The Concept of Bailey Chains

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0. Introduction

In his expository lectures on q-series [3] G. E. Andrews devotes a whole chapter to Bailey's Lemma (Th. 2.1, 3.1) and discusses some of its numerous possible applications in terms of the "Bailey chain" concept. This name was introduced by G. E. Andrews [2] to describe the iterative nature of Bailey's Lemma, which was not observed by W. N. Bailey himself.

This iteration mechanism allows to derive many q-series identities by "reducing" them to more elementary ones. As an example, the famous Rogers-Ramanujan identities can be reduced to the q-binomial theorem.

G. E. Andrews [2] observed this iteration mechanism in its full generality by an appropriate reformulation of Bailey's Lemma, whereas P. Paule discovered important special cases [23]. W. N. Bailey never formulated his lemma in that way and consequently missed the full power of its potential for iteration. In that paper [2] G. E. Andrews introduced the notions of "Bailey pairs" and "Bailey chains" and laid the foundations of a Bailey chain theory for discovering and proving q-identities.

The purpose of this article is to give an introduction to that concept. Therefore many theorems are not stated in full generality, for which we refer to the literature.

1. Definitions and Tools

A *hypergeometric* series (see e.g. W. N. Bailey [10]) is a series

$$\sum c_n,$$

where

$$\frac{c_{n+1}}{c_n}$$

is a rational function in n, i.e.

$$c_0 = 1 \quad \text{and} \quad \frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_i)}{(n+b_1)(n+b_2)\cdots(n+b_j)}\frac{x}{n+1}.$$

Thus

$$c_n = \frac{\langle a_1\rangle_n \cdots \langle a_i\rangle_n}{\langle b_1\rangle_n \cdots \langle b_j\rangle_n}\frac{x^n}{n!},$$

where

$$\langle a\rangle_n := a(a+1)\cdots(a+n-1) \quad \text{and} \quad \langle a\rangle_0 := 1$$

Notation:

$${}_iF_j\left(\begin{matrix} a_1,\ldots,a_i \\ b_1,\ldots,b_j \end{matrix};x\right) = \sum_{k=0}^{\infty} \frac{\langle a_1\rangle_k \ldots \langle a_i\rangle_k}{\langle b_1\rangle_k \ldots \langle b_j\rangle_k} \frac{x^k}{k!}$$

Examples:

1. *Wallis*:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{\langle 1\rangle_n \langle 1\rangle_n}{\langle \frac{1}{2}\rangle_n \langle \frac{3}{2}\rangle_n}.$$

2. The binomial series

$$\frac{1}{(1-x)^\alpha} = {}_1F_0(\alpha;—;x) = \sum c_n \quad \text{with } \frac{c_{n+1}}{c_n} = \frac{\alpha+n}{1+n}x \text{ and } c_0 = 1.$$

3. $y = {}_2F_1(a,b;c;x)$ is solution of

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$$

(*hypergeometric differential equation*).

4. The *Jacobi polynomials* (see e.g. R. Askey [7]) $(\alpha,\beta > -1)$

$$P_n^{(\alpha,\beta)}(x) = \frac{\langle \alpha+1\rangle_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right)$$

with orthogonality relation $(m \neq n)$:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \mathrm{d}x = 0.$$

For many reasons it is convenient to extend this definition by introducing an extra parameter q (see e.g. L. Slater [30]):

Definition. *A basic- (or q-) hypergeometric series is a series* $\sum_{n=-\infty}^{\infty} c_n$*, where* $\frac{c_{n+1}}{c_n}$ *is a rational function of* q^n*.*

Example: The *theta-function*

$$\sum c_n = \sum_{n=-\infty}^{\infty} q^{n^2} x^n \quad (q = \mathrm{e}^{\pi i r},\ x = \mathrm{e}^{2iz})$$

where

$$\frac{c_{n+1}}{c_n} = q^{2n+1}x, c_0 = 1.$$

Now we define

$$(a;q)_n := (a)_n := (1-a)(1-qa)\cdots(1-q^{n-1}a) \quad \text{for } n=1,2,\dots$$

and

$$(a;q)_0 := (a)_0 := 1, \quad (a;q)_\infty := (a)_\infty := \prod_{k=0}^{\infty}(1-q^k a)$$

and

$$(a)_n := \frac{(a)_\infty}{(q^n a)_\infty} \quad \text{for integer } n.$$

Observe that $\dfrac{1}{(q)_n} = 0$ for $n=-1,-2,\dots$

All following q-identities can be treated as formal power series identities. If one likes to consider them as analytic ones, in most cases it will suffice to take q as real with $|q|<1$.

Notation:

$${}_i\varphi_j\left(\begin{matrix} a_1,\dots,a_i \\ b_1,\dots,b_j \end{matrix}; q, x\right) := \sum_{k=0}^{\infty} \frac{(a_1;q)_k \dots (a_i;q)_k}{(b_1;q)_k \dots (b_j;q)_k} \frac{x^k}{(q)_k}$$

Example: The q-analogue of the binomial series is

$$\sum c_n = {}_1\varphi_0(a;—;q;x) \quad \text{with } \frac{c_{n+1}}{c_n} = \frac{1-aq^n}{1-q^{n+1}}x,\ c_0 = 1.$$

Setting $a=q^\alpha$ we obtain

$$\frac{c_{n+1}}{c_n} = \frac{1-q^{\alpha+n}}{1-q} \cdot \frac{1-q}{1-q^{1+n}}x,$$

which is for $q=1$ equal to

$$\frac{\alpha+n}{1+n}x,$$

as in ${}_1F_0(\alpha;—;x)$.

In order to emphasize the analogy to the $q=1$ case, we introduce the *Gaussian polynomials* (or *q-binomial coefficients*) (see e.g. G. Andrews [4]):

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \dfrac{(q)_n}{(q)_k(q)_{n-k}} & \text{if } 0 \le k \le n \\ 0 & \text{else} \end{cases}.$$

We shall also write

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}, \quad \text{where } [n]! := [n][n-1]\dots[1],$$

$$[0]! := 1 \quad \text{and} \quad [n] := \frac{1-q^n}{1-q} = 1+q+\dots+q^{n-1}, \quad [0] := 1.$$

From this definition it obvious that

$$\begin{bmatrix} n \\ k \end{bmatrix}\Big|_{q=1} = \binom{n}{k}.$$

Now we introduce the q-binomial theorem in the notion of J. Cigler (cf. the survey article [15]):

Theorem 1.1. *Let $\mathcal{R}$ denote the ring of all power series in the variable x over the reals (or formal power series in x, respectivly). For linear operators A, B on $\mathcal{R}$ with $BA = qAB$ the following formula holds ($n = 0, 1, 2, \ldots$):*

$$(A+B)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} A^k B^{n-k}. \tag{1}$$

Proof. The proof is an easy induction exercise using the recursive formula

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = q^k \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}. \tag{2}$$

$\square$

Examples:

