ON A DECOMPOSITION OF SQUARE MATRICES OVER A RING WITH IDENTITY

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Let R be a not necessarily commutative ring with 1 and let P be an $(n \times n)$-matrix over R. Then P is called a *permutation matrix* if, and only if, the following conditions are satisfied:

(1) $P_{ij} \in \{0, 1\}$ for all $i, j \in \{0, 1, \ldots, n-1\}$.
(2) Each row of P contains exactly one 1.
(3) Each column of P contains exactly one 1.

Denote by S_n the symmetric group on the set $\{0, 1, \ldots, n-1\}$. If $\pi \in S_n$ then we define $P(\pi)$ by

$$P(\pi)_{ij} := \begin{cases} 1 & \text{if } \pi(j) = i, \\ 0 & \text{else.} \end{cases}$$

Then $P(\pi)$ is a permutation matrix and all permutations matrices are obtained in this way, as is well-known.

The set $\mathrm{Mat}_n(K)$ of all $(n \times n)$-matrices over the field K forms a vector space of dimension n^2 over K and it belongs to the folklore of permutation matrices that

$$\dim(\mathrm{span}(\{P(\pi) : \pi \in S_n\})) = (n-1)^2 + 1.$$

Linear Algebra tells us that there exists a basis of the span of permutation matrices consisting entirely of permutation matrices. Searching for such a basis yields a much more general theorem.

Theorem 1. *Let R be a not necessarily commutative ring with 1 and let n by a positive integer. Consider the set $\mathrm{Mat}_{n+1}(R)$ of all $((n+1) \times (n+1))$-matrices over R as a left R-module. Define the submodules V_1, V_2, V_3 of $\mathrm{Mat}_{n+1}(R)$ as follows:*

(1) *V_1 consists of all $a \in \mathrm{Mat}_{n+1}(R)$ such that $a_{ni} = a_{in} = 0$ for $0 \leq i \leq n-1$ and $a_{nn} = \sum_{j=0}^{n-1} a_{n-1,j}$.*

(2) *Let a be an n- and b be an $(n-1)$-tuple over R. Define the matrices $C(a)$ and $D(b)$ by*

$$\begin{pmatrix} 0 & a_n & \ldots & a_3 & a_2 & a_1 \\ a_1 & 0 & a_n & \ldots & a_3 & a_2 \\ a_2 & a_1 & 0 & a_n & \ldots & a_3 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ a_{n-1} & \ldots & a_2 & a_1 & 0 & a_n \\ a_n & \ldots & a_3 & a_2 & a_1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} X & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & b_{n-1} & \dots & b_2 & b_1 \\ 0 & b_1 & 0 & b_{n-1} & \dots & b_2 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & b_2 & b_1 & 0 & b_{n-1} \\ 0 & b_{n-1} & \dots & b_2 & b_1 & 0 \end{pmatrix}$$

respectively, where X is the sum of the b_i's. Then V_2 consists of all matrices of the form $C(a) + D(b)$.

(3) *V_3 is the set of all $a \in \mathrm{Mat}_{n+1}(R)$ with $a_{ij} = 0$ for all (i,j) different from $(n,0)$ and (n,n).*

Then $\mathrm{Mat}_{n+1}(R)$ is the direct sum of V_1, V_2, V_3. Moreover, V_1 is, as an R-module, isomorphic to $\mathrm{Mat}_n(R)$.

The proof is left as an exercise to the reader.

Theorem 2. *Same assumptions and notations as in Theorem 1. Define the permutations $\alpha, \beta \in S_{n+1}$ by $\alpha := (0,1,2,\dots,n)$ and $\beta := (1,2,\dots,n)$ and set $B(i) := P(\alpha^i)$ for $i = 1,2,\dots,n$ and $B(n+i) := P(\beta^i)$ for $i = 1,2,\dots,n-1$. Then $\{B(i) : 1 \le i \le 2n-1\}$ is a basis for V_2.*

Proof. Straightforward.

As a consequence of Theorems 1 and 2 we get the following theorem.

Theorem 3. *Denote by $\mathrm{const}_{n+1}(R)$ the set of all $a \in \mathrm{Mat}_{n+1}(R)$ such that there exists an $r \in R$ with $\sum_{k=0}^n a_{kj} = r = \sum_{l=0}^n a_{il}$ for all i and j. Then $\mathrm{const}_{n+1}(R)$ is a direct summand of $\mathrm{Mat}_{n+1}(R)$ having a basis consisting of n^2+1 permutation matrices.*

Theorems 1 and 2 give a recursion for a basis of $\mathrm{const}_{n+1}(R)$ as well as for a basis of a complement of $\mathrm{const}_{n+1}(R)$. As an example, we list the 17 permutations whose permutation matrices form a basis of $\mathrm{const}_5(R)$. The /'s indicate the steps in the recursion. Moreover, we list a set of 8 matrices forming a basis of a complement of $\mathrm{const}_5(R)$.

$$\begin{gathered}(0) \quad / \quad (0,1) \quad / \quad (0,1,2),\ (0,2,1),\ (1,2) \quad / \quad (0,1,2,3),\ (0,2)(1,3), \\ (0,3,2,1),\ (1,2,3),\ (1,3,2), \quad / \quad (0,1,2,3,4),\ (0,2,4,1,3),\ (0,3,1,4,2), \\ (0,4,3,2,1),\ (1,2,3,4),\ (1,3)(2,4),\ (1,4,3,2).\end{gathered}$$

$$\begin{matrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{matrix} \qquad \begin{matrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{matrix} \qquad \begin{matrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{matrix} \qquad \begin{matrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{matrix}$$

$$
\begin{matrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \end{matrix}
\qquad
\begin{matrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{matrix}
\qquad
\begin{matrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \end{matrix}
\qquad
\begin{matrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \end{matrix}.
$$

The recursion for the basis of $\mathrm{const}_{n+1}(R)$ clearly shows that $\mathrm{const}_{n+1}(R)$ has a basis consisting of n^2+1 permutation matrices.

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