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LINEARIZATION COEFFICIENTS FOR THE JACOBI POLYNOMIALS

BY

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RÉSUMÉ. — Une formule explicite pour les coefficients de linéarisation des polynômes de Jacobi a été donnée par RAHMAN, d'où l'on tire, sans calcul, les propriétés de positivité. L'obtention de la formule de RAHMAN par des méthodes combinatoires semble malaisée. On peut cependant donner plusieurs interprétations combinatoires de l'intégrale du produit de polynômes de Jacobi $\prod_i P_{n_i}^{(\alpha,\beta)}(x)$ et en déduire une évaluation dans le cas particulier où $n_1 = n_2 + \cdots + n_m$.

ABSTRACT. — The explicit non-negative representation of the linearization coefficients of the Jacobi polynomials obtained by RAHMAN seems to be difficult to be derived by combinatorial methods. However several combinatorial interpretations can be provided for the integral of the product of Jacobi polynomials $\prod_i P_{n_i}^{(\alpha,\beta)}(x)$ and furnish an evaluation of this integral in the particular case where $n_1 = n_2 + \cdots + n_m$.

1. Introduction. — Standard definition for the Jacobi polynomials reads :

$$\begin{aligned}P_n^{(\alpha,\beta)}(x) &= \sum_{j=0}^{n} \binom{n+\alpha}{n-j}\binom{n+\beta}{j}\left(\frac{x-1}{2}\right)^j\left(\frac{x+1}{2}\right)^{n-j}\\ &= \frac{2^{-n}}{n!}\sum_{j=0}^{n}\binom{n}{j}(\alpha+1+n-j)_j\,(\beta+1+j)_{n-j}\,(x-1)^{n-j}(x+1)^j.\end{aligned}$$

(See, e.g., [Er], [Sz]). Let $\mathbf{n} = (n_1, \ldots, n_m)$ and consider the integral

$$I_{\mathbf{n}} = \int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta \prod_{i=1}^{m} P_{n_i}^{(\alpha,\beta)}(x)\,dx.$$

Using the classical evaluation

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta\,dx = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$

it is readily seen that

$$(1.1)\quad I_{\mathbf{n}} = \frac{2^{\alpha+\beta+1}}{\prod_i n_i!\,(\alpha+\beta+2)_{\Sigma n_i}} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} L_{\mathbf{n}},$$

with

$$(1.2)\quad L_{\mathbf{n}} = \sum_{\mathbf{k}} (-1)^{\Sigma(n_i-k_i)}\,(\alpha+1)_{\Sigma(n_i-k_i)}\,(\beta+1)_{\Sigma k_i} \times \prod_i \binom{n_i}{k_i} (\alpha+1+n_i-k_i)_{k_i}\,(\beta+1+k_i)_{n_i-k_i}.$$

The linearization problem consists of finding an appropriate representation for $I_{\mathbf{n}}$ in such a way that non-negative properties of $I_{\mathbf{n}}$ are directly apparent from the representation itself. Along those lines RAHMAN [Ra] found the following fantastic formula involving the series ${}_9F_8$: let $s+1 \le n$, $0 \le j \le 2n-2s$ and let

$$(1.3)\quad I_{s+j,n-s,n} = \frac{(\alpha+1)_{s+j}(\alpha+1)_{n-s}(\alpha+1)_n}{(s+j)!\,(n-s)!\,n!} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2^{-(\alpha+\beta+1)}\Gamma(\alpha+\beta+1)} \times \frac{(s+j)!\,(\beta+1)_{s+j}}{(\alpha+\beta)_{s+j}(\alpha+\beta+1)_{s+j}(2s+2j+\alpha+\beta+1)} g(s+j,n-s,n).$$

Then, for j even

$$\begin{aligned} g(s+j,n-s,n) = {} & \frac{\alpha+\beta+1+2s+2j}{\alpha+\beta+1}(\alpha+\beta+1+n-s)_{n-s} \\ & \times \frac{(\alpha+1)_{s+j}(\beta+1)_n(\alpha+\beta+1)_{2s+j}(\alpha+\beta+1)_j n!}{(\alpha+1)_s(\alpha+1)_{n-s}(\beta+1)_{s+j}(\alpha+\beta+2)_{2n+j}\, s!\, j!} \\ & \times \frac{(s-n)_{j/2}(\alpha+\beta+n+1)_{j/2}}{\left(s-n-\dfrac{\alpha+\beta}{2}\right)_{j/2}(s+1)_{j/2}(\alpha+1)_{j/2}} \\ & \times \frac{(s-n-\alpha)_{j/2}(\beta+n+1)_{j/2}(1/2)_{j/2}}{\left(\dfrac{1}{2}+s-n-\dfrac{\alpha+\beta}{2}\right)_{j/2}(s+1)_{j/2}(\alpha+1)_{j/2}} \\ & \times {}_9F_8\left[\begin{matrix} \alpha, 1+\dfrac{\alpha}{2}, \alpha+\dfrac{1}{2}, \dfrac{\alpha-\beta}{2}\dfrac{\alpha-\beta+1}{2}, \\ \dfrac{\alpha}{2}, \dfrac{1}{2}, \dfrac{\alpha+\beta}{2}+1, \dfrac{\alpha+\beta+1}{2}, \end{matrix}\right. \\ & \left.\begin{matrix} \alpha+\beta+n+1+\dfrac{j}{2}, s-n+\dfrac{j}{2}, -s-\dfrac{j}{2}, -\dfrac{j}{2} \\ -\beta-n-\dfrac{j}{2}, \alpha+n+1-s-\dfrac{j}{2}, \alpha+s+1+\dfrac{j}{2}, \alpha+1+\dfrac{j}{2} \end{matrix}\right] \end{aligned}$$

and a corresponding formula for j odd. Note that in RAHMAN's paper [Ra, p. 917, formula (1.7)] the factor $(\alpha+\beta+1+n-s)_{n-s}$ occurring after the first fraction above is missing. As proved by RAHMAN [Ra], the foregoing formula shows that if j, s, n are non-negative integers with $s+1 \leq n$, $0 \leq j \leq 2n-2s$, then $g(s+j, n-s, n) \geq 0$ whenever $\alpha \geq \beta > -1$ and $\alpha+\beta+1 \geq 0$.

