totally ordered basis of Λ^n :

$$m_{(1^n)}(=e_n),\dots,m_\lambda,\dots,m_{(n)}(=p_n).$$

Now suppose we are given a (positive definite) scalar product $\langle\ ,\ \rangle$ on the space $\Lambda^n_{\mathbb{R}}$. Then by the Gram-Schmidt process we can derive a unique basis (u_λ) of $\Lambda^n_{\mathbb{R}}$ with the following two properties :

(A$'$) $u_\lambda = m_\lambda +$ a linear combination of the m_μ for partitions μ that precede λ in L_n ;

(B$'$) $\langle u_\lambda, u_\mu\rangle_\alpha = 0$ if $\lambda \neq \mu$.

If we replace L_n by some other total ordering of $\mathcal{P}_n$, and apply Gram-Schmidt as before, we should expect in general to end up with a different basis (u_λ). What in fact happens in each of the cases (5) — (8) is that, for the appropriate scalar product, the basis obtained is *independent* of the total ordering chosen, provided only that it is compatible with the partial ordering (1.1). (The lexicographical order L_n satisfies this condition : if $\mu < \lambda$ then μ precedes λ in L_n.) In other words, the conditions (A) and (B) (in each of the cases (5) — (8)) *overdetermine* the corresponding family of symmetric functions.

2. — Let V be a vector space (over some field F) and let $S = S(V)$ be the symmetric algebra of V. (If x_1, x_2, $\dots$ is a basis of V then $S = F[x_1, x_2, \dots]$, the polynomial algebra over F generated by the x_i.) Now suppose we are given a scalar product $\langle u, v\rangle$ on V, with values in F. This scalar product has a natural extension to S, defined by

$$\langle u_1 \dots u_m, v_1 \dots v_n\rangle = \begin{cases} 0, & \text{if } m \neq n,\\ \mathrm{per}(\langle u_i, v_j\rangle), & \text{if } m = n,\end{cases}$$

where $\mathrm{per}(\langle u_i, v_j\rangle)$ is the permanent of the matrix of scalar products $\langle u_i, v_j\rangle$ $(1 \le i, j \le n)$. Here the u's and v's are arbitrary elements of V.

In particular, let V be the vector subspace of Λ_F spanned by the power sums p_r $(r \ge 1)$, so that $S = F[p_1, p_2, \dots] = \Lambda_F$. Suppose that $\langle\ ,\ \rangle$ is a scalar product on V for which the p_r are mutually orthogonal, say

$$\langle p_r, p_s\rangle = \delta_{rs} a_r.$$

Then the natural extension of this scalar product to $S = \Lambda_F$ is such that

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda a_\lambda,$$

where $a_\lambda = a_{\lambda_1} a_{\lambda_2} \dots$ for any partition λ.

For the scalar products (5) — (8) above, the a_r are respectively 1, 2, α, and $(1-t^r)^{-1}$.

2. The symmetric functions $P_\lambda(q,t)$

Let q, t be independent indeterminates and let $F = \mathbb{Q}(q,t)$ be the field of rational functions in q and t. We shall now change the scalar product yet again, and define

$$\langle p_\lambda, p_\mu \rangle_{(q,t)} = \delta_{\lambda\mu} z_\lambda(q,t) \tag{2.1}$$

where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{2.1}$$

It is perhaps better to think of the parameters q and t as real variables lying in the interval $(0,1)$ of $\mathbb{R}$, so that the scalar product (2.1) is positive definite. The main result of this section is the following existence theorem.

(2.3) THEOREM. — *For each partition λ there is a unique symmetric function $P_\lambda = P_\lambda(q,t) \in \Lambda_F$ such that*

(A) $\quad P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu$

with coefficients $u_{\lambda\mu} \in F$;

(B) $\quad \langle P_\lambda, P_\mu \rangle_{(q,t)} = 0 \quad \text{if } \lambda \neq \mu.$

As remarked in the previous section, these two conditions overdetermine the P_λ, and their existence therefore requires proof. Before embarking on the proof, let us consider some particular cases.

(1) When $q = t$, the scalar product (2.1) reduces to the 'usual' scalar product (1.9), and hence $P_\lambda(q,q)$ is the Schur function s_λ.

(2) When $q = 0$, (2.1) reduces to (1.13), and hence $P_\lambda(0,t)$ is the Hall-Littlewood function $P_\lambda(t)$.

(3) Let $q = t^\alpha$ ($\alpha \in \mathbb{R},\ \alpha > 0$) and let $t \to 1$, so that $q \to 1$ also. Then

$$\frac{1-q^m}{1-t^m} = \frac{1-t^{\alpha m}}{1-t^m} \to \alpha,$$

as $t \to 1$, for all m. Hence the scalar product (2.1) tends to (1.11) as $t \to 1$ and hence

$$\lim_{t \to 1} P_\lambda(t^\alpha, t)$$

is the Jack symmetric function $P_\lambda(\alpha)$.

(4) When $t = 1$ (and q is arbitrary) we have $P_\lambda(q, 1) = m_\lambda$.

(5) When $q = 1$ (and t is arbitrary) we have $P_\lambda(1, t) = e_{\lambda'}$.

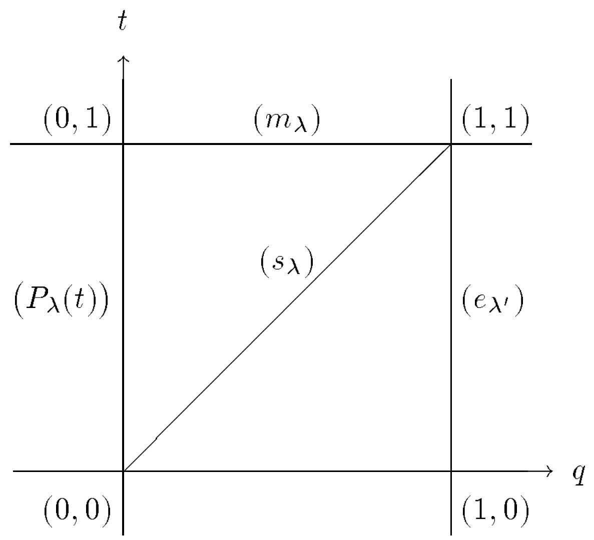
(6) Finally, it is clear from (2.2) that

$$z_\lambda(q^{-1}, t^{-1}) = (q^{-1}t)^n z_\lambda(q, t),$$

if λ is a partition of n. Hence on each Λ_F^n, the scalar products $\langle\ ,\ \rangle_{(q,t)}$ and $\langle\ ,\ \rangle_{q^{-1},t^{-1}}$ are proportional, and therefore

$$P_\lambda(q^{-1}, t^{-1}) = P_\lambda(q, t).$$

We may summarize these special cases in the following diagram, in which the point (q, t) represents the basis $(P_\lambda(q, t))$ of Λ_F (or of $\Lambda_\mathbb{R}$, since we are regarding q, t as real numbers).



At each point (q, q) on the diagonal of the square we have the Schur functions s_λ, at each point on the upper edge ($t = 1$) the monomial symmetric functions m_λ, and so on. In this scheme the Jack functions $P_\lambda(\alpha)$, for varying α, correspond to the points in the infinitesimal neighbourhood of the point $(1, 1)$, and more precisely $P_\lambda(\alpha)$ corresponds to the direction through $(1, 1)$ with slope $1/\alpha$. Notice that the bottom edge ($t = 0$) of the square remains unmarked; I do not know if the $P_\lambda(q, 0)$ have any reasonable interpretation (except that they can be derived from the Hall-Littlewood functions $P_\lambda(0, t)$ by duality (§ 3)).

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two sequences of independent indeterminates over $F = \mathbb{Q}(q,t)$, and define

$$\Pi = \Pi(x, y; q, t) = \prod_{i,j} \frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty}, \tag{2.4}$$

where as usual

$$(a; q)_\infty = \prod_{r=0}^{\infty} (1 - aq^r)$$

for any a for which the product on the right makes sense.

Then we have

$$\Pi(x, y; q, t) = \sum_\lambda z_\lambda(q,t)^{-1} p_\lambda(x) p_\lambda(y). \tag{2.5}$$

Proof. — We compute $\exp(\log \Pi)$; first of all,

$$\begin{aligned}
\log \Pi &= \sum_{i,j} \sum_{r=0}^{\infty} \left(\log(1 - x_iy_jq^r)^{-1} - \log(1 - tx_iy_jq^r)^{-1} \right) \\
&= \sum_{i,j} \sum_{r \geq 0} \sum_{n \geq 1} \frac{1}{n} (x_iy_jq^r)^n (1 - t^n) \\
&= \sum_{n \geq 1} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x) p_n(y)
\end{aligned}$$

and therefore

$$\begin{aligned}
\Pi &= \prod_{n \geq 1} \exp\left(\frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x) p_n(y) \right) \\
&= \prod_{n \geq 1} \sum_{m_n = 0}^{\infty} \frac{1}{m_n!} \left(\frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x) p_n(y) \right)^{m_n}
\end{aligned}$$

in which the coefficient of $p_\lambda(x) p_\lambda(y)$, where $\lambda = (1^{m_1} 2^{m_2} \dots)$ is seen to be $z_\lambda(q,t)^{-1}$. $\square$

(2.6) *For each integer $n \geq 0$ let (u_λ), (v_λ) be F-bases of Λ_F^n, indexed by the partitions λ of n. Then the following statements are equivalent :*

(a) $\langle u_\lambda, v_\mu \rangle_{(q,t)} = \delta_{\lambda\mu}$ *for all λ, μ (i.e., (u_λ), (v_λ) are dual bases of Λ_F^n for the scalar product* (2.1)*) ;*

(b) $\sum_\lambda u_\lambda(x) v_\lambda(y) = \Pi(x, y; q, t)$.

