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AN UMBRAL CALCULUS FOR POLYNOMIALS CHARACTERIZING $U(n)$ TENSOR PRODUCTS

BY

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This talk reports on work to appear in [1] that has been done jointly with L.C. Biedenharn and R.A. Gustafson.

In this lecture we continue the study of the connection between the invariant polynomials

$$
(1)\qquad \begin{aligned} {}_\mu G_q^{(n)} &\equiv {}_\mu G_q^{(n)}\big((x_{ij}+\Delta_1)\big)\\ &\equiv {}_\mu G_q^{(n)}(\Delta_1,\ldots,\Delta_n; x_{12}, x_{23},\ldots,x_{n-1,n},x_{n,1}). \end{aligned}
$$

characterizing $U(n)$ tensor operators $\langle p,q,\ldots,q,0,\ldots,0\rangle$, and the classical theory of symmetric functions as presented in [9], that was established in [5]. The polynomial ${}_\mu G_q^{(n)}(X)$ arise naturally in the application of symmetry groups to mathematical physics. One such problem, with applications to spectroscopy at all levels, is the construction of a suitable basis for the set of all bounded operators mapping the set of all unitary irreducible representation spaces of the group into itself. The precise problems that give rise to ${}_\mu G_q^{(n)}(X)$ are motivated in more detail and put into a broader mathematical setting in [2–4].

The above irrep label $\langle p,q,\ldots,q,0,\ldots,0\rangle$ consists of one p, μ q's, and $n-\mu-1$ 0's. Furthermore, q determines p since $\Delta_1+\cdots+\Delta_n = p+\mu q$ and we are given q, $\Delta_1+\cdots+\Delta_n$ and μ.

Recently in [5–7] an alternate method for explicitly writing down ${}_\mu G_q^{(n)}(X)$ in polynomial form has been given. These methods provide a direct connection between the classical theory of symmetric functions as presented in [9], and the symmetries satisfied by a family of $U(n)$ invariant polynomials even more general then ${}_\mu G_q^{(n)}(X)$. In [5] it is shown that after the change of variables

$$
(2)\qquad \Delta_i = \gamma_i - \delta_i \qquad x_{i,i+1} = \delta_i - \delta_{i+1},
$$

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${}_{\mu}G_q^{(n)}(,\Delta_i,;,x_{i,i+1},)$ becomes an integral linear combination of products of Schur functions $S_\alpha(,\gamma_i,)\cdot S_\beta(,\delta_i,)$ in the variables $\{\gamma_1,\ldots,\gamma_n\}$ and $\{\delta_1,\ldots,\delta_n\}$, respectively. (See equation (6) for a definition of $S_\lambda(x_1,\ldots,x_n)$.) That is, it is directly proved that ${}_{\mu}G_q^{(n)}(,\Delta_i,;,x_{i,i+1},)$ is a bisymmetric polynomial in the variables $\{\gamma_1,\ldots,\gamma_n\}$ and $\{\delta_1,\ldots,\delta_n\}$ with integer coefficients. This motivated the study in [5] of the yet more general bisymmetric polynomials

$$(3)\qquad {}_{\mu}^{m}G_q^{(n)}(\gamma;\delta)\equiv {}_{\mu}^{m}G_q^{(n)}(\gamma_1,\ldots,\gamma_n;\delta_1,\ldots,\delta_m)$$

which are a common generalization of (2.2b) of [2] and equation (2.17) of [10] with no numerator parameters. These polynomials are given by