Putting back the value of $g(s+j, n-s, n)$ into (1.3) we deduce the following formula

$$
\begin{aligned}
L_{s+j,n-s,n} = {} & \frac{(\alpha+1)_n (s+j)!\,(\alpha+1)_{s+j}(\beta+1)_n(\alpha+\beta+1)_{2s+j}}{(\alpha+\beta+1)_{s+j}(\alpha+1)_s} \\
& \times \frac{(\alpha+\beta+1+n-s)_{n-s}(\alpha+\beta+1)_j\, n!}{s!\, j!} \\
& \times \frac{(s-n)_{j/2}(\alpha+\beta+n+1)_{j/2}}{\left(s-n-\dfrac{\alpha+\beta}{2}\right)_{j/2}(\alpha+s+1)_{j/2}} \\
& \times \frac{(s-n-\alpha)_{j/2}(\beta+n+1)_{j/2}(1/2)_{j/2}}{\left(\dfrac{1}{2}+s-n-\dfrac{\alpha+\beta}{2}\right)_{j/2}(s+1)_{j/2}(\alpha+1)_{j/2}} \\
& \times {}_9F_8\left[\begin{matrix} \alpha, 1+\dfrac{\alpha}{2}, \alpha+\dfrac{1}{2}, \dfrac{\alpha-\beta}{2}\dfrac{\alpha-\beta+1}{2}, \alpha+\beta+n+1+\dfrac{j}{2}, \\ \dfrac{\alpha}{2}, \dfrac{1}{2}, \dfrac{\alpha+\beta}{2}+1, \dfrac{\alpha+\beta+1}{2}, -\beta-n-\dfrac{j}{2}, \end{matrix}\right. \\
& \qquad\qquad \left.\begin{matrix} s-n+\dfrac{j}{2}, -s-\dfrac{j}{2}, -\dfrac{j}{2} \\ \alpha+n+1-s-\dfrac{j}{2}, \alpha+s+1+\dfrac{j}{2}, \alpha+1+\dfrac{j}{2} \end{matrix}\right],
\end{aligned}
$$

when j is even. When j is odd, formula (1.8) of RAHMAN [Ra] leads to :

$$
\begin{aligned}
L_{s+j,n-s,n} = {} & \frac{(\alpha+1)_n (s+j)!\,(\alpha+1)_{s+j}(\beta+1)_n}{(\alpha+\beta+1)_{s+j}(\alpha+1)_s} \\
& \times \frac{(\alpha+\beta+1)_{2s+j}(\alpha+\beta+1+n-s)_{n-s}(\alpha+\beta+1)_j n!}{s!\, j!} \\
& \times \frac{(s-n)_{(j+1)/2}(\alpha+\beta+n+1)_{(j+1)/2}}{\left(s-n-\dfrac{\alpha+\beta}{2}\right)_{(j+1)/2}(\alpha+s+1)_{(j+1)/2}} \\
& \times \frac{(s-n-\alpha)_{(j-1)/2}(\beta+n+1)_{(j-1)/2}(3/2)_{(j-1)/2}}{\left(\dfrac{1}{2}+s-n-\dfrac{\alpha+\beta}{2}\right)_{(j-1)/2}(s+1)_{(j-1)j/2}(\alpha+2)_{(j-1)/2}}
\end{aligned}
$$

$$\times \frac{\alpha-\beta}{\alpha+\beta+1}\,{}_9F_8\left[\begin{matrix}\alpha+1, \frac{\alpha+3}{2}, \alpha+\frac{1}{2}, \frac{\alpha-\beta}{2}+1, \frac{\alpha-\beta+1}{2},\\ \frac{\alpha+1}{2}, \frac{3}{2}, \frac{\alpha+\beta}{2}+1, \frac{\alpha+\beta+3}{2},\end{matrix}\right.$$

$$\left.\begin{matrix}\alpha+\beta+n+\frac{3}{2}+\frac{j}{2}, s-n+\frac{1}{2}+\frac{j}{2}, \frac{1}{2}-s-\frac{j}{2}, \frac{1-j}{2}\\ \frac{1-j}{2}-\beta-n, \alpha+n+\frac{3}{2}-s-\frac{j}{2}, \alpha+s+\frac{3}{2}+\frac{j}{2}, \alpha+\frac{3}{2}+\frac{j}{2}\end{matrix}\right].$$

It seems that a derivation of RAHMAN's formula by means of combinatorial methods is out of scope. It would first require an interpretation of the factor $(\alpha)_k(1+\alpha/2)_k(\alpha+1/2)_k/(\alpha/2)_k(1/2)_k$ occurring in the series ${}_9F_8$. But such a factor already occurs in each classical hypergeometric series identity involving ${}_pF_{p+1}$ for $p\ge 3$, for instance in the Dougall, Whipple and Bailey identities (see [Bai, Chap. 4]).