Proof. — Let $p_\lambda^* = z_\lambda(q,t)^{-1} p_\lambda$, so that

$$\langle p_\lambda^*, p_\mu \rangle_{(q,t)} = \delta_{\lambda\mu}.$$

Suppose that

$$u_\lambda = \sum_\rho a_{\lambda\rho} p_\rho^*, \qquad v_\lambda = \sum_\sigma a_{\mu\sigma} p_\sigma.$$

Then we have

$$\langle u_\lambda, v_\mu \rangle_{(q,t)} = \sum_\rho a_{\lambda\rho} b_{\mu\rho},$$

so that (a) is equivalent to

(a′) $$\sum_\rho a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu}.$$

On the other hand, by (2.5), (b) is equivalent to

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\rho p_\rho^*(x) p_\rho(y)$$

and hence to

(b′) $$\sum_\lambda a_{\lambda\rho} b_{\lambda\sigma} = \delta_{\rho\sigma}.$$

Since (a′) and (b′) are equivalent ((a′) says that $AB^t = 1$, where A is the matrix $(a_{\lambda\mu})$ and B the matrix $(b_{\lambda\mu})$, and (b′) says that $A^t B = 1$), it follows that (a) and (b) are equivalent. □

After these preliminaries we can embark on the proof of the existence theorem (2.3). The idea of the proof is as follows : we shall work initially with a finite set of variables $x = (x_1, \ldots, x_n)$ and construct an F-linear map (or operator)

$$D = D_{q,t} : \Lambda_{n,F} \to \Lambda_{n,F}$$

having the following properties :

(2.7.1) $$Dm_\lambda = \sum_{\mu \le \lambda} c_{\lambda\mu} m_\mu$$

for each partition λ of length $\le n$;

(2.7.2) $$\langle Df, g \rangle_{(q,t)} = \langle f, Dg \rangle_{(q,t)}$$

for all $f, g \in \Lambda_F$;

(2.7.3) $$\lambda \ne \mu \Rightarrow c_{\lambda\lambda} \ne c_{\mu\mu}.$$

These three properties say respectively that the matrix of D relative to the basis (m_λ) is triangular (2.7.1) ; that D is self-adjoint (2.7.2) ; and that the eigenvalues of D are distinct (2.7.3).

The P_λ are then just the eigenfunctions (or eigenvectors) of the operator D. Namely we have

(2.8) *For each partition* λ (*of length* $\leq n$) *there is a unique symmetric polynomial* $P_\lambda \in \Lambda_{n,F}$ *satisfying the two conditions*

$$\text{(A)} \quad P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu$$

where $u_{\lambda\mu} \in F$ *and* $u_{\lambda\lambda} = 1$ *;*

$$\text{(C)} \quad DP_\lambda = c_{\lambda\lambda} P_\lambda.$$

Proof. — From (A) and (2.7.1) we have

$$\begin{aligned} DP_\lambda &= \sum_{\mu \leq \lambda} u_{\lambda\mu} D m_\mu \\ &= \sum_{\nu \leq \mu \leq \lambda} u_{\lambda\mu} c_{\mu\nu} m_\nu \end{aligned}$$

and

$$c_{\lambda\lambda} P_\lambda = \sum_{\nu \leq \lambda} c_{\lambda\lambda} u_{\lambda\nu} m_\nu,$$

so that (A) and (C) are satisfied if and only if

$$c_{\lambda\lambda} u_{\lambda\nu} = \sum_{\nu \leq \mu \leq \lambda} u_{\lambda\mu} c_{\mu\nu},$$

that is to say, if and only if

$$(c_{\lambda\lambda} - c_{\nu\nu}) u_{\lambda\nu} = \sum_{\nu < \mu \leq \lambda} u_{\lambda\mu} c_{\mu\nu}$$

whenever $\nu < \lambda$. Since $c_{\lambda\lambda} \neq c_{\nu\nu}$ by (2.7.3), this relation determines $u_{\lambda\mu}$ uniquely in terms of the $u_{\lambda\mu}$ such that $\nu < \mu \leq \lambda$. Hence the coefficients $u_{\lambda\mu}$ in (A) are uniquely determined, given that $u_{\lambda\lambda} = 1$. □

With the P_λ as defined in (2.8) we have, by the self-adjointness of D,

$$\begin{aligned} c_{\lambda\lambda} \langle P_\lambda, P_\mu \rangle_{q,t} &= \langle DP_\lambda, P_\mu \rangle_{q,t} \\ &= \langle P_\lambda, DP_\mu \rangle_{q,t} = c_{\mu\mu} \langle P_\lambda, P_\mu \rangle_{q,t}. \end{aligned}$$

But $c_{\lambda\lambda} \neq c_{\mu\mu}$ if $\lambda \neq \mu$, hence $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$.

This establishes (2.3) when the number of variables is finite (i.e. in $\Lambda_{n,F}$ rather Λ_F). As explained in § 1, we may then compute the coefficients $u_{\lambda\mu}$ in (A) by Gram-Schmidt ; they will involve only the scalar products $\langle m_\mu, m_\nu \rangle_{q,t}$, which are independent of n. Hence the P_λ are well-defined as elements of Λ_F.

The proof of (2.3) therefore reduces to the construction of an operator D satisfying (2.7.1) — (2.7.3). Let

$$\begin{aligned}\Delta &= \prod_{1\le i<j\le n}(x_i-x_j)\\ &= \sum_{w\in\mathfrak{S}_n}\epsilon(w)x^{w\delta}\end{aligned} \tag{2.9}$$

be the Vandermonde determinant in $x_1, \ldots, x_n$, where $\epsilon(w)$ is the sign of the permutation w, and $\delta=(n-1,n-2,\ldots,1,0)$. Next, for any polynomial $f(x_1,\ldots,x_n)$, symmetric or not, we define

$$\begin{aligned}(T_{q,x_i}f)(x_1,\ldots,x_n) &= f(x_1,\ldots,qx_i,\ldots,x_n),\\ (T_{t,x_i}f)(x_1,\ldots,x_n) &= f(x_1,\ldots,tx_i,\ldots,x_n)\end{aligned}$$

for $1\le i\le n$. Then D is defined as follows :

$$\begin{aligned}D &= \Delta^{-1}\sum_{i=1}^{n}(T_{t,x_i}\Delta)T_{q,x_i}\\ &= \sum_{i=1}^{n}\left(\prod_{j\ne i}\frac{tx_i-x_j}{x_i-x_j}\right)T_{q,x_i}.\end{aligned} \tag{2.10}$$

A more useful expression for D is, from (2.9) above,

$$D=\Delta^{-1}\sum_{w\in\mathfrak{S}_n}\epsilon(w)\sum_{i=1}^{n}t^{(w\delta)_i}x^{w\delta}T_{q,x_i}. \tag{2.10'}$$

We must now verify that D satisfies (2.7.1) — (2.7.3). Let λ be a partition of length $\le n$, and let $\mathfrak{S}_n^\lambda$ be the subgroup of $\mathfrak{S}_n$ that fixes λ, so that

$$m_\lambda=\left|\mathfrak{S}_n^\lambda\right|^{-1}\sum_{w_1\in\mathfrak{S}_n}x^{w_1\lambda}.$$

From (2.10′) we have

$$\left|\mathfrak{S}_n^\lambda\right|Dm_\lambda=\Delta^{-1}\sum_{w,w_1}\epsilon(w)\sum_{i=1}^{n}t^{(w\delta)_i}q^{(w_1\lambda)_i}x^{w\delta+w_1\lambda}.$$

In this sum w and w_1 run independently through $\mathfrak{S}_n$. Put $w_1=ww_2$, and we obtain :

$$\begin{aligned}\left|\mathfrak{S}_n^\lambda\right|Dm_\lambda &= \Delta^{-1}\sum_{w,w_2}\epsilon(w)\left(\sum_{i=1}^{n}q^{(w_2\lambda)_i}t^{\delta_i}\right)x^{w(w_2\lambda+\delta)}\\ &= \sum_{w_2\in\mathfrak{S}_n}\left(\sum_{i=1}^{n}q^{(w_2\lambda)_i}t^{n-i}\right)s_{w_2\lambda}.\end{aligned}$$

Hence if we define

$$u(\mu) = \sum_{i=1}^{n} q^{\mu_i} t^{n-i}$$

for any $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$, we have

$$Dm_\lambda = \sum_{\mu} u(\mu) s_\mu$$

summed over all distinct derangements μ of $\lambda = (\lambda_1, \ldots, \lambda_n)$. In this sum μ is not a partition (unless $\mu = \lambda$), but s_μ is defined for all $\mu \in \mathbb{N}^n$ and is either zero or equal to $\pm s_\nu$ for some partition $\nu < \lambda$. Hence

$$Dm_\lambda = u(\lambda) s_\lambda + \cdots$$

where the terms not written are a linear combination of the Schur functions s_ν such that $\nu < \lambda$, and therefore finally

$$Dm_\lambda = \sum_{\mu \leq \lambda} c_{\lambda\mu} m_\mu$$

with

$$c_{\lambda\lambda} = u(\lambda) = \sum_{i=1}^{n} q^{\lambda_i} t^{n-i}. \tag{2.11}$$

This establishes (2.7.1) and also (2.7.3), since the eigenvalues $c_{\lambda\lambda}$ of D given by (2.11) are visibly all distinct.

It remains to show that D is self-adjoint (2.7.2). The proof is in several stages :

(2.12) *D is self-adjoint when $q = t$.*

Proof. — We have, when $q = t$,

$$D = \Delta^{-1} \sum_{i=1}^{n} (T_{t,x_i} \Delta) T_{t,x_i}$$

so that

$$Ds_\lambda = \Delta^{-1} \sum_{i=1}^{n} T_{t,x_i} (\Delta s_\lambda).$$

Since

$$\Delta s_\lambda = \sum_{w \in \mathfrak{S}_n} \epsilon(w) x^{w(\lambda+\delta)}$$

it follows that

$$Ds_\lambda = \left(\sum_{i=1}^{n} t^{\lambda_i + n - i} \right) s_\lambda$$

and hence that $\langle Ds_\lambda, s_\mu \rangle = 0$ if $\lambda \neq \mu$. So

$$\langle Ds_\lambda, s_\mu \rangle = \langle s_\lambda, Ds_\mu \rangle$$

for all λ, μ, and therefore D is self-adjont. $\square$

(2.13) *$D = D_{q,t}$ is self-adjoint if and only if*

$$D_x \Pi = D_y \Pi,$$

where $\Pi = \Pi(x, y; q, t)$ (2.4) and D_x (resp. D_y) means D operating on symmetric functions in the x (resp. y) variables.

Proof. — Let (u_λ), (v_λ) be dual bases of $\Lambda_{n,F}$ as in (2.6), and let

$$a_{\lambda\mu} = \langle Du_\lambda, u_\mu \rangle_{q,t}. \tag{2.14}$$

Clearly, D is self-adjoint if and only if $a_{\lambda\mu} = a_{\mu\lambda}$ for all λ, μ.