1. $(\underline{x}f)(x) := xf(x)$, $(\varepsilon f)(x) := f(qx)$ and $(\varepsilon^{-1}f)(x) := f(q^{-1}x)$ for $f \in \mathcal{R}$. Now we have $BA = qAB$ for e.g. $A = \underline{x}\varepsilon$, $B = \varepsilon$ or $A = \varepsilon^{-1}$, $B = \underline{x}$.
2. Observing that $(\underline{x}\varepsilon)^k 1 = q^{\binom{k}{2}} x^k$ and $(\underline{x}\varepsilon + \varepsilon)^k 1 = (x+1)(qx+1)\cdots(q^{k-1}x+1)$ we obtain by setting $A = \underline{x}\varepsilon$ and $B = \varepsilon$ in (1):

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k = (1+x)(1+qx)\cdots(1+q^{n-1}x). \tag{3}$$

3. A further consequence of Theorem 1.1 is the infinite form of the q-binomial theorem (cf. [15]):

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k = \frac{(ax)_\infty}{(x)_\infty}. \tag{4}$$

Now for $a = q^\alpha$ the q-analogy becomes evident:

$${}_1F_0(\alpha; —; x) = \sum_{k=0}^{\infty} \frac{\langle \alpha \rangle_k}{k!} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} x^k = \frac{1}{(1-x)^\alpha}$$

and

$${}_1\varphi_0(q^\alpha; —; x) = \sum_{k=0}^{\infty} \frac{(q^\alpha)_k}{(q)_k} x^k = \sum_{k=0}^{\infty} \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix} x^k = \frac{1}{(x)_\alpha}.$$

4. The q-binomial theorem (3) gives ($x \neq 0$):

$$\sum_{k=-\infty}^{\infty} \begin{bmatrix} 2j \\ j-k \end{bmatrix} (-1)^k x^k q^{\frac{1}{2}k^2} = (x^{-1}q^{\frac{1}{2}})_j (xq^{\frac{1}{2}})_j \tag{5}$$

(The sum on the left is actually finite!), which in the limit $j \to \infty$ becomes:

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{1}{2}k^2} x^k = (q)_\infty (x^{-1}q^{\frac{1}{2}})_\infty (xq^{\frac{1}{2}})_\infty. \tag{6}$$

(Note that

$$\begin{bmatrix} 2j \\ j-k \end{bmatrix} = \frac{(q)_{2j}}{(q)_{j-k}(q)_{j+k}} \to \frac{1}{(q)_\infty} \quad \text{for } j \to \infty.)$$

Identity (6) is called *Jacobi triple product identity* and serves as a fundamental tool for transforming sums into products and vice versa.

We give a prominent example, one of the *Rogers-Ramanujan identities* (see e.g. G. Andrews [4]):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}. \tag{7}$$

By (6) (q replaced by q^5 and $x = -q^{\frac{1}{2}}$) the product on the right is equal to

$$\frac{1}{(q)_\infty}(q^5;q^5)_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty = \frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{5}{2}k^2 - \frac{1}{2}k}.$$

Now to prove (7) means to prove

$$\frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{5}{2}k^2 - \frac{1}{2}k} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}. \tag{8}$$

This is exactly the point where the Bailey chain concept enters the stage. In the following we shall see how identities of this type can be reduced by Bailey chain iteration to identities of a simpler form or to well-known ones, respectively. In particular we shall demonstrate, how the Rogers-Ramanujan identity (8) is iterated to a special case of the q-binomial theorem (5). Further we shall derive the iteration mechanism for that and many other important applications as a consequence of the q-binomial theorem in the form (1). This is followed by a closer investigation of the inner structure of that mechanism, i.e. how to "walk along" Bailey chains.

2. Bailey Pairs and Bailey Chains

2.1 Bailey's Lemma In distilling some of the work of L. J. Rogers [27, 28] and others W. N. Bailey formulated the following fundamental q-series transform [(3.1), 9]:

Theorem 2.1 *(Bailey's Lemma).*

$$\sum_{k=0}^{\infty} (r_1)_k (r_2)_k \left(\frac{xq}{r_1 r_2}\right)^k b_k = \frac{\left(\frac{xq}{r_1}\right)_\infty \left(\frac{xq}{r_2}\right)_\infty}{(xq)_\infty \left(\frac{xq}{r_1 r_2}\right)_\infty} \sum_{k=0}^{\infty} \frac{(r_1)_k (r_2)_k}{\left(\frac{xq}{r_1}\right)_k \left(\frac{xq}{r_2}\right)_k} \left(\frac{xq}{r_1 r_2}\right)^k a_k, \tag{9}$$

where

$$b_n = \sum_{k=0}^{n} \frac{a_k}{(q)_{n-k}(xq)_{n+k}} \quad n = 0, 1, 2, \ldots \tag{10}$$

Following G. E. Andrews [2] we say, sequences $a = (a_n)$, $b = (b_n)$ related like (10) form a *Bailey pair* (a, b).

Using Bailey's Lemma as a tool for proving identities of the Rogers-Ramanujan type, like identity (7) or e.g. one of the *Göllnitz-Gordon* identities

$$\frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2-n} = \sum_{k=0}^{\infty} \frac{q^{2k^2}}{(q^2;q^2)_k(-q;q^2)_k}, \tag{11}$$

one has to look for a suitable Bailey pair (a, b), which after insertion into (9) with special chosen parameters r_1, r_2 yields the desired identity.

By a skillful application of this procedure L. J. Slater gave a list of 130 identities of that type in 1950 [29]. As we shall see, by using the Bailey chain concept the search for appropriate Bailey pairs and the problem of proving or discovering such identities are far easier to handle and to solve.

In 1972 [6] G. E. Andrews showed how Bailey's Lemma fits into the frame of a connection-coefficient problem: Let

$$P_k(x; \alpha, \beta|q) := {}_2\varphi_1\left(\begin{matrix} q^{-k}, \alpha\beta q^{k+1} \\ \alpha\beta \end{matrix}; q, xq\right)$$

(*q-Jacobi polynomials*).

Now Bailey's Lemma is essentially equivalent to the following expansion:

$$p_n(x) = \sum_{k=0}^{n} c_{nk} P_k(x; \alpha, \beta|q),$$

where

$$p_n(x) = {}_3\varphi_2\left(\begin{matrix} r_1, r_2, q^{-n} \\ \alpha q, \frac{r_1 r_2 q^{-n}}{\alpha\beta q} \end{matrix}; q, xq\right)$$

and $(\alpha\beta q := x)$

$$c_{nk} = \frac{\left(\frac{xq}{r_1}\right)_n \left(\frac{xq}{r_2}\right)_n (xq)_{k-1}(1 - xq^{2k})(r_1)_k(r_2)_k(q^{-n})_k}{(xq)_n \left(\frac{xq}{r_1 r_2}\right)_n \left(\frac{xq}{r_1}\right)_k \left(\frac{xq}{r_1}\right)_k (q)_k(xq^{n+1})_k} \left(\frac{xq^{n+1}}{r_1 r_2}\right)^k.$$

2.1 Bailey chains. In order to describe its potential for iteration we consider the following special case ($r_1 = q^{-m}$, $r_2 = q^{-n}$, a_k replaced by $q^{-k^2} a_k$ and $m \to \infty$) of Bailey's Lemma:

$$\sum_{k=0}^{n} \frac{a_k x^k}{(q)_{n-k}(xq)_{n+k}} = \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q)_{n-j}} \sum_{k=0}^{j} \frac{a_k q^{-k^2}}{(q)_{j-k}(xq)_{j+k}}. \tag{12}$$

Because of its importance we give a separate proof of this identity (cf. Paule [25]).