When $\alpha=\beta$ (the case of ultraspheric polynomials), the factor ${}_9F_8$ vanishes and RAHMAN's formula greatly simplifies. For instance, for j even we get :

if $0\le j\le n-s$

$$\begin{aligned}L_{s+j,n-s,n} = &\binom{s+j}{s, j/2, j/2} s!\,\left(\frac{j}{2}\right)!\binom{n}{s+j/2}(n-s)!\\ &\times(\alpha+1+j/2)_{n-j/2}(\alpha+1+s+j/2)_{j/2}\\ &\times(\alpha+1+n-s-j/2)_{j/2}(\beta+1)_{n+j/2}\\ &\times(\alpha+\beta+1+s+j)_s(\alpha+\beta+1+n-s)_{n-s-j}\\ &\times(\alpha+\beta+1)_j(\alpha+\beta+n+1)_{j/2}\,;\end{aligned}$$

if $n-s\le j\le 2n-2s$

$$\begin{aligned}L_{s+j,n-s,n} = &\binom{s+j}{s, j/2, j/2} s!\,\left(\frac{j}{2}\right)!\binom{n}{s+j/2}(n-s)!\\ &\times(\alpha+1+j/2)_{n-j/2}(\alpha+1+s+j/2)_{j/2}\\ &\times(\alpha+1+n-s-j/2)_{j/2}(\beta+1)_{n+j/2}\\ &\times(\alpha+\beta+1+s+j)_s(\alpha+\beta+1+n-s)_{n-s}\\ &\times(\alpha+\beta+1)_{2n-2s-j}(\alpha+\beta+n+1)_{j/2}.\end{aligned}$$

In particular, with $j=0$ we get

$$(1.5)\quad L_{s,n-s,n} = (\alpha+\beta+s+1)_s(\alpha+\beta+1+n-s)_{n-s}n!\,(\alpha+1)_n(\beta+1)_n,$$

a formula that will be extended further in the paper.

The purpose of this article is to give a combinatorial interpretation to $L_{\mathbf{n}}$ and to deduce from it several analytic consequences (sections 2 and 3). As mentioned previously, we cannot derive RAHMAN's formula, but we can, at least, evaluate an extension of (1.5), that is,

$$(1.6)\quad L_{\mathbf{n}} = (\alpha+\beta+n_2+1)_{n_2}\cdots(\alpha+\beta+n_m+1)_{n_m} n_1!\,(\alpha+1)_{n_1}(\beta+1)_{n_1},$$

when m is arbitrary and $n_1 = n_2+\cdots+n_m$. This is presented in section 4.

2. Weighted bipermutations. — Consider formulas (1.1) and (1.2). The expression $L_{\mathbf{n}}$ will first be proved to be the generating function for certain combinatorial objects, called *weighted bipermutations*, as follows. Let $N = N_1 + \cdots + N_m$ be an ordered partition of a set N with $|N_i| = n_i$ $(i = 1, 2, \ldots, m)$. If K is a subset of N, let $k_i = K \cap N_i$ and $|K_i| = k_i$ $(i = 1, 2, \ldots, m)$. Next consider a permutation π of the set N. An element x of N is said to be *π-incestuous*, if both x and $\pi(x)$ belong to the same component N_i. Denote by $\operatorname{Inc}\pi$ the set of all π-incestuous elements of N. Finally, define a *weighted bipermutation* of $N = N_1 + \cdots + N_m$ as being a triple (π_1, π_2, K), where π_1 and π_2 are permutations of N and K is a subset of N that satisfies the properties :

$$K \subset \operatorname{Inc}\pi_1 \qquad \text{and} \qquad N \setminus K \subset \operatorname{Inc}\pi_2.$$

Define the *weights* of a weighted bipermutation (π_1, π_2, K) to be :

$$w(\alpha, \beta; \pi_1, \pi_2, K) = (\alpha+1)^{\operatorname{cyc}\pi_1}(\beta+1)^{\operatorname{cyc}\pi_2}\,;$$
$$w'(\alpha, \beta; \pi_1, \pi_2, K) = (-1)^{|N\setminus K|}(\alpha+1)^{\operatorname{cyc}\pi_1}(\beta+1)^{\operatorname{cyc}\pi_2}\,;$$

where $\operatorname{cyc}\pi$ designates the number of cycles of the permutation π.

THEOREM 1. — *The polynomial $L_{\mathbf{n}}$ defined in* (1.2) *is the generating function for the weighted bipermutations by the weight w'. In other words,*

$$\begin{aligned} L_{\mathbf{n}}(\alpha,\beta) := L_{\mathbf{n}} &= \sum w'(\alpha,\beta;\pi_1,\pi_2,K) \\ &= \sum (-1)^{|N\setminus K|}(\alpha+1)^{\operatorname{cyc}\pi_1}(\beta+1)^{\operatorname{cyc}\pi_2}. \end{aligned}$$

Proof. — Let (π_1, π_2, K) be a weighted bipermutation of $N = N_1+\cdots+N_m$. To the pair (π_1, K) we can associate a sequence $(\pi_{11}, \ldots, \pi_{1m}, \sigma_1)$, where each π_{1i} is an injection of K_i into N_i $(i = 1, 2, \ldots, m)$ and σ_1 is a permutation of the set $N \setminus K = \sum_i (N_i \setminus K_i)$. Moreover, $\operatorname{cyc}\pi_1 = \sum_i \operatorname{cyc}\pi_{1i} + \operatorname{cyc}\sigma_1$ and the mapping $(\pi_1, K) \mapsto (\pi_{i1}, \ldots, \pi_{im}, \sigma_1)$ is bijective. Such a mapping has been described in [Fo-Ze]. Fig. 1 indicates the construction of such a bijection In the same manner, we associate a