From (2.14) we have

$$Du_\lambda = \sum_\mu a_{\lambda\mu} v_\mu$$

and therefore, since $\Pi = \sum u_\lambda(x) v_\lambda(y)$,

$$D_x \Pi = \sum_{\lambda,\mu} a_{\lambda\mu} v_\mu(x) v_\lambda(y).$$

Likewise

$$D_y \Pi = \sum_{\lambda,\mu} a_{\lambda\mu} v_\mu(y) v_\lambda(x)$$

and therefore $D_x \Pi = D_y \Pi$ if and only if $a_{\lambda\mu} = a_{\mu\lambda}$ for all λ, μ. ∎

From the definition (2.4) of Π we have

$$\Pi^{-1} T_{q,x_i} \Pi = \prod_{j=1}^n \frac{1 - x_i y_j}{1 - t x_i y_j}$$

which is independent of q. Hence $\Pi^{-1} D_x \Pi$ is independent of q. Hence we may assume that $q = t$. But then by (2.13) and (2.14) we have $\Pi^{-1} D_x \Pi = \Pi^{-1} D_y \Pi$, and so by (2.14) again $D_{q,t}$ is self-adjoint for all q, t. This completes the proof of (2.3). ∎

We shall see in the subsequent sections that the formal properties of the symmetric functions $P_\lambda(x; q, t)$ generalize to a remarkable extent familiar properties of Schur functions. What is lacking (at any rate at present) is any sort of usable "closed formula" for P_λ. This has the effect that the proofs we shall give are usually indirect and quite complicated in detail, compared to the usual proofs of the corresponding properties of Schur functions.

3. Duality

It is well known (see, e.g., [M_1], ch. I, (3.8)) that the involution $\omega : \Lambda \to \Lambda$ of § 1 permutes the Schur functions, namely that

$$\omega(s_\lambda) = s_{\lambda'}.$$

The duality theorem to be stated below generalizes this fact. Let

$$b_\lambda = b_\lambda(q,t) = \langle P_\lambda, P_\mu \rangle_{q,t}^{-1} \in F. \tag{3.1}$$

(We shall later (§ 5) obtain an explicit formula for $b_\lambda(q,t)$ in terms of λ, q and t.) Now define

$$Q_\lambda = b_\lambda P_\lambda \tag{3.2}$$

so that

$$\langle P_\lambda, P_\mu \rangle_{q,t} = \delta_{\lambda\mu},$$

i.e., (P_λ), (Q_λ) are dual bases of Λ_F for the scalar product $\langle\ ,\ \rangle_{q,t}$.

We also define an automorphism

$$\omega_{q,t} : \Lambda_F \to \Lambda_F$$

by

$$\omega_{q,t}(p_r) = (-1)^{r-1} \frac{1-q^r}{1-t^r} p_r. \tag{3.3}$$

Clearly $\omega_{q,t}^{-1} = \omega_{t,q}$, and $\omega_{t,t} = \omega$. Also we have

$$\langle \omega_{q,t} f, g \rangle_{t,q} = \langle \omega f, g \rangle \tag{3.4}$$

for all $f, g \in \Lambda_F$, where the scalar product on the right is that defined by (1.9).

It is enough to check (3.4) when $f = p_\lambda$ and $g = p_\mu$, and then it is immediate from the definitions.

We can now state the duality theorem :

(3.5) Theorem. — *For all partitions λ we have*

$$\omega_{q,t} P_\lambda(q,t) = Q_{\lambda'}(t,q)$$

or equivalently

$$\omega_{q,t} Q_\lambda(q,t) = P_{\lambda'}(t,q).$$

(The equivalence of these two statements follows from the fact that $\omega_{q,t}^{-1} = \omega_{t,q}$.)

For the proof of (3.5) we require the following lemma, whose proof we leave as an exercise.

(3.6) *Let λ be a partition, thought of as an infinite sequence, and let*

$$f_\lambda(q,t) = (1-t)\sum_{i=1}^{\infty} q^{\lambda_i} t^{i-1}.$$

Then $f_\lambda(q,t) = f_{\lambda'}(t,q)$.

Let $D = D_{q,t}$ be the operator defined in § 2, acting on symmetric polynomials in n variables $x_1, \ldots, x_n$. We need to modify it slightly : we define

$$E = E_{q,t} = t^{-n}\big(1 + (t-1)D_{q,t}\big). \tag{3.7}$$

From (2.11) we have, for any partition λ of length $\leq n$,

$$\begin{aligned} EP_\lambda(q,t) &= t^{-n}\big(1 + (t-1)\sum_{i=1}^{n} q^{\lambda_i} t^{n-i}\big)P_\lambda(q,t) \\ &= f_\lambda(q,t^{-1})P_\lambda(q,t). \end{aligned} \tag{3.8}$$

Next we have

$$\omega_{q,t} E_{q,t} \omega_{q,t}^{-1} = E_{t^{-1},q^{-1}}. \tag{3.9}$$

The only proof I have of (3.9) at present is rather messy, and I shall not give it here. Assuming (3.9), the proof of (3.5) proceeds as follows : we have

$$\begin{aligned} E_{t^{-1},q^{-1}}\omega_{q,t}P_\lambda(q,t) &= \omega_{q,t}E_{q,t}P_\lambda(q,t) \\ &= f_\lambda(q,t^{-1})\omega_{q,t}P_\lambda(q,t) \\ &= f_{\lambda'}(t^{-1},q)\omega_{q,t}P_\lambda(q,t) \end{aligned}$$

by (3.9), (3.8) and (3.6). Hence $\omega_{q,t}P_\lambda(q,t)$ is an eigenfunction of $E_{t^{-1},q^{-1}}$ with eigenvalue $f_{\lambda'}(t^{-1},q)$. Hence it must be a scalar multiple of $P_{\lambda'}(t^{-1},q^{-1}) = P_{\lambda'}(t,q)$. We want to show that it is actually $Q_{\lambda'}(t,q)$, so we must show that

$$\langle \omega_{q,t}P_\lambda(q,t), P_{\lambda'}(t,q)\rangle_{t,q} = 1.$$

By (3.4) this is equivalent to showing that

$$\langle \omega P_\lambda(q,t), P_{\lambda'}(t,q)\rangle = 1 \tag{3.10}$$

for the "usual" scalar product (1.9).

To prove (3.10), we shall express P_λ and $P_{\lambda'}$ as linear combinations of Schur functions : say

$$P_\lambda(q,t) = s_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} s_\mu,$$
$$P_{\lambda'}(t,q) = s_{\lambda'} + \sum_{\nu'<\lambda'} b_{\lambda\nu} s_{\nu'}.$$

Since $\omega s_\mu = s_{\mu'}$ it follows that

$$\omega P_\lambda(q,t) = s_{\lambda'} + \sum_{\mu'>\lambda'} a_{\lambda\mu} s_{\mu'}$$

and therefore $P_{\lambda'}(t,q)$ and $\omega P_\lambda(q,t)$ have only $s_{\lambda'}$ in common, so that

$$\langle \omega P_\lambda(q,t), P_{\lambda'}(t,q)\rangle = \langle s_{\lambda'}, s_{\lambda'}\rangle = 1$$

as required. This completes the proof of (3.5). $\square$

Since (P_λ), (Q_λ) are dual bases of Λ_F, we have from (2.6)

(3.11) $$\sum_\lambda P_\lambda(x;q,t) Q_\lambda(y;q,t) = \Pi(x,y;q,t).$$

Apply $\omega_{q,t}$ to the y-variables; from (2.5) it is easily computed that

$$\omega_{q,t}\Pi(x,y;q,t) = \prod_{i,j}(1+x_iy_j).$$

Hence it follows from (3.5) that

(3.12) $$\sum_\lambda P_\lambda(x;q,t) P_{\lambda'}(y;t,q) = \prod_{i,j}(1+x_iy_j).$$

When $q=t$, (3.11) and (3.12) reduce to the familiar identities

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j}(1-x_iy_j)^{-1}$$

and

$$\sum_\lambda s_\lambda(x) s_{\lambda'}(y) = \prod_{i,j}(1+x_iy_j)$$

respectively.

Finally, we can now identify the symmetric functions $P_\lambda(q,t)$ when either $q=1$ or $t=1$. Suppose first that $t=1$. Then from (2.10) we have

$$D_{q,1}=\sum_{i=1}^{n} T_{q,x_i}$$

so that (for any partition λ of length $\leq n$)

$$D_{q,1}m_\lambda=\left(\sum_{i=1}^{n} q^{\lambda_i}\right)m_\lambda.$$

Hence the m_λ are the eigenfunctions of the operator $D_{q,1}$, and therefore

$$P_\lambda(q,1)=m_\lambda \tag{3.13}$$

for all partitions λ.

Next, it follows from (3.5) that

$$\langle \omega_{q,t}P_\lambda(q,t),P_{\mu'}(t,q)\rangle_{t,q}=\delta_{\lambda\mu}$$

and hence by (3.4) that

$$\langle P_\lambda(q,t),\omega P_{\mu'}(t,q)\rangle=\delta_{\lambda\mu}.$$

By (3.13) this gives

$$\langle m_\lambda,\omega P_{\mu'}(1,q)\rangle=\delta_{\lambda\mu}$$

when $t=1$. But the basis dual to (m_λ) for the usual scalar product is (h_λ) ([M$_1$], ch. I). Hence $\omega P_{\mu'}(1,q)=h_\mu$, i.e., $P_{\mu'}(1,q)=\omega h_\mu=e_\mu$. Hence (replacing q, μ' by t, λ)

$$P_\lambda(1,t)=e_{\lambda'}. \tag{3.14}$$

4. Skew P and Q functions

Let μ and ν be two partitions. Then the product $P_\mu P_\nu$ is a linear combination of the P_λ, say

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu} P_\mu \tag{4.1}$$

where

$$f^\lambda_{\mu\nu} = f^\lambda_{\mu\nu}(q,t) = \langle Q_\lambda, P_\mu P_\nu\rangle_{q,t} \in F.$$

In particular

(1) $f^\lambda_{\mu\nu}(t,t)$ is the coefficient $c^\lambda_{\mu\nu}$ of s_λ in $s_\mu s_\nu$, which may be calculated by the Littlewood-Richardson rule.