Proof. We need Theorem 1.1 together with the following facts, which are easily checked: for all $f, g \in \mathcal{R}$

$$\varepsilon(fg) = (\varepsilon f)(\varepsilon g), \tag{i}$$
$$(\varepsilon^{-1} + \underline{x})(xq)_\infty = (xq)_\infty. \tag{ii}$$

Now we apply the q-binomial Theorem (1) with $A = \varepsilon^{-1}$ and $B = \underline{x}$ as follows:

$$\begin{aligned}
\sum_{k=0}^{n} \frac{a_k x^k}{(q)_{n-k}(xq)_{n+k}} &= \frac{1}{(xq)_\infty}\left(\sum_{k=0}^{n} \frac{a_k x^k}{(q)_{n-k}} \varepsilon^{n+k}(xq)_\infty\right) \\
&\overset{\text{(ii)}}{=} \frac{1}{(xq)_\infty}\left(\sum_{k=0}^{n} \frac{a_k x^k}{(q)_{n-k}} \varepsilon^{n+k}(\varepsilon^{-1}+\underline{x})^{n-k}(xq)_\infty\right) \\
&\overset{(1)}{=} \frac{1}{(xq)_\infty}\left(\sum_{k=0}^{n} \frac{a_k x^k}{(q)_{n-k}} \varepsilon^{n+k} \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix} (\varepsilon^{-1})^{n-k-j}\underline{x}^j (xq)_\infty\right) \\
&\overset{\text{(i)}}{=} \frac{1}{(xq)_\infty}\left(\sum_{k=0}^{n} \frac{a_k x^k}{(q)_{n-k}} \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix} q^{j(j+2k)} x^j (q^{j+2k+1}x)_\infty\right) \\
&= \sum_{k=0}^{n} \frac{a_k x^k}{(q)_{n-k}} \sum_{j=k}^{n} \begin{bmatrix} n-k \\ j-k \end{bmatrix} q^{j^2-k^2} \frac{x^{j-k}}{(xq)_{j+k}} \\
&= \sum_{j=0}^{n} \frac{q^{j^2}x^j}{(q)_{n-j}} \sum_{k=0}^{j} \frac{a_k q^{-k^2}}{(q)_{j-k}(xq)_{j+k}}. \qquad \square
\end{aligned}$$

If $x = 1$ or q for many applications it is of advantage to symmetrize (12) as follows: (Observe that all sums are finite!)

$$\sum_{k=-\infty}^{\infty} \frac{c_k}{(q)_{n-k}(q)_{n+k}} = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_{n-j}} \sum_{k=-\infty}^{\infty} \frac{c_k q^{-k^2}}{(q)_{j-k}(q)_{j+k}} \tag{13}$$

(in (1): $x = 1$, $a_0 = c_0$ and $a_k = c_k + c_{-k}$ for $k \geq 1$),

$$\sum_{k=-\infty}^{\infty} \frac{c_k}{(q)_{n-k}(q)_{n+1+k}} = \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q)_{n-j}} \sum_{k=-\infty}^{\infty} \frac{c_k q^{-k^2-k}}{(q)_{j-k}(q)_{j+1+k}} \tag{14}$$

(in (1): $x = q$ and $a_k = \frac{q^{-k}}{1-q}(c_k + c_{-k-1})$ for $k \geq 0$).

Writing Bailey's Lemma as (12) (or (13), (14), respectively) its potential for iteration now leaps to our eyes, namely:

The second sums of the right-hand sides are of the same form as the corresponding sums on the left-hand sides. Thus we may iterate them substituting the whole formula (modified e.g. by taking $c_k q^{-k^2}$ instead of c_k) in the place of the second sum of the right-hand side as often as we want, in order to reduce the initial sum on the left to a simpler or well-known one.

Example: In the limit $n \to \infty$ (13) becomes

$$\frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} c_k = \sum_{j=0}^{\infty} q^{j^2} \sum_{k=-\infty}^{\infty} \frac{c_k q^{-k^2}}{(q)_{j-k}(q)_{j+k}}. \tag{15}$$

With the above iteration-algorithm the Rogers-Ramanujan identity (8) now is easily *computed* as follows:

$$\begin{aligned}
\frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{5}{2}k^2 - \frac{1}{2}k} &\overset{(15)}{=} \sum_{j=0}^{\infty} q^{j^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\frac{3}{2}k^2 - \frac{1}{2}k}}{(q)_{j-k}(q)_{j+k}} \\
&\overset{(15)}{=} \sum_{j=0}^{\infty} q^{j^2} \sum_{l=0}^{\infty} \frac{q^{l^2}}{(q)_{j-l}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k^2 - \frac{1}{2}k}}{(q)_{l-k}(q)_{l+k}} \\
&\overset{(5)}{=} \sum_{j=0}^{\infty} q^{j^2} \sum_{l=0}^{\infty} \frac{q^{l^2}}{(q)_{j-l}} \frac{(q)_l (1)_l}{(q)_{2l}} \\
&= \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j}.
\end{aligned}$$

The last line follows from the fact that $(1)_0 = 1$ and $(1)_l = (1-1)(1-q)\cdots(1-q^{l-1}) = 0$ for all $l \geq 1$. $\square$

Now we formulate as a

Theorem 2.2. *Given a Bailey pair $(a, b) = \big((a_n), (b_n)\big)$ a new Bailey pair $(a', b') = \big((a'_n), (b'_n)\big)$ is constructed by*

$$a'_n = q^{n^2} x^n a_n \quad \text{and} \quad b'_n = \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q)_{n-j}} b_j \quad n = 0, 1, 2, \ldots$$

Proof. The proof is an immediate consequence of equation (12) with a_k replaced by $a_k q^{k^2}$. $\square$

Definition. *The sequence*

$$\big((a_n), (b_n)\big) \to \big((a'_n), (b'_n)\big) \to \big((a''_n), (b''_n)\big) \to \cdots$$

is called a Bailey chain *(cf. Andrews [2]).*

Remark. To include the symmetrized versions (13) and (14) among that concept, we call sequences $a = (a_n)_{n=-\infty}^{\infty}$, $b = (b_n)_{n \geq 0}$ with

$$b_n = \sum_{k=-\infty}^{\infty} \frac{a_k}{(q)_{n-k}(q)_{n+k}} \quad \text{or} \quad b_n = \sum_{k=-\infty}^{\infty} \frac{a_k}{(q)_{n-k}(q)_{n+1+k}}$$

also a *Bailey pair* (a,b).

With this definition Theorem 2.2 remains valid with $x=1$ or q according the equations (13) and (14).