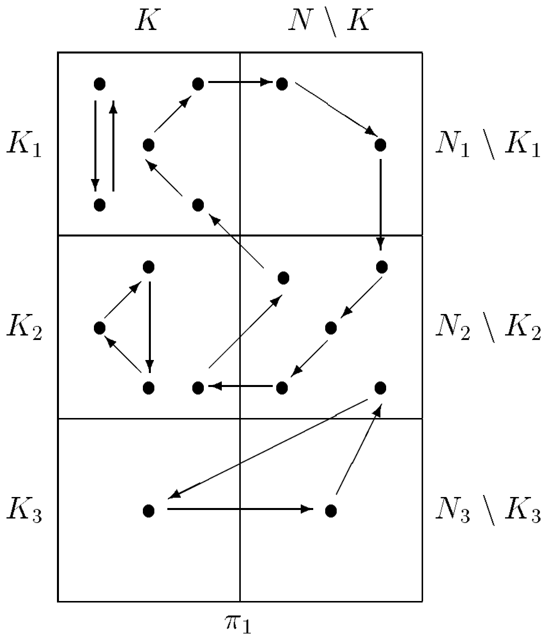


Fig. 1

sequence $(\pi_{21}, \ldots, \pi_{2m}, \sigma_2)$ to π_2, where each π_{2i} is an injection of $N_i \setminus K_i$ into N_i $(i = 1, 2, \ldots, m)$ and σ_2 is a permutation of K.

Therefore, $(\alpha+1)_{\Sigma(n_i-k_i)} \prod_i (\alpha+1+n_i-k_i)_{k_i}$ is the generating function for permutations π_1 by number of cycles satisfying $K \subset \operatorname{Inc} \pi_1$. In the same manner, $(\beta+1)_{\Sigma k_i} \prod_i (\beta+1+k_i)_{n_i-k_i}$ is the generating function for permutations π_2 by number of cycles satisfying $N \setminus K \subset \operatorname{Inc} \pi_2$. Thus, to calculate $L_{\mathbf{n}}$ we can first fix $\mathbf{k}$, then a sequence $\mathbf{K} = (K_1, \ldots, K_m)$ with $|K_i| = k_i$ $(i = 1, 2, \ldots, m)$ and finally sum over all weighted bipermutations (π_1, π_2, K). $\square$

As an application of this combinatorial interpretation we can state the following corollary and also obtain another combinatorial interpretation in terms of pairs of permutations with prescribed incestuous element sets.

Corollary 1. — *If $|N| = n$, then $L_{\mathbf{n}}(\beta, \alpha) = (-1)^{|N|} L_{\mathbf{n}}(\alpha, \beta)$. In particular, when n is odd, $L_{\mathbf{n}}(\alpha, \alpha) = 0$.*

Proof. — Consider the transformation $(\pi_1, \pi_2, K) \mapsto (\pi_2, \pi_1, N \setminus K)$. Then

$$\begin{aligned} w'(\beta, \alpha; \pi_2, \pi_1, N \setminus K) &= (-1)^{|K|} (\beta+1)^{\operatorname{cyc} \pi_2} (\alpha+1)^{\operatorname{cyc} \pi_1} \\ &= (-1)^{|N|} (-1)^{|N \setminus K|} (\alpha+1)^{\operatorname{cyc} \pi_1} (\beta+1)^{\operatorname{cyc} \pi_2} \\ &= (-1)^{|N|} w'(\alpha, \beta; \pi_1, \pi_2, K). \end{aligned}$$

Thus $L_{\mathbf{n}}(\beta, \alpha) = (-1)^{|N|} L_{\mathbf{n}}(\alpha, \beta)$. $\square$

COROLLARY 2 (Second combinatorial interpretation). — *One has :*

$$L_{\mathbf{n}} = \sum (-1)^{|N\setminus K|} (\alpha+1)^{\operatorname{cyc}\pi_1} (\beta+1)^{\operatorname{cyc}\pi_2},$$

where the summation is over all triples (π_1, π_2, K) *with* π_1 *and* π_2 *permutations of* N, *and* K *a subset of* N *with the property that* $K = \operatorname{Inc}\pi_1$ *and* $N \setminus K = \operatorname{Inc}\pi_2$.

Proof. — Let (π_1, π_2, K) be a weighted bipermutation. If $\operatorname{Inc}\pi_1 \cap \operatorname{Inc}\pi_2$ is non-empty, look at the smallest element ξ in that set. Then define $\phi(\pi_1, \pi_2, K) = (\pi_1, \pi_2, K \setminus \{\xi\})$ or $(\pi_1, \pi_2, K + \{\xi\})$, depending on whether ξ is in K or not. In both cases $\phi(\pi_1, \pi_2, K)$ is a weighted bipermutation and

$$w'\phi(\alpha, \beta; \pi_1, \pi_2, K) = -w'(\alpha, \beta; \pi_1, \pi_2, K).$$

Therefore the summation $\sum w'(\alpha, \beta; \pi_1, \pi_2, K)$ over all pairs (π_1, π_2) such that $\operatorname{Inc}\pi_1 \cap \operatorname{Inc}\pi_2 = \emptyset$ equals 0. Now as $K \subset \operatorname{Inc}\pi_1$ and $N \setminus K \subset \operatorname{Inc}\pi_2$, the condition $\operatorname{Inc}\pi_1 \cap \operatorname{Inc}\pi_2 = \emptyset$ means that $K = \operatorname{Inc}\pi_1$ and $N \setminus K = \operatorname{Inc}\pi_2$. □

3. Weighted derangements. — The polynomial $L_{\mathbf{n}}$ can also be expressed in terms of derangement polynomials as follows. Keep the same notations as in the beginning of section 2 for N, K, N_i, K_i and define a K-*derangement* to be a permutation σ of K such that for every x in K the elements x and $\sigma(x)$ belong to different components K_i and K_j $(i \neq j)$. Set

$$D(\mathbf{k}; \alpha) = \sum (\alpha+1)^{\operatorname{cyc}\sigma},$$

where σ ranges over all K-derangements.