(2) $f^\lambda_{\mu\nu}(0,t)$ is the *Hall polynomial* $f^\lambda_{\mu\nu}(t)$ ([M$_1$], ch. II). It may be written as a sum

$$f^\lambda_{\mu\nu}(t) = \sum_T f_T(t)$$

of monic polynomials, where T runs through the set of LR-tableaux of shape $\lambda' - \mu'$ and weight ν' (*loc. cit.*)

(3) $f^\lambda_{\mu\nu}(q,1)$ is the coefficient of m_λ in $m_\mu m_\nu$, and so is independent of q.

(4) $f^\lambda_{\mu\nu}(1,t) = 1$ if $\lambda = \mu + \nu$, and is zero otherwise. For $P_\mu(1,t) = e_{\mu'}$, and $e_{\mu'}e_{\nu'} = e_{\mu'\cup\nu'} = e_{(\mu+\nu)'}$.

(5) $f^\lambda_{\mu\nu}(q^{-1},t^{-1}) = f^\lambda_{\mu\nu}(q,t)$, since $P_\lambda(q^{-1},t^{-1}) = P_\lambda(q,t)$ (§ 2).

(6) By duality (3.5) we have

$$f^\lambda_{\mu\nu}(q,t) = f^{\lambda'}_{\mu'\nu'}(t,q) b_{\mu'}(t,q) b_{\nu'}(t,q)/b_{\lambda'}(t,q).$$

In view of (1) and (2) above, it is natural to ask whether it is possible to attach to each LR-tableau T a non zero rational function $f_T(q,t)$ so that

$$f^\lambda_{\mu\nu}(q,t) = \sum_T f_T(q,t),$$

where T runs through the set of LR-tableaux of shape $\lambda - \mu$ and weight ν. (If so, then by duality ((6) above) there will be likewise a decomposition $f^\lambda_{\mu\nu}$ over the LR-tableaux of shape $\lambda' - \mu'$ and weight ν'.) Again, is it true that $f^\lambda_{\mu\nu}(q,t) \neq 0$ if and only if $c^\lambda_{\mu\nu} \neq 0$? The answers to these questions are not known, at any rate to the author.

Clearly we shall have $f^\lambda_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$. In fact, more is true :

(4.2) $f^{\lambda}_{\mu\nu} = 0$ *unless* $\lambda \supset \mu$ *and* $\lambda \supset \nu$ *(i.e., the diagram of* λ *contains those of* μ *and* ν*).*

Stanley [S] gives a proof of (4.2) in the context of Jack's symmetric functions. His proof can be transposed to the present context without difficulty.

We now define skew Q-functions as follows. If λ, μ are partitions, define

$$Q_{\lambda/\mu} = \sum_{\nu} f^{\lambda}_{\mu\nu} Q_{\nu} \tag{4.3}$$

so that

$$\langle Q_{\lambda/\mu}, P_{\nu} \rangle_{q,t} = \langle Q_{\lambda}, P_{\mu} P_{\nu} \rangle_{q,t}.$$

From (4.2) if follows that $Q_{\lambda/\mu} = 0$ unless $\lambda \supset \mu$.

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two sequences of independent indeterminates. If f is any symmetric function, $f(x, y)$ shall mean $f(x_1, y_1, x_2, y_2, \dots)$, and likewise we define $f(x, y, z, \dots)$ for three or more sequences $x, y, z, \dots$ Then we have

$$Q_{\lambda/\mu}(x, y) = \sum_{\nu} Q_{\lambda/\nu}(x) Q_{\nu/\mu}(y) \tag{4.4}$$

summed over partitions ν *such that* $\lambda \supset \nu \supset \mu$.

Proof. — We have

$$\begin{aligned}\sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda}(y) &= \sum_{\lambda,\nu} f^{\lambda}_{\mu\nu} Q_{\nu}(x) P_{\lambda}(y) \\ &= \sum_{\nu} Q_{\nu}(x) P_{\mu}(y) P_{\nu}(y) \\ &= P_{\mu}(y) \Pi(x, y)\end{aligned}$$

by (3.11). Hence

$$\begin{aligned}\sum_{\lambda,\mu} Q_{\lambda/\mu}(x) P_{\lambda}(y) Q_{\mu}(z) &= \Pi(x, y) \Pi(z, y) \\ &= \sum_{\lambda} P_{\lambda}(y) Q_{\lambda}(x, z)\end{aligned}$$

by (3.11) again. By comparing the coefficients of $P_{\lambda}(y)$ on either side, we obtain

$$Q_{\lambda}(x, z) = \sum_{\mu} Q_{\lambda/\mu}(x) Q_{\mu}(z). \tag{4.5}$$

If we now replace x by x, y in (4.5), we have

$$\begin{aligned}\sum_\mu Q_{\lambda/\mu}(x,y)Q_\mu(z) &= Q_\lambda(x,y,z)\\ &= \sum_\nu Q_{\lambda/\nu}(x)Q_\nu(y,z)\\ &= \sum_{\mu,\nu} Q_{\lambda/\nu}(x)Q_{\nu/\mu}(y)Q_\mu(z)\end{aligned}$$

by two applications of (4.5). If we now equate the coefficients of $Q_\mu(s)$ at either end of this string of equalities, we shall obtain (4.4). ∎

The identity (4.4) clearly generalizes to n sets of variables $x^{(1)}, \ldots, x^{(n)}$: if λ, μ are partitions, then

(4.6) $$Q_{\lambda/\mu}(x^{(1)}, \ldots, x^{(n)}) = \sum_{(\nu)} \prod_{i=1}^{n} Q_{\nu^i/\nu^{i-1}}(x^{(i)})$$

summed over all sequences $(\nu) = (\nu^0, \nu^1, \ldots, \nu^n)$ *of partitions such that* $\mu = \nu^0 \subset \nu^1 \subset \cdots \subset \nu^n = \lambda$.

Let us apply (4.6) in the case where each set of variables $x^{(i)}$ consists of a single element x_i. For a single x, we have

(4.7) $$Q_{\lambda/\mu}(x) = \varphi_{\lambda/\mu}\, x^{|\lambda-\mu|}$$

say, where $\varphi_{\lambda/\mu} = \varphi_{\lambda/\mu}(q,t) \in F$, and in fact

(4.8) $\varphi_{\lambda/\mu} = 0$ *unless* $\lambda \supset \mu$ *and* $\lambda - \mu$ *is a horizontal strip, i.e., unless the partitions* λ, μ *are interlaced :*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$$

Again we refer to [S] for a proof of this. As in the case of (4.2), STANLEY's proof (for Jack's functions) can be transposed without difficulty (and indeed (4.2) is a a consequence of (4.8)).

From (4.6) and (4.7) we obtain

(4.9) $$Q_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{(\nu)} \prod_{i=1}^{n} \varphi_{\nu^i/\nu^{i-1}}\, x_i^{|\nu^i-\nu^{i-1}|}$$

with (ν) as in (4.6) and each skew diagram $\nu^i - \nu^{i-1}$ a horizontal strip. The sequence (ν) of partitions determines a (column strict) *tableau* T of shape $\lambda - \mu$, in which the symbol i occurs in each square of $\nu^i - \nu^{i-1}$, for $1 \le i \le n$. If we now define

$$\varphi_T = \prod_{i \ge 1} \varphi_{\nu^i/\nu^{i-1}} \tag{4.10}$$

then we shall have

$$Q_{\lambda/\mu} = \sum_T \varphi_T \, x^T \tag{4.11}$$

summed over all tableaux T of shape $\lambda - \mu$, where x^T is the monomial determined by the tableau T, i.e., $x^T = x^\alpha$, where α is the weight of T.

Later (§ 5) we shall derive an explicit formula for $\varphi_{\lambda/\mu}$ and hence also for φ_T, and then (4.11) will provide an explicit (if complicated) expression for $Q_{\lambda/\mu}$ as a sum of monomials.

Finally, one can define skew P-functions $P_{\lambda/\mu}$ by interchanging the roles of the P's and the Q's throughout. The relation between the two is

$$P_{\lambda/\mu} = b_\lambda^{-1} b_\mu Q_{\lambda/\mu} \tag{4.12}$$

with b_λ as defined by (3.1).

5. Explicit formulas

We have introduced various scalars b_λ, $\varphi_{\lambda/\mu}$, φ_T in the preceding sections, but so far we have no way of computing them explicitly. The key to this is a specialization theorem, which will be stated in a moment.

Let u be a new indeterminate, and define a homomorphism (or specialization)

$$\epsilon_{u,t} : \Lambda_F \to F[u]$$

by

$$\epsilon_{u,t}(p_r) = \frac{1-u^r}{1-t^r} \tag{5.1}$$

for each $r \ge 1$.

To motivate this definition, suppose that $u = t^n$, n a positive integer. Then

$$\begin{aligned} \epsilon_{t^n,t}(p_r) &= \frac{1-t^{nr}}{1-t^r} \\ &= 1 + t^r + \cdots + t^{(n-1)r} \\ &= p_r(1, t, \ldots, t^{n-1}) \end{aligned}$$

so that

$$\epsilon_{t^n,t}(f) = f(1, t, \ldots, t^{n-1})$$

for any symmetric function f : i.e., the effect of $\epsilon_{t^n,t}$ is to evaluate at $(x_1, \ldots, x_n, x_{n+1}, \ldots) = (1, t, \ldots, t^{n-1}, 0, 0, \ldots)$.

There is a nice formula for $\epsilon_{u,t}(P_\lambda(q,t))$. In order to state it in a convenient form, let us introduce the following notation. For each square $s = (i,j)$ in the diagram of a partition λ, let

$$\begin{cases} a(s) = \lambda_i - j, & a'(s) = j - 1, \\ l(s) = \lambda'_j - i, & l'(s) = i - 1, \end{cases} \tag{5.2}$$

so that $l'(s)$, $l(s)$, $a(s)$ and $a'(s)$ are respectively the numbers of squares in the diagram of λ to the north, south, east and west of the square s. The numbers $a(s)$ and $a'(s)$ may be called respectively the *arm-length* and the *arm-colength* of s, and $l(s)$, $l'(s)$ the *leg-length* and *leg-colength*. The *hook-length* at s is $a(s) + l(s) + 1$.