Example: for the Rogers-Ramanujan identity (8) the Bailey chain corresponding to our iteration (cf. the example above) is as follows: We start with the simplest pair

$$(a,b)=\big((a_n)_{n=-\infty}^{\infty},(b_n)_{n\geq 0}\big),$$

where

$$a_n=(-1)^n q^{\binom{n}{2}},\quad b_n=\delta_{n0}\quad (:=\begin{cases}1 & \text{if } n=0\\ 0 & \text{else}\end{cases}),$$

then

$$(a',b')=\big((a'_n)_{n=-\infty}^{\infty},(b'_n)_{n\geq 0}\big),$$

where (Theorem 2.2 with $x=1$)

$$a'_n=q^{n^2}a_n=(-1)^n q^{\frac{3}{2}n^2-\frac{1}{2}n},\quad b'_n=\sum_{j=0}^{n}\frac{q^{j^2}}{(q)_{n-j}}b_j=\frac{1}{(q)_n},$$

and

$$(a'',b'')=\big((a''_n)_{n=-\infty}^{\infty},(b''_n)_{n\geq 0}\big),$$

where (Theorem 2.2 with $x=1$)

$$a''_n=q^{n^2}a'_n=(-1)^n q^{\frac{5}{2}n^2-\frac{1}{2}n},\quad b''_n=\sum_{j=0}^{n}\frac{q^{j^2}}{(q)_{n-j}}b'_j=\sum_{j=0}^{n}\frac{q^{j^2}}{(q)_{n-j}(q)_j}.$$

It is interesting to look at the corresponding Bailey pair identities:

$$(a,b)\iff\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^{\frac{1}{2}k^2-\frac{1}{2}k}}{(q)_{n-k}(q)_{n+k}}=\delta_{n0},\tag{16}$$

$$(a',b')\iff\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^{\frac{3}{2}k^2-\frac{1}{2}k}}{(q)_{n-k}(q)_{n+k}}=\frac{1}{(q)_n},\tag{17}$$

$$(a'',b'')\iff\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^{\frac{5}{2}k^2-\frac{1}{2}k}}{(q)_{n-k}(q)_{n+k}}=\sum_{j=0}^{\infty}\frac{q^{j^2}}{(q)_{n-j}(q)_j}.\tag{18}$$

In the limit $n\to\infty$ equation (18) becomes the Rogers-Ramanujan identity (8).

Thus starting with the special case (16) of the q-binomial theorem (5) and by walking along the Bailey chain

$$(a,b)\to(a'b')\to(a'',b''),$$

we have proved equation (8) and as a by-product identities (17) and (18).

Remark. Note that equation (18) is a *finite* sum identity, which yields in the limit $n\to\infty$ the Rogers-Ramanujan identity (8).

In the following section we shall have a closer look at the question how to construct Bailey pairs and to walk along Bailey chains.

2.3 Bailey pairs and walking along Bailey chains. An efficient use of the iteration mechanism of Bailey's Lemma depends on our knowledge of Bailey pairs. For that, one may consult L. Slater's list [29]. But in addition to those classical examples there exist many other interesting pairs, which naturally arise in different contexts, as G. Andrews pointed out in [3].

Now we shall describe some techniques how to find Bailey pairs and how to construct new Bailey pairs out of given ones as in Theorem 2.2. Further we shall see that there are several ways how to walk along Bailey chains, which introduces the more general concept of a Bailey lattice (cf. [13] and [1]).

First we observe:

Theorem 2.3. *A Bailey pair* $(a,b) = \big((a_n),(b_n)\big)$ *is uniquely determined by one of the sequences* (a_n) *or* (b_n)*, respectively.*

Proof. *This is proved by the inversion formula*

$$a_n = \begin{cases} (1-xq^{2n})\displaystyle\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\frac{(xq)_{n+k-1}}{(q)_{n-k}}b_k & \textit{if } n \geq 1 \\ b_0 & n = 0 \end{cases}$$

iff

$$b_n = \sum_{k=0}^{n}\frac{a_k}{(q)_{n-k}(xq)_{n+k}} \tag{19}$$

(cf. Andrews [(4.1), 2]). □

Example: For arbitrary parameter x we obtain with $b_n = \delta_{0,n}(n = 0,1,\cdots)$

$$a_n = (-1)^n q^{\binom{n}{2}}(1-xq^{2n})\frac{(xq)_{n-1}}{(q)_n} \quad (n \geq 1), a_0 = 1$$

(cf. equation (16) in case $x = 1$). As G. Andrews pointed out [3], this Bailey pair lies behind the majority of the identities in L. Slater's list [29].

Remark. Putting $x = q^N$, $B_n = \dfrac{(q)_{2n+N}}{(q)_N}b_n$ and $A_n = a_n$ this inverse relation reads as:

$$B_n = \sum_{k=0}^{n}\begin{bmatrix}2n+N\\ n-k\end{bmatrix}A_k$$

iff

$$A_n = \sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\frac{[2n+N]}{[n+N+k]}\begin{bmatrix}n+N+k\\ n-k\end{bmatrix}B_k, \tag{20}$$

which is a q-analogue of one of J. Riordon's inversion relations of the Legendre type (cf. e.g. Hofbauer [20]).

I. Gessel and D. Stanton have first pointed out [18] that the Bailey transform can be viewed as a matrix inversion, which contains (19) as a special case. Using a matrix inverse observed by D. Bressoud [12] this approch is generalized in [1] in order to establish the concept of Bailey lattices.

The inverse relation (20) independently was derived by C. Krattenthaler [22] as an application of his q-Lagrange formula (cf. also [26]).

A Bailey chain is in fact doubly infinite, for the pair (a, b) can be uniquely reconstructed from (a', b'). Thus we also can move to the left in a Bailey chain:

$$\cdots \leftarrow \big((a_n),(b_n)\big) \leftarrow \big((a'_n),(b'_n)\big) \leftarrow \cdots$$

With respect to the Bailey chain of Theorem 2.2 we explicitly have

Theorem 2.4.

$$b'_n = \sum_{k=0}^{n} \frac{q^{k^2} x^k}{(q)_{n-k}} b_k \iff b_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{q^{\binom{n-k}{2}-n^2}}{(q)_{n-k}} x^{-n} b'_k. \tag{21}$$

Proof. For a proof see equation (3.42) of Andrews [3], which contains (21) as a special case. □

Remark. Substituting $\rho_1 = q^{-a}$, $\rho_2 = q^{-b}$ and $a = q^N$ in equation (3.42) of Andrews [3], equation (21) results from the following inverse relation:

$$A_n = \sum_{k=0}^{n} \begin{bmatrix} p+k \\ k \end{bmatrix} B_{n-k} \iff B_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} p+1 \\ k \end{bmatrix} A_{n-k}. \tag{22}$$

This relation was proved by C. Krattenthaler in [21] as an application of his q-Lagrange formula.

Another important way to produce a new Bailey pair $\big((A_n),(B_n)\big)$ from a given one $\big((a_n),(b_n)\big)$ is the following (see Andrews [2]).

Theorem 2.5. *If* $a_n = a_n(x, q)$, $b_n = b_n(x, q)$ *and*

$$b_n = \sum_{k=0}^{n} \frac{a_k}{(q)_{n-k}(xq)_{n+k}}$$

then

$$B_n = \sum_{k=0}^{n} \frac{A_k}{(q)_{n-k}(xq)_{n+k}},$$

where $A_k := A_k(x,q) := x^k q^{k^2} a_k(x^{-1}, q^{-1})$ *and* $B_k := b_k(x,q) := x^{-k} q^{-k^2-k} b_k(x^{-1}, q^{-1})$.