THEOREM 3 (third combinatorial interpretation). — *One has :*

$$L_{\mathbf{n}} = \sum_{K\subset N} (-1)^{|K\setminus N|} D(\mathbf{n}-\mathbf{k}; \alpha) D(\mathbf{k}; \beta) \times \prod_i (\alpha+1+n_i-k_i)_{k_i} (\beta+1+k_i)_{n_i-k_i}. \tag{3.1}$$

Proof. — Consider a weighted bipermutation (π_1, π_2, K) with $K = \operatorname{Inc}\pi_1$ and $N \setminus K = \operatorname{Inc}\pi_2$. If we use the bijection described in the proof of THEOREM 1, the pair (π_1, K) is transformed into a sequence

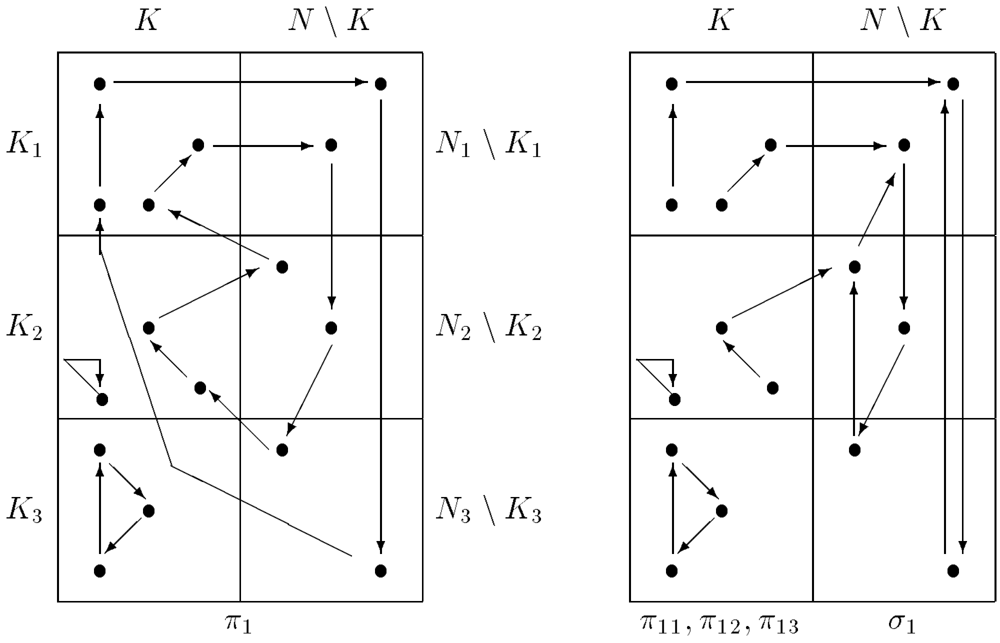


Fig. 2

$(\pi_{11}, \dots, \pi_{1m}, \sigma_1)$, but this time σ_1 is an $(N \setminus K)$-*derangement*, as shown in Fig. 2.

In the same way, $(\pi_2, N \setminus K)$ is transformed into $(\pi_{21}, \dots, \pi_{2m}, \sigma_2)$ with σ_2 being a K-derangement. Therefore

$$L_{\mathbf{n}} = \sum_{K \subset N} (-1)^{|N \setminus K|} \sum \Bigl(\prod_i (\alpha+1)^{\mathrm{cyc}\,\pi_{1i}} (\beta+1)^{\mathrm{cyc}\,\pi_{2i}} \Bigr) \times (\alpha+1)^{\mathrm{cyc}\,\sigma_1} (\beta+1)^{\mathrm{cyc}\,\sigma_2}.$$

As π_{1i} (resp. π_{2i}) is an injection of K_i (resp. $N_i \setminus K_i$) into N_i and σ_1 (resp. σ_2) is an $(N \setminus K)$-derangement (resp. a K-derangement), the summation over the π_{1i}'s, the π_{2i}'s and the derangements σ_1 and σ_2 yields (3.1). $\square$

There are several consequences of this interpretation when $n_1 \geq n_2 + \cdots + n_m$. First, study the case of the strict inequality.

Lemma 1. — *If $k_1 > k_2 + \cdots + k_m$, then $D(\mathbf{k}, \alpha) = 0$.*

Proof. — If π is a K-derangement, then $\pi(K_1) \subset K_2 + \cdots + K_m$. But $|K_i| = k_1 > k_2 + \cdots + k_m$ and there do not exist any K-derangements under this hypothesis. $\square$

Lemma 2. — *Suppose $n_1 > n_2 + \cdots + n_m$ and let $0 \leq k_i \leq n_i$ for $i = 1, \dots, m$. Then*

$$\text{either} \quad k_1 > k_2 + \cdots + k_m, \quad \text{or} \quad (n_1 - k_1) > (n_2 - k_2) + \cdots + (n_m - k_m).$$

Proof. — If $k_1 \le k_2 + \cdots + k_m$, then $n_1 - k_1 > n_2 + \cdots + n_m - k_1 > n_2 - k_2 + \cdots + n_m - k_m$. $\square$

PROPOSITION 1. — *If $n_1 > n_2 + \cdots + n_m$, then $L_{\mathbf{n}} = 0$.*

Proof. — With the foregoing hypothesis, either $k_1 > k_2 + \cdots + k_m$ or $(n_1 - k_1) > (n_2 - k_2) + \cdots + (n_m - k_m)$. Then, either $D(\mathbf{k};\beta) = 0$, or $D(\mathbf{n}-\mathbf{k};\alpha) = 0$. Therefore, (3.1) shows that $L_{\mathbf{n}} = 0$. $\square$