(5.3) THEOREM. — *We have*

$$\epsilon_{u,t}\big(P_\lambda(q,t)\big) = \prod_{s \in \lambda} \frac{q^{a'(s)}u - t^{l'(s)}}{q^{a(s)}t^{l(s)+1} - 1}.$$

Proof. — Since by (5.1) $\epsilon_{u,t}(p_r)$ is a polynomial of degree r in u with coefficients in F, it follows that $\epsilon_{u,t}(P_\lambda)$ is a polynomial of degree $\leq |\lambda|$, say

$$\epsilon_{u,t}\big(P_\lambda(q,t)\big) = \Phi_\lambda(u; q, t) \in F[u].$$

The idea of the proof is first to locate the zeros of this polynomial, which will give the numerator of the expression in (5.3). For this we require two relations. The first comes from the formula (4.5) (with Q's replaced by P's), which gives

$$P_\lambda(x_1, \ldots, x_n) = \sum_\mu P_{\lambda/\mu}(x_1) P_\mu(x_2, \ldots, x_n). \tag{5.4}$$

Set $(x_1, \ldots, x_n) = (1, t, \ldots, t^{n-1})$ and let

$$\psi_{\lambda/\mu} = P_{\lambda/\mu}(1; q, t). \tag{5.5}$$

Then (5.4) becomes

$$\Phi_\lambda(t^n; q, t) = \sum_\mu \psi_{\lambda/\mu}(q,t) t^{|\mu|} \Phi_\mu(t^{n-1}; q, t)$$

for all $n \geq 1$, and hence we have

$$\text{(A)} \qquad \Phi_\lambda(u;q,t) = \sum_\mu \psi_{\lambda/\mu}(q,t)t^{|\mu|}\Phi_\mu(ut^{-1};q,t).$$

Next we have

$$\text{(5.6)} \qquad \epsilon_{u,t}\omega_{t,q}(f) = (-q)^{-n}\epsilon_{u,q^{-1}}(f)$$

if $f \in \Lambda_F^n$. (Since both ϵ and ω are ring homomorphisms, it is enough to check (5.6) when $f = p_n$.) Hence by duality (3.5) we have

$$\begin{aligned}\epsilon_{u,t}P_\lambda(q,t) &= \epsilon_{u,t}\omega_{t,q}Q_{\lambda'}(t,q)\\ &= (-q)^{-|\lambda|}\epsilon_{u,q^{-1}}Q_\lambda(t,q)\\ &= (-q)^{-|\lambda|}b_{\lambda'}(t,q)\epsilon_{u,q^{-1}}P_{\lambda'}(t^{-1},q^{-1})\end{aligned}$$

(since $P_{\lambda'}(t,q) = P_{\lambda'}(t^{-1},q^{-1})$) and therefore

$$\text{(B)} \qquad \Phi_\lambda(u;q,t) = (-q)^{-|\lambda|}b_{\lambda'}(t,q)\Phi_{\lambda'}(u;t^{-1},q^{-1}).$$

We observe next that

(5.7) $P_\lambda(x_1,\dots,x_n) = 0$ *if* $n < \ell(\lambda)$.

For P_λ is a linear combination of the m_μ such that $\mu \leq \lambda$, and

$$\begin{aligned}\mu \leq \lambda \Rightarrow \mu' \geq \lambda' \Rightarrow \ell(\mu) = \mu_1' \geq \lambda_1' = \ell(\lambda) > n\\ \Rightarrow \ell(\mu) > n \Rightarrow m_\mu(x_1,\dots,x_n) = 0. \quad \square\end{aligned}$$

It follows from (5.7) that

$$\Phi_\lambda(u;q,t) = 0 \quad \text{for } u = 1,t,\dots,t^{\ell(\lambda)-1}.$$

Hence by (B) the polynomial $\Phi_{\lambda'}(u;t^{-1},q^{-1})$ vanishes also for these values of u. By replacing (λ,q,t) by (λ',t^{-1},q^{-1}), we see that

$$\Phi_\lambda(u;q,t) = 0 \quad \text{for } u = 1,q^{-1},\dots,q^{1-\lambda_1}$$

and therefore

$$\text{(C)} \qquad \Phi_\lambda(u;q,t) \textit{ is divisible in } F[u] \textit{ by } \prod_{j=1}^{\lambda_1}(q^{j-1}u-1).$$

Now we consider the relation (A) above. By (C), each term on the right of (A) is divisible by

$$\prod_{j=1}^{\lambda_2}(q^{j-1}u-t)$$

(because the sum in (A) is over partitions μ that interlace λ). Hence $\Phi_\lambda(u;q,t)$ is divisible by this product. We now repeat the argument : each term on the right-hand side of (A) is divisible by

$$\prod_{j=1}^{\lambda_3}(q^{j-1}u-t^2)$$

(since $\mu_2 \geq \lambda_3$) and therefore $\Phi_\lambda(u;q,t)$ is also divisible by this product. Thus finally $\Phi_\lambda(u;q,t)$ is divisible in $F[u]$ by

$$\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}(q^{j-1}u-t^{i-1})=\prod_{s\in\lambda}(q^{a'(s)}u-t^{l'(s)}).$$

(Observe that all these linear factors are distinct.) But we know that Φ_λ has degree at most $|\lambda|$ in u. Hence we have

(D) $$\Phi_\lambda(u;q,t)=v_\lambda(q,t)\prod_{s\in\lambda}(q^{a'(s)}u-t^{l'(s)}),$$

and it remains to identify the scalar factor $v_\lambda(q,t)$. For this purpose we require

(5.8) *Let λ be a partition of length n, and let $\lambda=(\lambda_1-1,\ldots,\lambda_n-1)$. Then*

$$P_\lambda(x_1,\ldots,x_n)=x_1\ldots x_nP_\mu(x_1,\ldots,x_n).$$

(Compute $D_{q,t}(x_1\ldots x_nP_\mu)$ and show that $x_1\ldots x_nP_\mu$ is an eigenfunction of $D_{q,t}$ with eigenvalue $\sum q^{\lambda_i}t^{n-i}$.)

From (5.8) it follows that

$$\Phi_\lambda(t^n;q,t)=t^{n(n-1)/2}\Phi_\mu(t^n;q,t)$$

if $\ell(\lambda)=n$, and hence from (D) that

$$\begin{aligned}\frac{v_\lambda(q,t)}{v_\mu(q,t)}&=t^{n(n-1)/2}\prod_{s\in\lambda-\mu}\left(q^{a'(s)}t^n-t^{l'(s)}\right)^{-1}\\&=\prod_{i=1}^{n}\left(q^{\lambda_i-1}t^{n-i+1}-1\right)^{-1}.\end{aligned}$$

Now the factors in this product are precisely $\left(q^{a(s)}t^{l(s)+1}-1\right)^{-1}$ for s in the first column of λ. Hence by induction on the number of columns of λ we conclude that

$$v_\lambda(q,t)=\prod_{s\in\lambda}\left(q^{a(s)}t^{l(s)+1}-1\right)^{-1}$$

and the proof of (5.3) is complete. $\square$

From (B) we have

$$b_{\lambda'}(t,q) = (-q)^{|\lambda|} \frac{\Phi_\lambda(u;q,t)}{\Phi_{\lambda'}(u;t^{-1},q^{-1})}$$

which together with (5.3) leads to the formula

$$b_\lambda(q,t) = \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}.$$

So we know now the value of $\langle P_\lambda, P_\mu\rangle_{q,t} = b_\lambda(q,t)^{-1}$.

Knowing the b_λ, it turns out that it is now quite straightforward to calculate the scalars

$$\varphi_{\lambda/\mu} = Q_{\lambda/\mu}(1)$$

defined in § 4. I will omit the details and merely state the result. For each square s and each partition λ, define

$$b_\lambda(s) = b_\lambda(s;q,t) = \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} \tag{5.10}$$

if $s\in\lambda$, and $b_\lambda(s)=1$ if $s\notin\lambda$.

Next, if S is any set of squares (contained in the diagram of λ or not), let

$$b_\lambda(S) = \prod_{s\in S} b_\lambda(s). \tag{5.11}$$

Now let λ, μ be partitions such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$ or equivalently such that $\lambda \supset \mu$ and $\lambda-\mu$ is a horizontal strip. Let $C_{\lambda/\mu}$ denote the union of the columns that contain squares of $\lambda-\mu$. Then the formula for $\varphi_{\lambda/\mu}$ is

$$\varphi_{\lambda/\mu} = b_\lambda(C_{\lambda/\mu})/b_\mu(C_{\lambda/\mu}) \tag{5.12}$$

and for a tableau T

$$\varphi_T = \prod_{i\geq 0} b_{\lambda^i}(C_i)/b_{\lambda^i}(C_{i+1}), \tag{5.13}$$

where $\lambda^0 \subset \lambda^1 \subset \cdots$ is the sequence of partitions defined by the tableau, and C_i is the union of the columns that contain a symbol i (so that C_0 is empty).

In the case of Jack's symmetric functions, the results of this section are all due to Richard Stanley [S], and the transposition of his proofs into the present context is quite straightforward. Also when $q = 0$ the formulas (5.9), (5.12) and (5.13) reduce respectively to [M$_1$], ch. III, (2.12), (5.8) and (5.9).