Definition 4. — Given that ${}_{\mu}^{m}G_0^{(n)}(\gamma;\delta)\equiv 1$, we uniquely determine ${}_{\mu}^{m}G_q^{(n)}(\gamma;\delta)$ by means of

$$(5)\quad {}_{\mu}^{m}G_q^{(n)}(,\gamma_i,;,\delta_i,)=\sum_{\substack{S\subset I_n\\ \|S\|=\mu+1}}(-1)^{\mu+1+\Sigma(S)}\prod_{\substack{i<j\\ i\in S,\,j\in S^c}}(\gamma_i-\gamma_j)^{-1}$$
$$\times\prod_{\substack{i<j\\ i\in S^c,\,j\in S}}(\gamma_i-\gamma_j)^{-1}\prod_{\substack{i=1\\ i\in S}}^{n}\prod_{l=1}^{m}(\gamma_i-\delta_l)\cdot {}_{\mu}^{m}G_{q-1}^{(n)}(,\gamma_i-\chi(i\in S),;,\delta_i,),$$

where $S\subset I_n$ is a $(\mu+1)$-element subset of $\{1,2,\ldots,n\}$, $\Sigma(S)$ denotes the sum of the elements in S, and $\chi(A)$ is 1 if statement A is true and 0, otherwise.

At this point we need to review some basic facts about the Schur functions S_λ. Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r,\ldots)$ be a *partition*, i.e., a (finite or infinite) sequence of nonnegative integers in decreasing order, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_r\cdots$ such that only finitely many of the λ_i are nonzero. The number of nonzero λ_i, denoted by $l(\lambda)$, is called the *length* of λ. If $\sum\lambda_i=n$, then λ is called a *partition of weight* n, denoted by $|\lambda|=n$. Given a partition $\lambda=(\lambda_1,\ldots,\lambda_n)$ of length $\leq n$, the Schur fonctions S_λ are defined by

$$(6)\qquad S_\lambda(x_1,\ldots,x_n)=\frac{\det\big(x_i^{\lambda_j+n-j}\big)_{1\leq i,j\leq n}}{\det\big(x_i^{n-j}\big)_{1\leq i,j\leq n}}.$$

The determinant in the numerator of (6) is divisible in $\mathbf{Z}[x_1,\ldots,x_n]$ by each of the differences (x_i-x_j), $1\leq i<j\leq n$, and hence by their product, which is the *Vandermonde determinant*

$$(7\text{a})\qquad \prod_{1\leq i<j\leq n}(x_i-x_j)=\det\big(x_i^{n-j}\big)_{1\leq i,j\leq n}$$
$$(7\text{b})\qquad \equiv V_n(x_1,\ldots,x_n).$$

Thus, the quotient in (6) is a symmetric polynomial in $x_1, \ldots, x_n$ with coefficient in $\mathbf{Z}$. For example, $S_{(n)} = h_n$ and $S_{(1^n)} = e_n$ where h_n and e_n are, respectively, the homogeneous and elementary symmetric functions of $x_1, \ldots, x_n$.

If $Z = m_1 + \cdots + m_n$, with m_i monomials, then

$$S_\lambda(Z) \equiv S_\lambda(m_1 + \cdots + m_n) \equiv S_\lambda(x_1, \ldots, x_n)\Big|_{x_i = m_i}. \tag{8}$$

On the other hand, $S_\lambda\big(-(Z)\big)$ is defined symbolically by

$$S_\lambda\big(-(Z)\big) \equiv S_\lambda\big(-(m_1 + \cdots + m_n)\big) \equiv (-1)^{|\lambda|} S_{\lambda'}(Z), \tag{9}$$

where $|\mu| = \lambda_1 + \cdots + \lambda_n$ is the sum of the parts of λ and λ' is the conjugate partition to λ. That is, $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{(\lambda_1)})$, with $\lambda'_i = \|\{j \mid \lambda_j \geq i\}\|$. For example $(5,2,1)$ is the conjugate partition of $(3,2,1,1,1)$. Note that $S_\lambda\big(-(m_1 + \cdots + m_n)\big)$ is *not* equal to $S_\lambda(x_1, \ldots, x_n)|_{x_i = -m_i}$. The definitions given by (8) and (9) are implicit in [9; see Remark (3.10) of p. 26].