Example: From the well-known Gaussian identity

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(q)_{n-k}(q)_{n+k}} = \frac{1}{(q^2-1)(q^4-1)\cdots(q^{2n}-1)} \tag{23}$$

(see, e.g., Cigler [15, (1.4.7)]) we obtain by replacing q by q^{-1}

$$\sum_{-\infty}^{\infty} \frac{(-1)^k q^{k^2}}{(q)_{n-k}(q)_{n+k}} = \frac{1}{(q^2-1)(q^4-1)\cdots(q^{2n}-1)} \tag{24}$$

In many applications the following Bailey pair generation is useful (cf. Paule [Lemma 2, 24])

Lemma 2.1. *If* $c \in \mathbf{R}$ *and*

$$b_n = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{ck^2-ck}}{(q)_{n-k}(q)_{n+k}}$$

then

$$q^{-n} b_n = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{ck^2-(c-1)k}}{(q)_{n-k}(q)_{n+k}}.$$

Thus given a Bailey pair $\left((a_n)_{-\infty}^{\infty}, (b_n)_0^{\infty}\right)$, where $a_n = (-1)^n q^{cn^2-cn}$, a new Bailey pair is constituted by $\left((q^n a_n)_{-\infty}^{\infty}, (q^{-n} b_n)_0^{\infty}\right)$.

One example of the wide range of application of the Bailey chain concept lies in the field of multiple series generalizations of identities of the Rogers-Ramanujan type (cf. Andrews [2]).

As an example we consider the analytic counterpart to Gordon's partition theorem [19] discovered by Andrews [5]:

$$\prod_{\substack{n=1 \\ n \not\equiv 0, \pm r \pmod{2s+1} \\ 1 \le r \le s}}^{\infty} (1-q^n)^{-1} = \sum_{n_1 \ge \cdots n_{s-1} \ge 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{s-1}^2+n_r+\cdots+n_{s-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{s-1}}}. \tag{25}$$

According Jacobi's identity (6) the left hand side of (25) is equal to

$$\frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{(s+\frac{1}{2})k^2-(s-r+\frac{1}{2})k} := A_{r,s}.$$

Now the case $r = s$ is immediately obtained by walking along a Bailey chain as far as we arrive at a simple special case (16) of the q-binomial theorem.

$$
\begin{aligned}
A_{s,s} &= \frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{(s+\frac{1}{2})k^2 - \frac{1}{2}k} \\
&\overset{(15)}{=} \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(s+\frac{1}{2})k^2-\frac{1}{2}k}}{(q)_{n_1-k}(q)_{n_1+k}} \\
&\overset{(13)}{=} \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{n_2=0}^{\infty} \frac{q^{n_2^2}}{(q)_{n_1-n_2}} \cdots \sum_{n_{s-1}=0} \frac{q^{n_{s-1}^2}}{(q)_{n_{s-2}-n_{s-1}}} \times \\
&\quad \times \sum_{nB_s=0}^{\infty} \frac{q^{n_s^2}}{(q)_{n_{s-1}-n_s}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k^2-\frac{1}{2}k}}{(q)_{n_s-k}(q)_{n_s+k}} \\
&\overset{(16)}{=} \sum_{n_1 \geq \cdots n_{s-1} \geq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{s-1}^2}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{s-1}}}.
\end{aligned}
$$

We observe that for $r \neq s$ this Bailey chain does not arrive at the q-binomial theorem. Now the way out of that problem is to leave the original chain at some point by switching to a new Bailey pair and continuing the Bailey chain walk with that new pair as a new starting point. If desired, we may repeat this process. Moving like that the authors of [1] would call a walk in a *Bailey lattice*.

As an example we look at the case $r \neq s$ of equation (25) (cf. Paule [24]). We have

$$
A_{r,s} \overset{(15)}{=} \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(s-\frac{1}{2})k^2-(s-r+\frac{1}{2})k}}{(q)_{n_1-k}(q)_{n_1+k}}.
$$

Now we apply (13) $(r-1)$-times, which gives

$$
\begin{aligned}
A_{r,s} &= \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{n_2=0}^{\infty} \frac{q^{n_2^2}}{(q)_{n_1-n_2}} \cdots \sum_{n_r=0}^{\infty} \frac{q^{n_r^2}}{(q)_{n_{r-1}-n_r}} \times \\
&\quad \times \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(s-r+\frac{1}{2})k^2-(s-r+\frac{1}{2})k}}{(q)_{n_r-k}(q)_{n_r+k}}.
\end{aligned}
$$

According to Lemma 2.1 we now switch to a new Bailey pair:

$$
\begin{aligned}
A_{r,s} &= \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{n_2=0}^{\infty} \frac{q^{n_2^2}}{(q)_{n_1-n_2}} \cdots \sum_{n_r=0}^{\infty} \frac{q^{n_r^2}}{(q)_{n_{r-1}-n_r}} \times \\
&\quad \times q^{n_r} \sum_{k=-\infty}^{\infty} \frac{(-1) q^{(s-r+\frac{1}{2})k^2-(s-r-\frac{1}{2})k}}{(q)_{n_r-k}(q)_{n_r+k}}.
\end{aligned}
$$

Now we move one step in the new Bailey chain by (13):

$$A_{r,s} = \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{n_2=0}^{\infty} \frac{q^{n_2^2}}{(q)_{n_1-n_2}} \cdots \sum_{n_r=0}^{\infty} \frac{q^{n_r^2+n_r}}{(q)_{n_{r-1}-n_r}} \times$$
$$\times \sum_{n_{r+1}=0}^{\infty} \frac{q^{n_{r+1}^2}}{(q)_{n_r-n_{r+1}}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(s-r-\frac{1}{2})k^2-(s-r-\frac{1}{2})k}}{(q)_{n_{r+1}-k}(q)_{n_{r+1}+k}}.$$

We repeat this process of applying Lemma 2.1 followed by (13) until we arrive at

$$A_{r,s} = \sum_{n_1\geq\cdots\geq n_s\geq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_s^2+n_r+\cdots+n_s}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{s-1}-n_s}} \times$$
$$\times \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k^2-\frac{1}{2}k}}{(q)_{n_s-k}(q)_{n_s+k}},$$

which now is reduced by the special case (16) of the q-binomial theorem to

$$A_{r,s} = \sum_{n_1\geq\cdots\geq n_{s-1}\geq 0} \frac{q^{n_1^2+\cdots+n_{s-1}^2+n_r+\cdots+n_{s-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{s-1}}}.$$

This proves equation (25). □

Now there is another Bailey chain walk to prove equation (25) which brings identity (14), the counterpart of (13), into play (cf. Agarwal-Andrews-Bressoud [1]). To demonstrate this we first need

Lemma 2.2. *If $c \in \mathbf{R}$ and*

$$b_n = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(c+1)k^2-ck}}{(q)_{n-k}(q)_{n+k}}$$

then

$$b_n = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(c+1)k^2+ck}}{(q)_{n-k}(q)_{n+1+k}},$$

and vice versa.