Finally we may remark that the identities of Schur and Littlewood

$$\sum_\lambda s_\lambda = \prod (1 - x_i)^{-1} \prod_{i<j} (1 - x_i x_j)^{-1},$$
$$\sum_\mu s_\mu = \prod_{i \le j} (1 - x_i x_j)^{-1},$$
$$\sum_\nu s_\nu = \prod_{i<j} (1 - x_i x_j)^{-1}$$

in which λ runs through all partitions, μ through all even partitions (i.e. with all parts even) and ν through all partitions such that ν' is even, generalize to identities for the $P_\lambda(q,t)$ as follows. For any partition λ, let

$$b_\lambda^{\mathrm{el}} = \prod_{\substack{s \in \lambda \\ l(s) \text{ even}}} b_\lambda(s), \qquad b_\lambda^{\mathrm{oa}} = \prod_{\substack{s \in \lambda \\ a(s) \text{ odd}}} b_\lambda(s),$$

(so the superscripts el and oa stand for "even legs" and "odd arms," respectively. Then there are product expressions for the four series

$$\text{(a)} \quad \sum_\lambda b_\lambda^{\mathrm{el}}(q,t) P_\lambda(q,t), \qquad \text{(b)} \quad \sum_\lambda b_\lambda^{\mathrm{oa}}(q,t) P_\lambda(q,t),$$
$$\text{(c)} \quad \sum_\mu b_\mu^{\mathrm{oa}}(q,t) P_\mu(q,t), \qquad \text{(d)} \quad \sum_\nu b_\nu^{\mathrm{el}}(q,t) P_\nu(q,t),$$

where as before λ runs through all partitions, μ through all even partitions and ν through partitions such that ν' is even. For example, the sum (a) is equal to the product

$$\prod_i \frac{(tx_i;q)_\infty}{(x_i;q)_\infty} \prod_{i<j} \frac{(tx_i x_j;q)_\infty}{(x_i x_j;q)_\infty}$$

and (d) is equal to

$$\prod_{i<j} \frac{(tx_i x_j;q)_\infty}{(x_i x_j;q)_\infty}.$$

The products for (b) and (c) are a little more complicated : they may be derived from (a) and (b) by duality (3.5). In the case of Jack's symmetric functions, (a) is due to K. Kadell.

6. The Kostka matrix

Let

$$h_\lambda(q,t) = \prod_{s\in\lambda}(1 - q^{a(s)}t^{l(s)+1}), \tag{6.1}$$

$$\begin{aligned} h'_\lambda(q,t) &= \prod_{s\in\lambda}(1 - q^{a(s)+1}t^{l(s)}), \\ &= h_{\lambda'}(t,q), \end{aligned} \tag{6.2}$$

so that by (5.9) we have

$$b_\lambda(q,t) = h_\lambda(q,t)/h'_\lambda(q,t).$$

Now define

$$\begin{aligned} J_\lambda(q,t) &= h_\lambda(q,t)P_\lambda(q,t) \\ &= h'_\lambda(q,t)Q_\lambda(q,t). \end{aligned} \tag{6.3}$$

It seems likely that when the $J_\lambda(q,t)$ are expressed in terms of the monomial symmetric functions, the coefficients are *polynomials*, i.e. elements of $\mathbb{Z}[q,t]$. I shall make a more precise conjecture later.

When $q = t$, we have

$$J_\lambda(t,t) = H_\lambda(t)s_\lambda,$$

where

$$H_\lambda(t) = \prod_{s\in\lambda}(1 - t^{h(s)})$$

is the hook-length polynomial ($h(s) = a(s) + l(s) + 1$).

When $q = 0$,

$$J_\lambda(0,t) = Q_\lambda(0,t) \tag{6.4}$$

because $h'_\lambda(0,t) = 1$.

Duality (3.5) now takes the form

$$\omega_{q,t}J_\lambda(q,t) = J_{\lambda'}(t,q) \tag{6.5}$$

and the specialization theorem (5.3) takes the form

$$\epsilon_{u,t}J_\lambda(q,t) = \prod_{s\in\lambda}(t^{l'(s)} - q^{a'(s)}u).$$

From the fact (§ 2) that $P_\lambda(q,t) = P_\lambda(q^{-1},t^{-1})$ we deduce that

$$J_\lambda(q^{-1},t^{-1}) = (-1)^{|\lambda|} q^{-n(\lambda')} t^{-n(\lambda)-|\lambda|} J_\lambda(q,t), \tag{6.6}$$

where

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda_i'}{2}.$$

Recall ([M$_1$], ch. I, § 6) that the *Kostka numbers* $K_{\lambda\mu}$ are defined by

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu. \tag{6.7}$$

We have $K_{\lambda\mu} = 0$ unless $\mu \leq \lambda$, and $K_{\lambda\lambda} = 1$. They generalize to the *Kostka-Foulkes polynomials* $K_{\lambda\mu}(t)$ ([M$_1$], ch. III) defined as follows :

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(t) P_\mu(x;t), \tag{6.8}$$

where the $P_\mu(x;t) = P_\mu(0;x,t)$ are the Hall-Littlewood functions. Let $S_\lambda(x;t)$ denote the Schur functions associated with the product

$$\prod (1-tx_i)/(1-x_i)\,;$$

they form a basis of $\Lambda_{\mathbb{Q}(t)}$ dual to the basis $(s_\lambda(x))$, relative to the scalar product (1.13), and therefore (6.8) is equivalent to

$$Q_\mu(x;t) = \sum_\lambda K_{\lambda\mu}(t) S_\lambda(x;t). \tag{6.8$'$}$$

Since $P_\mu(x;1) = m_\mu$ it follows from (6.7) and (6.8) that $K_{\lambda\mu}(1) = K_{\lambda\mu}$. Now $K_{\lambda\mu}$ is the number of tableaux of shape λ and weight μ. Foulkes conjectured, and Lascoux and Schützenberger proved, that $K_{\lambda\mu}(t)$ is a polynomial in t with positive integral coefficients, and more precisely that

$$K_{\lambda\mu}(t) = \sum_T t^{c(T)}$$

summed over all tablelaux T of shape λ and weight μ, where $c(T)$ (the *charge* of T) is a well-defined $\mathbb{N}$-valued function of the tableau T.

In the present context we now define $K_{\lambda\mu}(q,t) \in F$ by

$$J_\mu(x;q,t) = \sum_\lambda K_{\lambda\mu}(q,t) S_\lambda(x;t). \tag{6.9}$$

Computations of the $K_{\lambda\mu}(q,t)$ suggest the

CONJECTURE. — *$K_{\lambda\mu}(q,t)$ is a polynomial in q and t with positive integral coefficients.*

Here are some partial results which tend to confirm this conjecture.

(1) We have $K_{\lambda\mu}(0,t) = K_{\lambda\mu}(t)$ (by (6.4) and (6.8′)). In particular, $K_{\lambda\mu}(0,0) = \delta_{\lambda\mu}$ and $K_{\lambda\mu}(0,1) = K_{\lambda\mu}$.

(2) From (6.5) and (6.6) we deduce that

$$\begin{aligned} K_{\lambda\mu}(q,t) &= K_{\lambda'\mu'}(t,q), \\ K_{\lambda\mu}(q,t) &= q^{n(\mu')}t^{n(\mu)}K_{\lambda'\mu}(q^{-1},t^{-1}). \end{aligned}$$

(3) Another special case is (λ, μ partitions of n)

$$K_{\lambda\mu}(1,1) = \frac{n!}{h(\lambda)},$$

where $h(\lambda)$ is the product of the hook lengths of λ. In other words, $K_{\lambda\mu}(1,1)$ is the number of standard tableaux of shape λ (and in particular does not depend on μ). This prompts the following question, which refines the conjecture above : can one find $\mathbb{N}$-valued functions $a_\mu(T)$, $b_\mu(T)$, defined for standard tableaux T, such that

$$K_{\lambda\mu}(q,t) = \sum_T q^{a_\mu(T)}t^{b_\mu(T)}$$

summed over the standard tableaux T of shape λ ? Of course this question, as I have stated it, is not well-posed. The point is to find some "natural" bijection between the monomials $q^a t^b$ that occur in $K_{\lambda\mu}(q,t)$ (assuming the truth of the conjecture) and the standard tableaux of shape λ.

(4) When $q = t$ we have $K_{\lambda\mu}(t,t) \in \mathbb{Z}[t]$, by a result of STANLEY.

(5) $K_{\lambda\mu}(1,t)$ is a polynomial in t with positive integer coefficients. (Recall that $P_\lambda(1,t) = e_{\lambda'}$; this makes it feasible to compute $K_{\lambda\mu}(1,t)$.) By duality ((2) above) $K_{\lambda\mu}(q,1)$ is a polynomial in q with positive integral coefficients.

(6) Let $\lambda = (r, 1^s)$. Then for each partition μ of $r+s$, $K_{\lambda\mu}(q,t)$ is the coefficient of u^s in the product

$$\prod (t^{i-1} + q^{j-1}u)$$

over all $(i,j) \in \mu$ with the exception of (1,1).

The symmetric functions $S_\lambda(x,t)$ are linear combinations of the $m_\mu(x)$ with coefficients in $\mathbb{Z}[t]$, from the table on p. 128 of [M$_1$]. Hence the conjecture would imply that

$$J_\lambda(q,t) = \sum_{\mu \leq \lambda} v_{\lambda\mu}(q,t) m_\mu$$

with coefficients $v_{\lambda\mu} \in \mathbb{Z}[q,t]$, divisible by $(1-t)^{l(\mu)}$. Another consequence of the conjecture is that

$$\langle J_\lambda, J_\mu J_\nu \rangle_{q,t} \in \mathbb{Z}[q,t]$$

for any three partitions λ, μ, ν.

Let $K_n(q,t)$ denote the matrix $\big(K_{\lambda\mu}(q,t)\big)_{\lambda,\mu\in\mathcal{P}_n}$. For $n = 1, \ldots, 6$ the matrices K_n (or rather their transposes K'_n) are shown in the appendix.

7. Another scalar product

We have seen that the symmetric functions $P_\lambda(q,t)$ are pairwise orthogonal with respect to the scalar product $\langle\ ,\ \rangle_{q,t}$. It will appear that they are also pairwise orthogonal with respect to *another* scalar product $\langle\ ,\ \rangle'_{q,t}$, which I shall now define.

We shall work throughout with a finite number of indeterminates $x = (x_1, \ldots, x_n)$ (i.e., in $\Lambda_{n,F}$ rather Λ_F). Moreover we shall assume (although it is not strictly necessary) that $t = q^k$ where $k \in \mathbb{N}$.

Let

$$L_n = F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

be the F-algebra of Laurent polynomials in $x_1, \ldots, x_n$, i.e. of polynomials in the x_i and x_i^{-1}. If $f \in L_n$ let $\overline{f} = f(x_1^{-1}, \ldots, x_n^{-1})$ and let $[f]_1$ denote the constant term in f. Moreover let

$$\Delta = \Delta(x;q,t) = \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{(x_i x_j^{-1};q)_\infty}{(t x_i x_j^{-1};q)_\infty} = \prod_{i\neq j}\prod_{r=0}^{k-1}(1 - q^r x_i x_j^{-1}) \tag{7.1}$$

so that $\Delta \in L_n$.