COROLLARY. — *If $n_1 \neq n_2$, then $L_{n_1,n_2} = 0$.*

This is precisely the *orthogonality relation.*

4. The evaluation of $L_{\mathbf{n}}$ for $n_1 = n_2 + \cdots + n_m$. — Consider again the summation (3.1). When $n_1 = n_2 + \cdots + n_m$, the inequality $k_1 < k_2 + \cdots + k_m$ implies $n_1 - k_1 > (n_2 - k_2) + \cdots + (n_m - k_m)$. Therefore, the factor $D(\mathbf{n}-\mathbf{k};\alpha)$ vanishes for such a sequence $\mathbf{k}$. In the same manner, if $k_1 > k_2 + \cdots + k_m$, then $D(\mathbf{k};\beta) = 0$. The summation (3.1) can then be restricted to those sequences $\mathbf{k}$ satisfying

$$0 \le k_1 = k_2 + \cdots + k_m \le n_1 = n_2 + \cdots + n_m. \tag{4.1}$$

In particular, for $m = 2$ and $n_1 = n_2 = n$ we obtain :

$$\begin{aligned} L_{n,n} = \sum_{k=0}^{n} & D(n-k, n-k, \alpha) D(k,k,\beta) \\ & \times \left(\binom{n}{k} (\alpha+1+n-k)_k (\beta+1+k)_{n-k} \right)^2. \end{aligned} \tag{4.2}$$

We will have a more precise evaluation further in the paper. For the time being, let us compare (4.2) with the classical evaluation of the integral :

$$\begin{aligned} I_{n,n} &= \int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta \left(P_n^{(\alpha,\beta)}(x) \right)^2 dx \\ &= \frac{2^{1+\alpha+\beta}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!\,(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}. \end{aligned}$$

(See, e.g., [Rai, p. 260 (11)].) By comparison with the definition of $L_{n,n}$ (formula (1.2)),

$$I_{n,n} = \frac{2^{1+\alpha+\beta}}{n!\,n!\,(\alpha+\beta+2)_{2n}} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} L_{n,n},$$

so that

$$L_{n,n} = (\alpha+\beta+n+1)_n n!\,(\alpha+1)_n(\beta+1)_n. \tag{4.3}$$

This formula will be a consequence of the next theorem.

THEOREM 4. — *When* $n_1 = n_2 + \cdots + n_m$, *then*

$$L_{\mathbf{n}} = (\alpha+\beta+n_2+1)_{n_2} \cdots (\alpha+\beta+n_m+1)_{n_m} n_1!\,(\alpha+1)_{n_1}(\beta+1)_{n_1}.$$

Proof. — As just noticed, the summation (3.1) can be restricted to the sequences $\mathbf{k}$ satisfying (4.1). But if (π_1, π_2, K) is a triple with $|K_1| = |K_2 + \cdots + K_m|$, then $|N \setminus K| = |N_1 + \cdots + N_m \setminus (K_1 + \cdots + K_m)| = |N_1 \setminus K_1| + \cdots + |N_m \setminus K_m| = 2|N_1 \setminus K_1|$, so that the sign $(-1)^{|N \setminus K|}$ is always equal to 1. Therefore,

$$L_{\mathbf{n}} = \sum (\alpha+1)^{\operatorname{cyc} \pi_1} (\beta+1)^{\operatorname{cyc} \pi_2},$$

where π_1 and π_2 are permutations of N and $\pi_1(K_i) \subset N_i$, $\pi_1(N_i \setminus K_i) \subset N \setminus N_i$, $\pi_2(N_i \setminus K_i) \subset N_i$, $\pi_2(K_i) \subset N \setminus N_i$ for $i = 1, 2, \ldots, m$.

Let $M_1 = K_1 + \sum_{j \geq 2} (N_j \setminus K_j)$ and $M_2 = N_1 \setminus K_1 + \sum_{j \geq 2} K_j$. Note that $|M_1| = |M_2| = |N_1|$. Our purpose is now to construct a bijection that will explain the occurrence of each factor in the right-hand side of the THEOREM 4 formula. The reader is advised to follow the construction by looking at the geometric representations of the mappings in Fig. 3

(i) For each $i = 2, \ldots, m$ let f_i be the mapping of N_i into itself defined by :

$$f_i \big|_{K_i} = \pi_1 \big|_{K_i} \qquad \text{and} \qquad f_i \big|_{N_i \setminus K_i} = \pi_2 \big|_{N_i \setminus K_i}.$$

As $\pi_1(K_i) \subset N_i$ and $\pi_2(N_i \setminus K_i) \subset N_i$, the pair (K_i, f_i) is a so-called *Jacobi endofunction* (see [Fo-Le]). Denote by $a(f_i)$ (resp. $b(f_i)$) the number of cycles of f_i all vertices of which are in K_i (resp. $(N_i \setminus K_i)$ and let the weight of (K_i, f_i) be defined by