Proof. To prove Lemma 2.2 is equivalent to show that

$$\sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix} (-1)^k q^{(c+1)k^2+ck} = (1-q^{2n+1}) \sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-k \end{bmatrix} (-1)^k q^{(c+1)k^2-ck}.$$

Using the recurrence formula

$$\begin{bmatrix} r+1 \\ s \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix} + q^{r-s+1} \begin{bmatrix} r \\ s-1 \end{bmatrix}$$

the left hand side of this equation becomes

$$\sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-k \end{bmatrix} (-1)^k q^{(c+1)k^2+ck} - q^{n+1} \sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-k \end{bmatrix} (-1)^k q^{(c+1)k^2-(c+1)k}$$
$$= (1-q^{2n+1}) \sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-k \end{bmatrix} (-1)^k q^{(c+1)k-ck}.$$

The last line follows by Lemma 2.1. $\square$

Behind Lemma 2.2 lies the following important observation:
It is possible to pass from a Bailey pair $\big((a_n),(b_n)\big)$ relative to x, *i.e.*

$$b_n = \sum_{k=0}^{n} \frac{a_k}{(q)_{n-k}(xq)_{n+k}},$$

to a Bailey pair $\big((a'_n),(b'_n)\big)$ relative to y, *i.e.*

$$b'_n = \sum_{k=0}^{n} \frac{a'_k}{(q)_{n-k}(yq)_{n+k}}.$$

In practice y will be x times an integer power of q.

Example: By Lemma 2.2 we immediately get

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\frac{3}{2}k^2+\frac{1}{2}k}}{(q)_{n-k}(q)_{n+1+k}} = \frac{1}{(q)_n}, \tag{26}$$

the counterpart of the Bailey pair corresponding to equation (17).
Remark. This problem was first considered by D. Bressoud [1] and Agarwal-Andrews-Bressoud [13] in the general context of matrix inversion and motivated these authors to introduce the notion of a *Bailey lattice*.

Now we present a second proof of equation (25) by a different Bailey chain walk:

$$A_{r,s} \overset{(15)}{=} \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(s-\frac{1}{2})k^2-(s-r+\frac{1}{2})k}}{(q)_{n_1-k}(q)_{n_1+k}}$$
$$\overset{(13)}{=} \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{n_2=0}^{\infty} \frac{q^{n_2^2}}{(q)_{n_1-n_2}} \cdots \sum_{n_{r-1}=0}^{\infty} \frac{q^{n_{r-1}^2}}{(q)_{n_{r-2}-n_{r-1}}} \times$$
$$\times \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(s-r+\frac{3}{2})k^2-(s-r+\frac{1}{2})k}}{(q)_{n_{r-1}-k}(q)_{n_{r-1}+k}}$$
$$\overset{(\text{Lemma 2.2})}{=} \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{n_2=0}^{\infty} \frac{q^{n_2^2}}{(q)_{n_1-n_2}} \cdots \sum_{n_{r-1}=0}^{\infty} \frac{q^{n_{r-1}^2}}{(q)_{n_{r-2}-n_{r-1}}} \times$$
$$\times \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(s-r+\frac{3}{2})k^2+(s-r+\frac{1}{2})k}}{(q)_{n_{r-1}-k}(q)_{n_{n-r}+1+k}}.$$

Now we walk along the Bailey chain by repeated application of (14) as far as we arrive at

$$\begin{aligned}
A_{r,s} &= \sum_{n_1=0}^{\infty} q^{n_1^2} \sum_{n_2=0}^{\infty} \frac{q^{n_2^2}}{(q)_{n_1-n_2}} \cdots \sum_{n_{r-1}=0}^{\infty} \frac{q^{n_{r-1}^2}}{(q)_{n_{r-2}-n_{r-1}}} \times \\
&\quad \times \sum_{n_r=0}^{\infty} \frac{q^{n_r^2+n_r}}{(q)_{n_{r-1}-n_r}} \cdots \sum_{n_{s-1}=0}^{\infty} \frac{q^{n_{s-1}^2+n_{s-1}}}{(q)_{n_{s-2}-n_{s-1}}} \times \\
&\quad \times \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\frac{3}{2}k^2+\frac{1}{2}k}}{(q)_{n_{s-1}-k}(q)_{n_{s-1}+1+k}} \\
&= \sum_{n_1 \geq \cdots \geq n_{s-1} \geq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{s-1}^2+n_r+\cdots+n_{s-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_1}\cdots(q)_{n_{s-1}}}.
\end{aligned}$$

The last line follows from equation (26). This proves equation (25). □

In this proof we first walked along the Bailey chain corresponding to identity (13). After that, we made a side-step changing the parameter $x = 1$ to $x = q$ and continued our walk along the Bailey chain corresponding to identity (14). Finally we arrived at a Bailey pair, which we recognized as the $(x = q)$-counterpart (26) of (17).

Thus we have seen that identity (25) has been proved by two different walks in a Bailey lattice.

In Agarwal-Andrews-Bressoud [1] identity (25) is proved as a special case of a very general theorem [Theorem 3.1, 1], which is based on a Bailey lattice walk with one parameter change of x to xq^{-1}.

Concluding this section we make some remarks on changing the parameter x.

Given a Bailey pair $\big((a_n),(b_n)\big)$, i.e.

$$b_n = \sum_{k=0}^{n} \frac{a_k}{(q)_{n-k}(xq)_{n+k}},$$

we substitute $x = q^N$. This motivates the following

Definition. *Two sequences (A_n), (B_n) form a q-*binomial *(or simply* binomial*)* Bailey pair *relative to N, if*

$$B_n = \sum_{k=0}^{n} \begin{bmatrix} 2n+N \\ n-k \end{bmatrix} A_k.$$

Each Bailey pair $\big((a_n),(b_n)\big)$, where $b_n = \sum_{k=0}^{n} \frac{a_k}{(q)_{n-k}(xq)_{n+k}}$ with $x = q^N$, can be turned into a binomial Bailey pair (by defining $A_n := a_n$ and $B_n := \frac{(q)_{2n+N}}{(q)_N} b_n$) and vice versa.

Now changing the parameter x to xq^{-1} is expressed as

Theorem 2.6. *If* $((A_n),(B_n))$ *is a binomial Bailey pair relative to* N, *i.e.*

$$B_n = \sum_{k=0}^{n} \begin{bmatrix} 2n+N \\ n-k \end{bmatrix} A_k,$$

then $((A'_n),(B'_n))$ *is a binomial Bailey pair relative to* $N-1$, *i.e.*

$$B'_n = \sum_{k=0}^{n} \begin{bmatrix} 2n+N-1 \\ n-k \end{bmatrix} A'_k,$$

where

$$A'_n := \frac{A_k}{[2k+N]} - q^{2k+N-2}\frac{A_{k-1}}{[2k+N-2]} \qquad (A_{-1} := 0)$$

and

$$B'_n := \frac{B_n}{[2n+N]}.$$

Proof. For $k \neq n$ we conclude from

$$\begin{bmatrix} 2n+N \\ n-k \end{bmatrix} = \frac{[2n+N]}{[n-k]} \begin{bmatrix} 2n+N-1 \\ n-k-1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2n+N-1 \\ n-k \end{bmatrix} = \frac{[n+k+N]}{[n-k]} \begin{bmatrix} 2n+N-1 \\ n-k-1 \end{bmatrix} = \left(\frac{[2k+N]}{[n-k]} + q^{2k+N} \right) \begin{bmatrix} 2n+N-1 \\ n-k-1 \end{bmatrix}$$

that

$$\begin{bmatrix} 2n+N \\ n-k \end{bmatrix} = \frac{[2n+N]}{[2k+N]} \left(\begin{bmatrix} 2n+N-1 \\ n-k \end{bmatrix} - q^{2k+N} \begin{bmatrix} 2n+N-1 \\ n-k-1 \end{bmatrix} \right).$$