If $\rho = (\rho_1, \ldots, \rho_k)$, where ρ_i are integers, then denote by $(X)^\rho$ the monomial $x_1^{\rho_1} \ldots x_k^{\rho_k}$ where $X = \{x_1, \ldots, x_k\}$. Let the permutations $w \in S_k$ act on subscripts. For example, $w(X)^\rho = x_{w(1)}^{\rho_1} \cdots x_{w(k)}^{\rho_k}$. Finally, let δ_k be the partition $(k-1, k-2, \ldots, 0)$ and m^l the partition consisting of l parts equal to m.

Fix μ and assume that $m - n \equiv \nu$ is a constant. Denote the sets of variables $\{\gamma_1, \ldots, \gamma_{\mu+1}\}$, $\{\gamma_{\mu+2}, \ldots, \gamma_n\}$, $\{\gamma_1, \ldots, \gamma_n\}$, and $\{\delta_1, \ldots, \delta_m\}$ by A, B, E and F, respectively. We then have from [5] the following fundamental

THEOREM 10. — *Let ${}^{m}_{\mu}G_q^{(n)}(E;F)$ be defined as in (5). We then have*

$$\begin{aligned} {}^{m}_{\mu}G_q^{(n)}(E;F) = \frac{(-1)^{\binom{\mu+1}{2}}}{V_n(E)} & \sum_{\substack{\lambda = (\lambda_1, \ldots, \lambda_{\mu+1}) \\ \lambda_1 \leq m}} S_\lambda(-F) \\ & \cdot \Bigg\{ \sum_{w \in S_n} \epsilon(w) \cdot w \Bigg[(A)^{m^{\mu+1} + \delta_{\mu+1} - (\lambda_{\mu+1}, \ldots, \lambda_1)} (B)^{\delta_{n-(\mu+1)}} \\ & \qquad \cdot {}^{m}_{\mu}G_{q-1}^{(n)}(\gamma_1 - 1, \ldots, \gamma_{\mu+1} - 1, B; F) \Bigg] \Bigg\}, \end{aligned} \tag{11}$$

where, without loss of generalitly, w acts only on $\gamma_1, \ldots, \gamma_{\mu+1}$ when applied to ${}^{m}_{\mu}G_{q-1}^{(n)}$, and $\epsilon(w)$ is the sign of the permutation w.

Starting with (11) and making direct use of the new symmetries discovered in [5], it is shown in [1] that the bisymmetric polynomials ${}^{m}_{\mu}G_q^{(n)}(E;F)$ are an integral linear commbination of Schur functions $S_\lambda(E-F)$ in the symbol $E-F$, where $E-F$ denotes the difference of the two sets of variables E and F. Making use of properties of skew Schur functions $S_{\lambda/\mu}$ and $S_\lambda(E-F)$ one puts together an umbral calculus for ${}^{m}_{\mu}G_q^{(n)}(E;F)$. That is, working entirely with polynomials, one uniquely determines ${}^{m}_{\mu}G_q^{(n)}$ from ${}^{m}_{\mu}G_{q-1}^{(n)}$ and combinatorial rules (such as the Littlewood-Richardson rule [9]) involving Ferrers diagrams (i.e. partitions). The deepest part of this umbral calculus is a summation theorem, involving many different aspects of the theory of Schur functions, which essentially reduces the double sum in (11) to the single term in which λ is the empty partition and w is the identity permutation. Here, we recall a general theorem form [1] which illustrates how the structure of ${}^{m}_{\mu}G_q^{(n)}(E;F)$ "stabilizes" as ν increases while $m-n=\nu$ remains fixed, state the umbral claculus for ${}^{m}_{\mu}G_q^{(n)}(E;F)$, and just give the final formulas for ${}^{m}_{\mu}G_1^{(n)}(E;F)$ and ${}^{m}_{\mu}G_2^{(n)}(E;F)$ which are a direct consequence of the umbral calculus and work in [5].