$$w(K_i, f_i) = (\alpha+1)^{a(f_i)} (\beta+1)^{b(f_i)}.$$

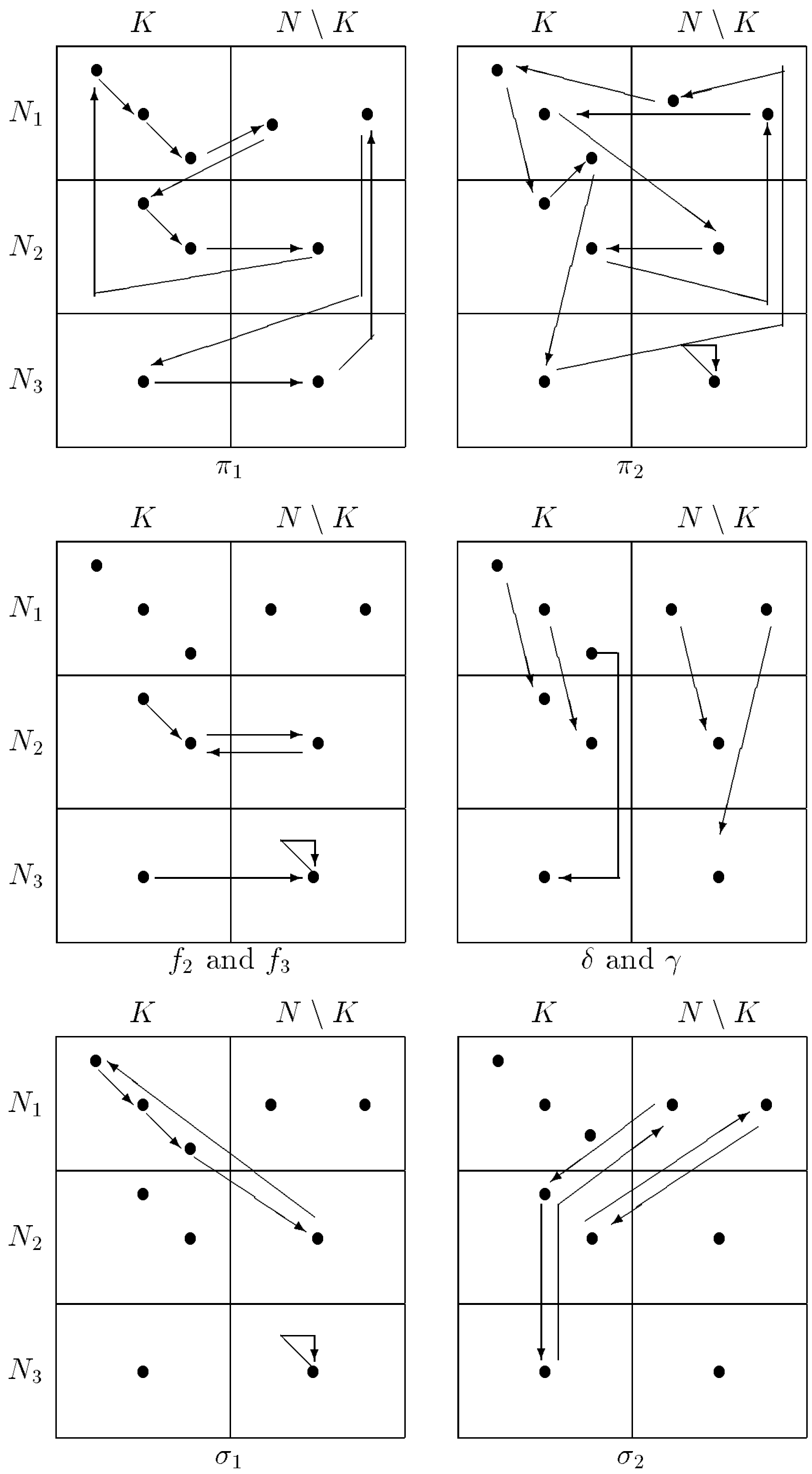
As shown in [Fo-Le, théorème 1]

$$\sum w(K_i, f_i) = (\alpha+\beta+n_i+1)_{n_i}, \tag{4.4}$$

the summation being over all Jacobi endofunctions on N_i.

(ii) Consider $x \in N_1 \setminus K_1$. As $\pi_1(N_1 \setminus K_1) \subset N_2 + \cdots + N_m$ and $\pi_1(K_i) \subset N_i$ $(i = 2, \ldots, m)$, sooner or later the iterates $\pi_1^k(x)$ will hit the set $\sum_{j \geq 2} (N_j \setminus K_j)$. Let $k(x)$ be the smallest integer satisfying $\pi_1^{k(x)}(x) \in \sum_{j \geq 2} (N_j \setminus K_j)$ and define $\gamma(x) = \pi_1^{k(x)}(x)$. Clearly

$$\gamma : N_1 \setminus K_1 \to \sum_{j \geq 2} (N_j \setminus K_j)$$



K
$N \setminus K$
N_1
N_2
N_3
π_1
π_2
f_2 and f_3
δ and γ
σ_1
σ_2

Fig. 3

is a *bijection*. In the same manner, define a *bijection* $\delta : K_1 \to \sum_{j\ge 2} K_j$ by means of π_2.

(iii) With the pair (γ,δ) we can then make up a bijection of N_1 onto the union $\sum_{j\ge 2} N_j$.

(iv) Consider the cycles of π_1. All the cycles that intersect M_1 but sometimes leave M_1 will be made purely M_1 by cancelling all the portions that leave M_1. This gives a permutation $\sigma_1 : M_1 \to M_1$.

(v) In the same manner, obtain a permutation $\sigma_2 : M_2 \to M_2$.

Remembering the definitions of $a(f_i)$ and $b(f_i)$ given in (i) we see that to each triple (π_1, π_2, K) there corresponds a sequence

$$(K_2, f_2, \ldots, K_m, f_m, \gamma, \delta, \sigma_1, \sigma_2)$$

with

$$\begin{aligned}\operatorname{cyc}\pi_1 &= a(f_2) + \cdots + a(f_m) + \operatorname{cyc}\sigma_1\,;\\ \operatorname{cyc}\pi_2 &= b(f_2) + \cdots + b(f_m) + \operatorname{cyc}\sigma_2.\end{aligned}$$

Accordingly,

$$w(\alpha,\beta;\pi_1,\pi_2,K) = w(K_2,f_2)\cdots w(K_m,f_m)(\alpha+1)^{\operatorname{cyc}\sigma_1}(\beta+1)^{\operatorname{cyc}\sigma_2}.$$

It can be verified that the correspondence between triples and sequences defined by (i)–(v) is one-to-one. Hence

$$\begin{aligned}&\sum w(\alpha,\beta;\pi_1,\pi_2,K)\\ &\quad= \sum w(K_2,f_2)\cdots \sum w(K_m,f_m) \sum_{\gamma,\delta} 1 \sum_{\sigma_1}(\alpha+1)^{\operatorname{cyc}\sigma_1} \sum_{\sigma_2}(\beta+1)^{\operatorname{cyc}\sigma_2}\\ &\quad= (\alpha+\beta+n_2+1)_{n_2}\cdots(\alpha+\beta+n_m+1)_{n_m}\, n_1!\,(\alpha+1)_{n_1}(\beta+1)_{n_1}. \quad\square\end{aligned}$$

Remark. — Note that for $m=2$ and $n_1 = n_2 = n$ THEOREM 4 yields identity (4.3).