For $f, g \in \Lambda_{n,F}$ we now define

$$\langle f, g\rangle'_{q,t} = \frac{1}{n!}[f\overline{g}\Delta]_1. \tag{7.2}$$

This is a symmetric, positive definite scalar product on $\Lambda_{n,F}$. It is not difficult (in fact, it is a good deal easier than it was in § 2) to show that

(7.3) *The operator $D_{q,t}$ (2.10) is self-adjoint for this scalar product :*

$$\langle Df, g\rangle'_{q,t} = \langle f, Dg\rangle'_{q,t}$$

for all $f, g \in \Lambda_{n,F}$.

From (7.3) it follows, just as in § 2, that

$$\langle P_\lambda, P_\mu \rangle'_{q,t} = 0$$

if $\lambda \neq \mu$.

Remark. — When $q = t$ (i.e., when $k = 1$) the two scalar products are the same. Otherwise they are different.

It remains to compute $\langle P_\lambda, P_\lambda \rangle'_{q,t}$, and the answer is as follows :

$$\begin{aligned} \langle P_\lambda, P_\lambda \rangle'_{q,t} &= \prod_{1 \le i < j \le n} \prod_{r=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_j + r} t^{j-i}}{1 - q^{\lambda_i - \lambda_j - r} t^{j-i}} \\ &= c_n \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t^{n-l'(s)}}{1 - q^{a'(s)+1} t^{n-l'(s)-1}}, \end{aligned} \tag{7.4}$$

where

$$c_n = \langle 1, 1 \rangle'_{q,t} = \frac{1}{n!}[\Delta]_1 = \prod_{i=2}^{n} \begin{bmatrix} ik-1 \\ k-1 \end{bmatrix}, \tag{7.5}$$

a product of q-binomial coefficients. (Notice that when $\lambda = 0$, (7.4) reduces to a special case of the q-Dyson conjecture.)

8. Conclusion

In the definition (7.1) of Δ, and in the first of the two products (7.4) the structure of the root system of type A_{n-1} is clearly visible. In fact this aspect of the theory generalizes to other root systems, and I shall conclude these lectures with a brief and simplified account of this generalization. For full details and proofs, see [M$_3$].

So let R be a reduced root system, R^+ a system of positive roots in R; let Q be the root lattice of R, and Q^+ the positive cone in Q, spanned by R^+; let P be the weight lattice, and P^{++} the cone of dominant weights; and let W be the Weyl group of R. Define a partial order on P by

$$\lambda \ge \mu \iff \lambda - \mu \in Q^+.$$

Now let q and t be indeterminates, and F as before the field $\mathbb{Q}(q,t)$. Let $A = F[P]$ be the group algebra of the lattice (or free abelian group) P over F. To each $\lambda \in P$ there corresponds an element e^λ of A, such that $e^\lambda . e^\mu = e^{\lambda+\mu}$, and $e^0 = 1$ is the identity element of A. The Weyl group

W acts on P, hence on $A : w(e^\lambda) = e^{w\lambda}$ ($w \in W$, $\lambda \in P$). Let A^W be the subalgebra of W-invariants in A.

We can easily describe two F-bases of A^W. One consists of the *orbit-sums*

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu \qquad (\lambda \in P^{++})$$

and is the counterpart of the monomial symmetric functions denoted by the same symbols. The other basis consists of the *Weyl characters* : let

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$$

and let

$$\delta = \prod_{\alpha \in R^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right) = \sum_{w \in W} \epsilon(w) e^{w\rho}$$

(where $\epsilon(w) = \det(w) = \pm 1$). For each $\lambda \in P^{++}$ we define the *Weyl character*

$$\chi_\lambda = \delta^{-1} \sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)}$$

which is the counterpart of the Schur function s_λ. We have

$$\chi_\lambda = m_\lambda + \sum_{\substack{\mu < \lambda \\ \mu \in P^{++}}} K_{\lambda\mu} m_\mu$$

with coefficients $K_{\lambda\mu} \in \mathbb{N}$.

If $f \in A$, say $f = \sum_{\lambda \in P} f_\lambda e^\lambda$, let

$$\overline{f} = \sum f_\lambda e^{-\lambda}$$

and let $[f]_1$ denote the constant term f_0 of f.

For simplicity we shall assume as in § 7 that $t = q^k$, $k \in \mathbb{N}$, and define, in analogy with (7.1),

$$\begin{aligned} \Delta = \Delta(q,t) &= \prod_{\alpha \in R} \frac{(e^a; q)_\infty}{(te^\alpha; q)_\infty} \\ &= \prod_{\alpha \in R} (e^\alpha; q)_k \end{aligned}$$

so that $\Delta \in A^W$. Now define a scalar product on A^W by

$$\langle f, g \rangle = |W|^{-1} [f\overline{g}\Delta]_1$$

for $f, g \in A^W$. Then we have

THEOREM. — *There exists a unique basis $(P_\lambda)_{\lambda\in P^{++}}$ of A^W such that*

$$\text{(A)}\quad P_\lambda = m_\lambda + \sum_{\substack{\mu<\lambda\\ \mu\in P^{++}}} u_{\lambda\mu} m_\mu$$

with coefficients $u_{\lambda\mu} \in F$;

(B) $\langle P_\lambda, P_\mu\rangle = 0$ if $\lambda \neq \mu$.

When R is of type A_{n-1}, these P_λ are essentially the symmetric functions $P_\lambda(q,t)$, restricted to n variables $x_1, \ldots, x_n$. For arbitrary R, when $k = 0$ (i.e., $t = 1$) we have $P_\lambda = m_\lambda$, and when $k = 1$ (i.e., $q = t$) we have $P_\lambda = \chi_\lambda$.

I will conclude with two conjectures which generalize (7.4) and the specialization theorem (5.3) respectively. For each root $\alpha \in R$ let $\alpha^\vee$ be the corresponding co-root, and let

$$\sigma = \frac{1}{2}\sum_{\alpha\in R^+} \alpha^\vee.$$

CONJECTURE 1. — *For all $\lambda \in P^{++}$,*

$$\langle P_\lambda, P_\mu\rangle = \prod_{\alpha\in R^+}\prod_{i=1}^{k-1} \frac{1-q^{\alpha^\vee(\lambda+k\rho)+i}}{1-q^{\alpha^\vee(\lambda+k\rho)-i}}.$$

This is non trivial even when $\lambda = 0$ (so that $P_\lambda = 1$); in that case it reduces to the constant term conjectures of [M$_2$].

CONJECTURE 2. — *Let $P_\lambda(k\sigma)$ denote the image of P_λ under the mapping $e^\mu \mapsto q^{k\sigma(\mu)} = t^{\sigma(\mu)}$ ($\mu \in P$). Then for all $\lambda \in P^{++}$,*

$$P_\lambda(k\sigma) = q^{-k\sigma(\lambda)} \prod_{\alpha\in R^+} \frac{(q^{\alpha^\vee(\lambda+k\rho)};q)_k}{(q^{\alpha^\vee(k\rho)};q)_k}.$$

REFERENCES

[M_1] MACDONALD (I.G.). — *Symmetric functions and Hall polynomials.* — Oxford University Press, 1979.

[M_2] MACDONALD (I.G.). — Some conjectures for root systems, *SIAM J. Math. Anal.*, t. **13**, 1982, p. 988–1007.

[M_3] MACDONALD (I.G.). — Orthogonal polynomials associated with root systems, preprint, 1988.

[S] STANLEY (Richard). — Some combinatorial properties of Jack symmetric functions, preprint, 1988.

I. G. MACDONALD,
School of Mathematical Sciences,
Queen Mary College,
University of London,
Mile End Road,
London E1 4NS,
United Kingdom.

9. Appendix

The matrices $K(q,t)'$, $n \le 6$

	1
1	1

	2	1^2
2	1	q
1^2	t	1

	3	21	1^3
3	1	$q+q^2$	q^3
21	t	$1+qt$	q
1^3	t^3	$t+t^2$	1

	4	31	2^2	21^2	1^4
4	1	$q+q^2+q^3$	q^2+q^4	$q^3+q^4+q^5$	q^6
31	t	$1+qt+q^2t$	$q+q^2t$	$q+q^2+q^3t$	q^3
2^2	t^2	$t+qt+qt^2$	$1+q^2t^2$	$q+qt+q^2t$	q^2
21^2	t^3	$t+t^2+qt^3$	$t+qt^2$	$1+qt+qt^2$	q
1^4	t^6	$t^3+t^4+t^5$	t^2+t^4	$t+t^2+t^3$	1

	5	41	32	31^2	2^21	21^3	1^5
5	1	$q+q^2+q^3+q^4$	$q^2+q^3+q^4+q^5+q^6$	$q^3+q^4+2q^5+q^6+q^7$	$q^4+q^5+q^6+q^7+q^8$	$q^6+q^7+q^8+q^9$	q^{10}
41	t	$1+qt+q^2t+q^3t$	$q+q^2+q^2t+q^3t+q^4t$	$q+q^2+q^3+q^3t+q^4t+q^5t$	$q^2+q^3+q^4+q^4t+q^5t$	$q^3+q^4+q^5+q^6t$	q^6
32	t^2	$t+qt+qt^2+q^2t^2$	$1+qt+q^2t+q^2t^2+q^3t^2$	$q+qt+2q^2t+q^3t+q^3t^2$	$q+q^2+q^2t+q^3t+q^4t^2$	$q^2+q^3+q^3t+q^4t$	q^4
31^2	t^3	$t+t^2+qt^3+q^2t^3$	$t+qt+qt^2+q^2t^2+q^2t^3$	$1+qt+q^2t+qt^2+q^2t^2+q^3t^3$	$q+qt+q^2t+q^2t^2+q^3t^2$	$q+q^2+q^3t+q^3t^2$	q^3
2^21	t^4	$t^2+t^3+qt^3+qt^4$	$t+t^2+qt^2+qt^3+q^2t^4$	$t+qt+2qt^2+qt^3+q^2t^3$	$1+qt+qt^2+q^2t^2+q^2t^3$	$q+qt+q^2t+q^2t^2$	q^2
21^3	t^6	$t^3+t^4+t^5+qt^6$	$t^2+t^3+t^4+qt^4+qt^5$	$t+t^2+t^3+qt^3+qt^4+qt^5$	$t+t^2+qt^2+qt^3+qt^4$	$1+qt+qt^2+qt^3$	q
1^5	t^{10}	$t^6+t^7+t^8+t^9$	$t^4+t^5+t^6+t^7+t^8$	$t^3+t^4+2t^5+t^6+t^7$	$t^2+t^3+t^4+t^5+t^6$	$t+t^2+t^3+t^4$	1

