SOME COMBINATORIAL PROPERTIES OF COMPLETE SEMI-THUE SYSTEMS

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A *reduction system* $(R, \longrightarrow)$ consists of a set R and a binary relation $\longrightarrow$ on R. Let $\xrightarrow{*}$ be the reflexive-transitive closure of $\longrightarrow$ and $[x]$ the class of an $x \in R$ with respect to the equivalence generated by $\longrightarrow$. x is called *irreducible* (or *in normal form*) if there is no $y \in R$ such that $x \longrightarrow y$. A reduction system can have the following properties:

- *Chain Condition:* There is no infinite chain $x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow \cdots$ in R. (Then $\longrightarrow$ is called *terminating* or *Noetherian.*)
- *Confluence:* $\forall w,\, x,\, y \in R$: $(w \xrightarrow{*} x \wedge w \xrightarrow{*} y \Rightarrow \exists z \in R: \; x \xrightarrow{*} z \wedge y \xrightarrow{*} z)$.
- *Completeness:* Chain condition and confluence.

If a reduction system is complete, normal forms always exist and are unique. See [3] for further details.

Let Σ be a finite alphabet. Σ^* denotes the free monoid over Σ and $\Box$ the empty word. A *semi-Thue system* (STS) on Σ is a subset $S \subseteq \Sigma^* \times \Sigma^*$. Each element (u, v) of S is called a *rule* and written in the form $u \longrightarrow v$. An STS S defines a reduction relation $\longrightarrow$ on Σ^* by $xuy \longrightarrow xvy \Leftrightarrow (u, v) \in S$.

Let $OV(u) = \{x \in \Sigma^* \mid \exists y, z \in \Sigma^* : \; u = yx = xz\} \backslash \{\Box, u\}$ be the set of *non-trivial self-overlaps* of $u \in \Sigma^*$. Generalizing results of Book [2], Otto and Wrathall [6], one obtains the following.

Theorem. *Let the single-rule STS $u \longrightarrow v$ satisfy the chain condition, and let $u = u_0 u_1 u_2 \ldots u_k$ $(k \geq 0)$, such that $OV(u) = \{u_1 u_2 \ldots u_k, u_2 \ldots u_k, \ldots, u_k\}$. The STS is confluent if and only if one of the following two conditions is satisfied:*

(a) *v has $u_1 u_2 \ldots u_k$ as a self-overlap, or*
(b) *there is a $j \in \{1, 2, \ldots, k+1\}$, such that $v = u_j u_{j+1} \ldots u_k$ (for $j = k+1$: $v = \Box$) and $u = u_{j-1}^j u_j u_{j+1} \ldots u_k$.*

For the case $v = \Box$, this means that u must be a power of a word y without proper self-overlap [2]. The classes $|w|$ of such complete systems $y^r \longrightarrow \Box$ are deterministic context-free languages [1]. One can show that the unambiguous grammar $(\Sigma \cup \{S\},\, \Sigma,\, P,\, S)$ with $P = \{S \longrightarrow \Box,\; S \longrightarrow (a_1 S a_2 S \ldots a_{k-1} S a_k)^r S\}$, where $y = a_1 a_2 \ldots a_k$ $(a_i \in \Sigma)$, generates $|\Box|$. (There is a similar grammar for the general case $|w|$.) From this presentation it follows that the structure generating function $S(z)$ (cf. [4]) of $|\Box|$ is the unique solution of the equation $S(z) = 1 + z^{rk}(S(z))^{r(k-1)+1}$ in $\mathbb{Z}[[z]]$, which is a variant of the well-known "trinomial equation" (T): $A(x) = 1 + x(A(x))^t$ $(t \in \mathbb{N})$. (T) has the unique solution $A(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_n = \frac{1}{n(t-1)+1}\binom{tn}{n}$ (the a_n having a lot of combinatorial interpretations, see, e.g., [5]). This is usually proved by the Lagrange inversion formula, but it can also be deduced from the set equation $[a^p] = [a^{p-1}]a + [a^{p+k-1}]b$ for the special STS $a^{k-1}b \longrightarrow \Box$. The enumeration of words in the classes $[w]$, w irreducible, can be carried out in the same manner.

References

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