5. Concluding remarks. — The problem of the linearization of the classical orthogonal polynomials has been studied again recently by means of combinatorial methods. ASKEY and his coauthors [As-Is, As-Is-Ko] have already obtained several significant results concerning the Hermite, Laguerre and Meixner polynomials. AZOR, GILLIS and VICTOR [Az-Gi-Vi] found an elegant set-up for the Hermite polynomials. The two authors of the present paper [Fo-Ze] have completed the work of ASKEY and his coauthors as far as the (generalized) Laguerre polynomials are concerned by exploiting a β-extension of the MacMahon Master Theorem. ZENG [Ze] has further extended the works of ASKEY and the two authors and proved new positivity results concerning the Meixner, Krawtchouk and Charlier polynomials. GILLIS and his coauthors [Gi-Je-Ze] reproved several positivity results on the Legendre polynomials. Some of their arguments have been implicitly used in the present paper. RAHMAN's formula, as said in the introduction, should discourage several researchers. Formula (3.1) seems to indicate that a new algebraic tool has to be found to handle the product of two derangement polynomials. There is also a linearization coefficient formula for the ultraspheric polynomials found by HSÜ [Hs], more compact than RAHMAN's formula for $\alpha = \beta$, but not so easy to be tackled by combinatorial methods. The only hope for the time being seems to be the symmetric function approach due to ZENG. He has already got an explicit formula for a single derangement polynomial that led to the positivity result for the Krawtchouk polynomials.

Note added in 1994. — The article [Ze] has been updated in the reference list. Notice that the linearization coefficients for the class of Sheffer polynomials has been thoroughly studied by Zeng [Ze92]. An interesting connection between linearization coefficeints for Jacobi polynomials and permutation pair counting has been derived in [Ze91]. The Rahman formula remains untamed in the combinatorial environment.

[Ze91] ZENG (Jiang). — Counting a pair of permutations and the linearization coefficients for Jacobi polynomials, *Actes de l'Atelier de Combinatoire franco-québecois*, Publ. LACIM, no. 10, Université du Québec à Montréal, 1991.

[Ze92] ZENG (Jiang). — Weighted derangements and Sheffer polynomials, *Proc. London Math. Soc.*, t. **65**, 1992, p. 1–22.

REFERENCES

[As-Is] ASKEY (Richard) and ISMAIL (Mourad E.H.). — Permutation problems and special functions, *Canad. J. Math.*, t. **28**, 1976, p. 853–874.

[As-Is-Ko] ASKEY (Richard), ISMAIL (Mourad E.H.) and KOORNWINDER (T.). — Weighted permutation problems and Laguerre polynomials, *J. Combinatorial Theory, Ser. A*, t. **25**, 1978, p. 277–287.

[Az-Gi-Vi] AZOR (R.), GILLIS (J.) and VICTOR (J.D.). — Combinatorial application of Hermite polynomials, *SIAM J. Math. Anal.*, t. **13**, 1982, p. 879–890.

[Bai] BAILEY (W.N.). — *Generalized Hypergeometric Series.* — Cambridge Univ. Press, 1936.

[Er] ERDÉLYI (A.) et al. — *Higher Transcendental Functions* (Bateman Manuscript Project), 3 vol. — New York, McGraw-Hill, 1953.

[Fo-Le] FOATA (Dominique) et LEROUX (Pierre). — Polynômes de Jacobi, interprétation combinatoire et fonction génératrice, *Proc. Amer. Math. Soc.*, t. **87**, 1983, p. 47–53.

[Fo-Ze] FOATA (Dominique) and ZEILBERGER (Doron). — Weighted derangements and Laguerre polynomials, in *Séminaire Lotharingien de combinatoire (Bayreuth, Erlangen, Strasbourg)*, 8^{e} session [D. Foata, éd.; Sainte-Croix-aux-Mines. 23–28 mai 1983], p. 17–25. — Publ. IRMA Strasbourg, 229/S–08, 1984.

[Gi-Je-Ze] GILLIS (J.), JEDWAB (J.) and ZEILBERGER (D.). — A combinatorial interpretation of the integral of the product of Legendre polynomials, manuscrit, Weizmann Inst., 1987.

[Hs] HSÜ (Hsien-Yü). — Certain integrals and infinite series involving ultraspheric polynomials and Bessel functions, *Duke Math. J.*, t. **4**, 1938, p. 374–383.

[Ra] RAHMAN (Mizan). — A non-negative representation of the linearization coefficients of the product of Jacobi polynomials, *Canad. J. Math.*, t. **33**, 1981, p. 915–928.

[Rai] RAINVILLE (E.D.). — *Special Functions.* — New York, Chelsea, 1960.

[Sz] SZEGÖ (Gabor). — *Orthogonal Polynomials.* — Providence, R.I., Amer. Math. Soc., 1939 (*Amer. Math. Soc. Colloq. Publ.*, **23**) [second printing of the fourth edition : 1978].

[Ze] ZENG (Jiang). — Linéarisation de produits de polynômes de Meixner, Krawtchouk et Charlier, *S.I.A.M. J. Math. Anal.*, t. **21**, 1989, p. 1349–1368.

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