	6	51	42
6	1	$q+q^2+q^3+q^4+q^5$	$q^2+q^3+2q^4+q^5+2q^6+q^7+q^8$
51	t	$1+qt+q^2t+q^3t+q^4t$	$q+q^2+q^3+q^2t+q^3t+2q^4t+q^5t+q^6t$
42	t^2	$t+qt+qt^2+q^2t^2+q^3t^2$	$1+qt+2q^2t+q^3t+q^2t^2+q^3t^2+2q^4t^2$
41^2	t^3	$t+t^2+qt^3+q^2t^3+q^3t^3$	$t+qt+q^2t+qt^2+q^2t^2+q^3t^2+q^2t^3+q^3t^3+q^4t^3$
3^2	t^3	$t^2+qt^2+q^2t^2+qt^3+q^2t^3$	$t+qt+q^2t+qt^2+q^2t^2+q^3t^2+q^2t^3+q^3t^3+q^4t^3$
321	t^4	$t^2+t^3+qt^3+qt^4+q^2t^4$	$t+t^2+2qt^2+qt^3+2q^2t^3+q^2t^4+q^3t^4$
31^3	t^6	$t^3+t^4+t^5+qt^6+q^2t^6$	$t^2+t^3+qt^3+t^4+qt^4+q^2t^4+qt^5+q^2t^5+q^2t^6$
2^3	t^6	$t^4+qt^4+t^5+qt^5+qt^6$	$t^2+t^3+qt^3+t^4+qt^4+q^2t^4+qt^5+q^2t^5+q^2t^6$
2^21^2	t^7	$t^4+t^5+t^6+qt^6+qt^7$	$2t^3+t^4+qt^4+t^5+2qt^5+qt^6+q^2t^7$
21^4	t^{10}	$t^6+t^7+t^8+t^9+qt^{10}$	$t^4+t^5+2t^6+t^7+qt^7+t^8+qt^8+qt^9$
1^6	t^{15}	$t^{10}+t^{11}+t^{12}+t^{13}+t^{14}$	$t^7+t^8+2t^9+t^{10}+2t^{11}+t^{12}+t^{13}$

	41^2	3^2
6	$q^3+q^4+2q^5+2q^6+2q^7+q^8+q^9$	$q^3+q^5+q^6+q^7+q^9$
51	$q+q^2+q^3+q^4+q^3t+q^4t+2q^5t+q^6t+q^7t$	$q^2+q^4+q^3t+q^5t+q^6t$
42	$q+qt+2q^2t+2q^3t+q^4t+q^3t^2+q^4t^2+q^5t^2$	$q+q^2t+q^3t+q^3t^2+q^5t^2$
41^2	$1+qt+q^2t+q^3t+qt^2+q^2t^2+q^3t^2+q^3t^3+q^4t^3+q^5t^3$	$qt+q^2t+q^2t^2+q^4t^2+q^3t^3$
3^2	$qt+q^2t+q^3t+qt^2+2q^2t^2+2q^3t^2+q^4t^2+q^3t^3$	$1+q^2t^2+q^3t^2+q^4t^2+q^3t^3$
321	$t+qt+2qt^2+q^2t^2+qt^3+2q^2t^3+q^3t^3+q^3t^4$	$t+qt^2+q^2t^2+q^2t^3+q^3t^4$
31^3	$t+t^2+t^3+qt^3+q^2t^3+qt^4+q^2t^4+qt^5+q^2t^5+q^3t^6$	$qt^2+t^3+qt^4+q^2t^4+q^2t^5$
2^3	$qt^2+t^3+2qt^3+q^2t^3+2qt^4+q^2t^4+qt^5+q^2t^5$	$qt^2+t^3+qt^3+qt^4+q^3t^6$
2^21^2	$t^2+t^3+qt^3+t^4+2qt^4+2qt^5+qt^6+q^2t^6$	$t^2+t^4+qt^4+qt^5+q^2t^6$
21^4	$t^3+t^4+2t^5+t^6+qt^6+t^7+qt^7+qt^8+qt^9$	$t^4+t^5+qt^6+t^7+qt^8$
1^6	$t^6+t^7+2t^8+2t^9+2t^{10}+t^{11}+t^{12}$	$t^6+t^8+t^9+t^{10}+t^{12}$

	321
6	$q^4+2q^5+2q^6+3q^7+3q^8+2q^9+2q^{10}+q^{11}$
51	$q^2+2q^3+2q^4+2q^5+q^6+q^4t+2q^5t+2q^6t+2q^7t+q^8t$
42	$q+2q^2+q^3+q^2t+3q^3t+3q^4t+q^5t+q^4t^2+2q^5t^2+q^6t^2$
41^2	$q+q^2+qt+2q^2t+2q^3t+q^4t+q^2t^2+2q^3t^2+2q^4t^2+q^5t^2+q^4t^3+q^5t^3$
3^2	$q+q^2+qt+2q^2t+2q^3t+q^4t+q^2t^2+2q^3t^2+2q^4t^2+q^5t^2+q^4t^3+q^5t^3$
321	$1+3qt+q^2t+qt^2+4q^2t^2+q^3t^2+q^2t^3+3q^3t^3+q^4t^4$
31^3	$t+qt+t^2+2qt^2+q^2t^2+2qt^3+2q^2t^3+qt^4+2q^2t^4+q^3t^4+q^2t^5+q^3t^5$
2^3	$t+qt+t^2+2qt^2+q^2t^2+2qt^3+2q^2t^3+qt^4+2q^2t^4+q^3t^4+q^2t^5+q^3t^5$
2^21^2	$t+2t^2+qt^2+t^3+3qt^3+3qt^4+q^2t^4+qt^5+2q^2t^5+q^2t^6$
21^4	$t^2+2t^3+2t^4+qt^4+2t^5+2qt^5+t^6+2qt^6+2qt^7+qt^8$
1^6	$t^4+2t^5+2t^6+3t^7+3t^8+2t^9+2t^{10}+t^{11}$

	31^3	2^3
6	$q^6+q^7+2q^8+2q^9+2q^{10}+q^{11}+q^{12}$	$q^6+q^8+q^9+q^{10}+q^{12}$
51	$q^3+q^4+2q^5+q^6+q^7+q^6t+q^7t+q^8t+q^9t$	$q^4+q^5+q^7+q^6t+q^8t$
42	$q^2+q^3+q^4+q^3t+2q^4t+2q^5t+q^6t+q^6t^2$	$q^2+q^4+q^4t+q^5t+q^6t^2$
41^2	$q+q^2+q^3+q^3t+q^4t+q^5t+q^3t^2+q^4t^2+q^5t^2+q^6t^3$	$q^3+q^2t+q^4t+q^4t^2+q^5t^2$
3^2	$q^3+q^2t+2q^3t+2q^4t+q^5t+q^3t^2+q^4t^2+q^5t^2$	$q^3+q^2t+q^3t+q^4t+q^6t^3$
321	$q+qt+2q^2t+q^3t+q^2t^2+2q^3t^2+q^3t^3+q^4t^3$	$q+q^2t+q^2t^2+q^3t^2+q^4t^3$
31^3	$1+qt+q^2t+qt^2+q^2t^2+qt^3+q^2t^3+q^3t^3+q^3t^4+q^3t^5$	$qt+qt^2+q^2t^2+q^3t^3+q^2t^4$
2^3	$qt+q^2t+qt^2+2q^2t^2+qt^3+2q^2t^3+q^3t^3+q^2t^4$	$1+q^2t^2+q^2t^3+q^3t^3+q^2t^4$
2^21^2	$t+qt+2qt^2+2qt^3+q^2t^3+qt^4+q^2t^4+q^2t^5$	$t+qt^2+qt^3+q^2t^3+q^2t^5$
21^4	$t+t^2+t^3+qt^3+t^4+qt^4+2qt^5+qt^6+qt^7$	$t^2+qt^3+t^4+qt^5+qt^6$
1^6	$t^3+t^4+2t^5+2t^6+2t^7+t^8+t^9$	$t^3+t^5+t^6+t^7+t^9$

	2^21^2	21^4	1^6
6	$q^7+q^8+2q^9+q^{10}+2q^{11}+q^{12}+q^{13}$	$q^{10}+q^{11}+q^{12}+q^{13}+q^{14}$	q^{15}
51	$q^4+q^5+2q^6+q^7+q^8+q^7t+q^8t+q^9t$	$q^6+q^7+q^8+q^9+q^{10}t$	q^{10}
42	$2q^3+q^4+q^5+q^4t+2q^5t+q^6t+q^7t^2$	$q^4+q^5+q^6+q^6t+q^7t$	q^7
41^2	$q^2+q^3+q^4+q^3t+q^4t+q^5t+q^4t^2+q^5t^2+q^6t^2$	$q^3+q^4+q^5+q^6t+q^6t^2$	q^6
3^2	$q^2+q^3+q^4+q^3t+q^4t+q^5t+q^4t^2+q^5t^2+q^6t^2$	$q^4+q^5+q^4t+q^5t+q^6t$	q^6
321	$q+q^2+2q^2t+q^3t+2q^3t^2+q^4t^2+q^4t^3$	$q^2+q^3+q^3t+q^4t+q^4t^2$	q^4
31^3	$q+qt+q^2t+qt^2+q^2t^2+q^3t^2+q^2t^3+q^3t^3+q^3t^4$	$q+q^2+q^3t+q^3t^2+q^3t^3$	q^3
2^3	$q+qt+q^2t+qt^2+q^2t^2+q^3t^2+q^2t^3+q^3t^3+q^3t^4$	$q^2+q^2t+q^3t+q^2t^2+q^3t^2$	q^3
2^21^2	$1+qt+2qt^2+q^2t^2+qt^3+q^2t^3+2q^2t^4$	$q+qt+q^2t+q^2t^2+q^2t^3$	q^2
21^4	$t+t^2+qt^2+t^3+qt^3+2qt^4+qt^5+qt^6$	$1+qt+qt^2+qt^3+qt^4$	q
1^6	$t^2+t^3+2t^4+t^5+2t^6+t^7+t^8$	$t+t^2+t^3+t^4+t^5$	1