Thus

$$\begin{aligned} B'_n &= \frac{B_n}{[2n+N]} = \frac{1}{[2n+N]} \sum_{k=0}^{n} \begin{bmatrix} 2n+N \\ n-k \end{bmatrix} A_k \\ &= \frac{A_n}{[2n+N]} + \sum_{k=0}^{n-1} \begin{bmatrix} 2n+N-1 \\ n-k \end{bmatrix} \frac{A_k}{[2k+N]} - \sum_{k=1}^{n} \begin{bmatrix} 2n+N-1 \\ n-k \end{bmatrix} \frac{q^{2k+N-2} A_{k-1}}{[2k+N-2]} \\ &= \begin{bmatrix} 2n+N-1 \\ n \end{bmatrix} \frac{A_0}{[N]} + \sum_{k=1}^{n} \begin{bmatrix} 2n+N-1 \\ n-k \end{bmatrix} A'_k, \end{aligned}$$

which proves Theorem 2.6. ∎

It is easily seen that the important Lemma 2.2 is contained in Theorem 2.6 as a special case: (set $N=1$ and $A_k = (-1)^k q^{(c+1)k^2+ck}$).

As we shall see in the next section a generalization of Theorem 2.6, which is the key result (Lemma 1.2) of the Agarwal-Andrews-Bressoud paper [1], is also a straight-forward application of Theorem 2.6.

What happens if we want to change the parameter x to xq? We treat this question as a transition from a binomial Bailey pair relative to $N-1$ to a binomial Bailey pair relative to N. The inversion of Theorem 2.6 yields

Theorem 2.7. *If $\big((A'_n),(B'_n)\big)$ is a binomial Bailey pair relative to $N-1$, i.e.*

$$B'_n = \sum_{k=0}^{n} \begin{bmatrix} 2n+N-1 \\ n-k \end{bmatrix} A'_k,$$

then $\big((A_n),(B_n)\big)$ is a binomial Bailey pair relative to N, i. e.

$$B_n = \sum_{k=0}^{n} \begin{bmatrix} 2n+N \\ n-k \end{bmatrix} A_k,$$

where

$$A_k := [2k+N] q^{k^2+(N-1)k} \sum_{j=0}^{k} q^{-j^2-(N-1)j} A'_j$$

and

$$B_n := [2n+N] B'_n.$$

Proof. *The proof is a simple matrix inversion using the inversion relation*

$$x_k = \sum_{j=0}^{k} y_j \iff y_k = x_k - x_{k-1} \quad (x_{-1} := 0),$$

with

$$x_k := q^{-k^2-(N-1)k} \frac{A_k}{[2k+N]}, y_k := q^{-k^2-(N-1)k} A'_k,$$

assuming that (A_k) and (A'_k) are connected as in Theorem 2.6. □

Example: Identity (24) can be rewritten as

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix} A'_k = B'_n,$$

where $A'_0 = 1$, $A'_k = 2(-1)^k q^{k^2} \quad k \geq 1$ and $B'_n = (q;q^2)_n$.
Now Theorem 2.7 with $N=1$ yields

$$A_k = [2k+1](-1)^k q^{k^2} \text{and} B_n = [2n+1](q;q^2)_n,$$

which gives

$$\sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix} (-1)^k q^{k^2} = (q;q^2)_{n+1}$$

as the $(N=1)$-counterpart of (24).

3. Generalizations

3.1 An extension of Bailey's Lemma. In this section we shall consider a generalization of Bailey's Lemma, which contains Theorem 2.1 as a limiting case.

Definition. *For $n_1 + n_2 + \cdots + n_j = n$ we define*

$$\begin{bmatrix} n \\ n_1, n_2, \cdots, n_j \end{bmatrix} := \frac{[n]!}{[n_1]![n_2]! \cdots [n_j]!}$$

(q-multinomial coefficient).

We formulate this extended form of Bailey's Lemma in the language of q-multinomial coefficients:

Theorem 3.1. *If $\big((A_n), (B_n)\big)$ is a binomial Bailey pair relative to N, then $\big((A'_n), (B'_n)\big)$ is also a binomial Bailey pair relative to N, where*

$$A'_k := \begin{bmatrix} 2a+N \\ a-k \end{bmatrix} \begin{bmatrix} 2b+N \\ b-k \end{bmatrix} q^{Nk+k^2} A_k$$

and

$$B'_n := \frac{[2a+N]![2b+N]![2n+N]!}{[a+b+N]![b+n+N]![n+a+N]!} \sum_{j=0}^{n} \begin{bmatrix} a+b+n+N-j \\ a-j, b-j, n-j, 2j+N \end{bmatrix} q^{N_j + j^2} B_j.$$

This is what G. Andrews denotes by "Baileys's Lemma" in [(2.3)-(2.6), 2] with $\rho_1 = q^{-a}, \rho_2 = q^{-b}$ and the parameter a replaced by $q^N (= x)$.

W. N. Bailey described in §4 of [9] how to get Theorem 3.1, but he never carried it out explicitly. The reason for that may lie in the fact that he did not observe the iteration mechanism hidden in his Lemma (Theorem 2.2). Bailey's Lemma was first stated in full generality by G. Andrews in [2] and P. Paule [23] independently observed the important special cases $N = 0$ and $N = 1$ of

$$\sum_{k=0} \begin{bmatrix} a+b+N \\ a-k \end{bmatrix} \begin{bmatrix} b+c+N \\ b-k \end{bmatrix} \begin{bmatrix} c+a+N \\ c-k \end{bmatrix} q^{k^2+Nk} A_k$$
$$= \sum_{j=0} \begin{bmatrix} a+b+c+N-j \\ a-j, b-j, c-j, 2j+N \end{bmatrix} q^{j^2+Nj} B_j, \tag{27}$$

where

$$B_j = \sum_{k=0}^{j} \begin{bmatrix} 2j+N \\ j-k \end{bmatrix} A_k$$

(see D1 and D2 of Paule [23]).

It is easily checked that (27) is an equivalent reformulation of Theorem 3.1.

