ON HALES–JEWETT'S THEOREM

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Summary. We prove that, for every finite semigroup S, there exist elements $a_1, a_2, \ldots, a_k, a_{k+1}$ of S and integers $i_1, i_2, \ldots, i_k$ such that

$$a_1 \cdot x^{i_1} \cdot a_2 \cdot x^{i_2} \cdot \ldots \cdot a_k \cdot x^{i_k} \cdot a_{k+1} = a_1 \cdot y^{i_1} \cdot a_2 \cdot y^{i_2} \cdot \ldots \cdot a_k \cdot y^{i_k} \cdot a_{k+1}$$

for each x, y of S.

We refer to [1] for the notion of (combinatorial) line as well as for the other combinatorial concepts we use in the sequel.

The following theorem of Hales–Jewett is well-known.

Theorem ([2]). *Given any finite set A and any integer r there exists an integer $N = N(A, r)$ such that for each $n \geq N(A, r)$ in any coloring of A^n there is always a monochromatic line.*

In [1], Graham presented an interesting algebraic application of the theorem: for every finite commutative semigroup S, there exist an element a of S and an integer n such that

$$a \cdot x^n = a \cdot y^n$$

for each x, y of S, i.e., $a \cdot x^n$ is independent of x (shortly, we speak of the constant word $a \cdot x^n$ for S).

Trivially, there are finite non-commutative semigroups without constant word of type $a \cdot x^n$, such as, for example, the semigroup D presented by the following Cayley table

	u	v
u	u	v
v	u	v

True, for this semigroup the word $x \cdot u$ is constant, i.e., for this semigroup a particular case of the word $x^n \cdot b$ ("dual" of $a \cdot x^n$) is constant. But the semigroup D' presented by the following Cayley table

	u	v
u	u	u
v	v	v

shows that $x^n \cdot b$ is not a constant word for each finite semigroup.

Again, the direct product $D \times D'$, where D and D' are as previously described, is a semigroup without constant word of type $a \cdot x^n$ and without a constant word of type $x^n \cdot b$.

After some other considerations like the previous ones, the following question naturally arises: given any finite semigroup, does there exist a sort of constant word for it?

The following proposition gives us the answer.

Proposition. *For any finite semigroup S there exist an integer k, a $(k+1)$-tuple $a_1, a_2, \ldots, a_k, a_{k+1}$ of elements of S, and a k-tuple $i_1, i_2, \ldots, i_k$ of integers such that*

$$a_1 \cdot x^{i_1} \cdot a_2 \cdot x^{i_2} \cdot \ldots \cdot a_k \cdot x^{i_k} \cdot a_{k+1}$$

is independent of x, i.e., it is a constant word for S.

Proof. Consider S both as an alphabet and as a set of colors. From Hales–Jewett's theorem there exists an integer $N(S, |S|)$ such that for each integer $n \geq N(S, |S|)$ and each $|S|$-coloring of S^n, a line of S^n is monochromatic.

This is true, in particular, when each element

$$(x_1, x_2, \ldots, x_n)$$

of S^n is colored by

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n,$$

i.e., the product in S of $x_1, x_2, \ldots, x_n$.

Now, if the monochromatic line is the diagonal, then the conclusion follows immediately.

If the monochromatic line is not the diagonal, then the lengths of the "lakes" of the non-fixed coordinates (there is always such a coordinate) give us the integer i_j, and the elements a_j are easily obtained by looking at the fixed coordinates.

References

1. R. L. Graham, *Rudiments of Ramsey theory*, CBMS Regional Conference Series in Mathematics, vol. 45, American Mathematical Society,, Providence, R.I., 1981.
2. A. W. Hales and R. I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222–229.

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