Now we come back to the key result Lemma 1.2 of Agarwal-Andrews-Bressoud [1], which can be rewritten as

$$\sum_{k=0}^{n} \begin{bmatrix} a+b+N-1 \\ a-k \end{bmatrix} \begin{bmatrix} b+n+N-1 \\ b-k \end{bmatrix} \begin{bmatrix} n+a+N-1 \\ n-k \end{bmatrix} q^{k^2+(N-1)k} A'_k$$
$$= \sum_{j=0}^{n} \begin{bmatrix} a+b+n+N-1 \\ a-j, b-j, n-j, 2j+N-1 \end{bmatrix} q^{j^2+(N-1)j} B'_j, \tag{28}$$

where

$$A'_k = \frac{A_k}{[2k+N]} - q^{2k+N-2} \frac{A_{k-1}}{[2k+N-2]} \quad (A_{-1} := 0)$$

and

$$B'_n = \frac{B_n}{[2n+N]}.$$

(See (1.9) of Agarwal-Andrews-Bressoud [1] with $\rho = q^{-a}, \delta = q^{-b}$ and $a = q^N$.)

Now, if $\big((A_n), (B_n)\big)$ forms a binomial Bailey pair, equation (28), i. e. Lemma 1.2 of [1], is immediately proved by (27) and Theorem 2.6.

If we let c tend to infinity in equation (27), we obtain a formula, which is equivalent to Theorem 2.1. (cf. Paule [24]). In this resulting formula it is possible to send b to infinity, which gives identity (12) with x replaced by q^N, and finally a to infinity, which yields a general equation (15) (with parameter $x = q^N$ instead $x = 1$).

G. Andrews observed [3] that also in this general case we can walk backwards in the Bailey chain. Thus is, because walking along the Bailey chain now means to construct a sequence of Bailey pairs

$$\cdots \to \big((A_n), (B_n)\big) \to \big((A'_n), (B'_n)\big) \to \cdots$$

according to Theorem 3.1, a process, which can be reversed by

$$A_k = \begin{bmatrix} 2a+N \\ a-k \end{bmatrix}^{-1} \begin{bmatrix} 2b+N \\ b-k \end{bmatrix}^{-1} q^{-k^2-Nk} A'_k$$

and

$$B_n = q^{-n^2-Nn} \frac{(q)_{a-n}(q)_{b-n}}{(q)_a (q)_b (q)_{a+N} (q)_{b+N}} \times$$
$$\times \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} a+b+N+1 \\ n-k \end{bmatrix} B'_k.$$

Also this extension of (21) is an application of the inverse relation (22).

If $N=0$ or $N=1$ it is possible to symmetrize identity (27) as follows:

$$\sum_{k=-\infty}^{\infty} \begin{bmatrix} a+c+\rho \\ a-k \end{bmatrix} \begin{bmatrix} b+c+\rho \\ b-k \end{bmatrix} \begin{bmatrix} c+a+\rho \\ c-k \end{bmatrix} c_k \tag{29}$$

$$= \sum_{j=0}^{\infty} \begin{bmatrix} a+b+c+\rho-j \\ a-j, b-j, c-j, 2j+\rho \end{bmatrix} q^{j^2+\rho j} \times$$

$$\times \sum_{k=-\infty}^{\infty} \begin{bmatrix} 2j+1 \\ j-k \end{bmatrix} q^{-k^2-\rho k} c_k,$$

where $\rho=0$ or 1 (cf. Paule [23]).

Example. In $[(B)$ and $(B'), 14]$ P. Cartier and D. Foata proved

$$\sum_{k=-\infty}^{\infty} \binom{b+c+\rho}{c+k} \binom{c+a+\rho}{a+k} \binom{a+b+\rho}{b+k} u^{p+k} v^{p-k+\rho}$$

$$= \sum_{n=0}^{p} \frac{(a+b+c-n+\rho)!}{(a-n)!(b-n)!(c-n)!} (uv)^{p-n} \frac{(u+v)^{2n+\rho}}{(2n+\rho)!}, \tag{30}$$

where $\rho=0$ or 1. (For the case $\rho=0$ see D. Foata [17])

By (29) and the q-binomial theorem in the form (5) or its $(N=1)$-counterpart

$$\sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix} (-1)^k x^k q^{\frac{1}{2}k^2} = (x^{-1} q^{\frac{1}{2}})_n (x q^{\frac{1}{2}})_{n+1}, \tag{31}$$

respectively, the following q-analogue of (30) can be given:

$$\sum_{k=-\infty}^{\infty} \begin{bmatrix} b+c+\rho \\ c+k \end{bmatrix} \begin{bmatrix} c+a+1 \\ a+k \end{bmatrix} \begin{bmatrix} a+b+\rho \\ b+k \end{bmatrix} q^{\frac{3}{2}k^2-\frac{1}{2}k} (\frac{u}{v})^k \tag{32}$$

$$= \sum_{n=0} \frac{[a+b+c-n+\rho]!}{[a-n]![b-n]![c-n]!} \frac{(-\frac{v}{u} q^{1-\rho})_n (-\frac{u}{v} q^{\rho})_{n+\rho}}{[2n+\rho]!},$$

where $\rho=0$ or 1.

Remark. In proving equation (30) P. Cartier and D. Foata gave a purely combinatorial proof of

$$\binom{b+c+\rho}{c+k} \binom{c+a+\rho}{a+k} \binom{a+b+\rho}{b+k} \tag{33}$$

$$= \sum_{n} \frac{(a+b+c+n-\rho)!}{(a-n)!(b-n)!(c-n)!(n+k)!(n-k+\rho)!},$$

where $\rho=0$ or 1 (a *Pfaff-Saalschütz* summation).

From that the proof of (29), the extension of Bailey's Lemma in the cases $N = 0$ or 1, for $q = 1$ is simple: one just has to multiply equation (32) by coefficients c_k and to sum over all possible k's and finally change the order of summation.

Recently [31] D. Zeilberger q-analogized the Cartier-Foata proof of (32) to show

$$\begin{bmatrix} b+c \\ c+k \end{bmatrix} \begin{bmatrix} c+a \\ a+k \end{bmatrix} \begin{bmatrix} a+b \\ b+k \end{bmatrix} = \tag{34}$$
$$= \sum_n q^{n^2-k^2} \frac{[a+b+c-n]!}{[a-n]![b-n]![c-n]![n+k]![n-k]!}.$$

This gives equation (29) with $\rho = 0$ as described above.

3.2 Concluding Remarks. Bailey's Lemma and the concept of Bailey chains have a wide range of application in additive number theory, combinatories and special functions, and physics (R. J. Baxter's hard hexagon model in statistical mechanics, see e.g. R. J. Baxter [11]). For these applications one should consult G. Andrews [2] and his q-series compendium [3], which contains also an extensive reference list.

In [2] G. Andrews pointed out that *all* of the 130 Rogers-Ramanujan type identities given in Slater's list [29] posess multiple series generalizations as equation (25), which contains the Rogers-Ramanujan identities. (e.g. $s = 2$ and $r = 2$ in (25) yields (7)).

Let us close with an example of H. Cohen [16], where two important special cases of equation (25) arise in an *algebraic* context:

$$\sum_{\substack{G \\ p^{k-1}G=0}} | AutG |^{-1} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm k \pmod{2k+1}}}^{\infty} (1-p^{-n})^{-1}$$

and

$$\sum_{\substack{G \\ p^{k-1}G=0}} \frac{| AutG |^{-1}}{| G |} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 1 \pmod{2k+1}}}^{\infty} (1-p^{-n})^{-1}$$

where the sums are over all isomorphism classes of abelian p-groups annihilated by p^{k-1} (p a prime).

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