Before giving these formulas we need some more notation. Let λ, ρ be partitions. The skew Schur function $S_{\lambda/\rho}$ is defined by

$$S_{\lambda/\rho}(E)=\sum_{\nu\subset\lambda}c^{\lambda}_{\rho\nu}\cdot S_\nu(E), \tag{12}$$

where $\nu\subset\lambda$ means $\nu_i\leq\lambda_i$ for all $i\geq 1$, and the integers $c^{\lambda}_{\rho\nu}$ are the Littlewood-Richardson rule coefficients determined by

$$S_\rho(E)\cdot S_\nu(E)=\sum_{\nu}c^{\lambda}_{\rho\nu}S_\lambda(E). \tag{13}$$

Note that $S_{\lambda/0}=S_\lambda$, where 0 denotes the zero partition. Also, $c^{\lambda}_{\rho\nu}=0$ unless $|\lambda|=|\rho|+|\nu|$, so that $S_{\lambda/\rho}$ is homogeneous of degree $|\lambda|-|\rho|$, and is zero if $|\lambda|<|\rho|$. In fact, $S_{\lambda/\rho}=0$ unless $\rho\subset\lambda$.

Given the sets of variables E and F, let $E+F$ be the set union $\{\gamma_1,\ldots,\gamma_n,\delta_1,\ldots,\delta_m\}$. There is then the following classical result

$$S_\lambda(E+F)=\sum_{\rho\subset\lambda}S_{\lambda/\rho}(E)S_\rho(F). \tag{14}$$

It is immediate from (9) and (14) that

$$S_\lambda(E-F)=\sum_{\rho\subset\lambda}S_{\rho/\lambda}(E)S_\rho(-F) \tag{15a}$$

$$=\sum_{\rho\subset\lambda}(-1)^{|\rho|}S_{\rho/\lambda}(E)\cdot S_{\rho'}(F), \tag{15b}$$

where ρ' is the conjugate partition to ρ. The above classical formulas can be found in Chapter I of [9].

We now state a fundamental result of [1].

THEOREM 16. — *Fix $\mu \geq 0$ and $k = m - n$. There are integers $a_{\lambda,q}$, λ a partition, q a non-negative integer such that*

$$ {}^{m}_{\mu}G^{(n)}_{q}(E;F) = \sum_{\lambda} a_{\lambda,q} S_{\lambda}(E-F), \tag{17}$$

where $|\lambda| \leq (\mu+1)(\mu+1+k)q$ and $\lambda \subset [(\mu+1+k)q]^{(\mu+1)q}$.

If $n \geq (\mu+1)q$ and $m \geq (m+1+k)q$, then $a_{\lambda,q}$ is independent of n. More generally, if $S_{\lambda}(E-F) \neq 0$, then $a_{\lambda,q}$ is independent of n.

A similar "stabilization" theorem involving $S_{\alpha}(E) \cdot S_{\beta}(F)$ instead of $S_{\lambda}(E-F)$ was proved in [5]. Just as in [5] it follows from (17) that the $n = (\mu+1)q$ and $m = (\mu+1+m-n)q$ case of (17) gives the correct formula for ${}^{m+l}_{\mu}G^{(n+l)}_{q}(E;F)$ for all integers l, when the sets E and F contain $(n+l)$ and $(m+l)$ variables, respectively.

Before stating the umbral calculus for ${}^{m}_{\mu}G^{(n)}_{q}(E;F)$ we need :

Definition 18. — Let α, β and γ be the three partitions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. The sum $\alpha + \beta$ and direct sum $\alpha \oplus \gamma$ are defined by

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n) \tag{19a}$$

and

$$\alpha \oplus \gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma_1, \gamma_2, \ldots, \gamma_m). \tag{19b}$$

Note that $\alpha + \beta$ is a partition while $\alpha \oplus \gamma$ may not be. In general $\alpha \oplus \gamma$ is just a $(n+m)$-tuple.

Definition 20. — Let $\phi \equiv (\phi_1, \phi_2, \ldots, \phi_n)$ be an n-tuple of non-negative integers that is not necessarily a partition. If all the coordinates of $(\phi + \delta_n) = (\phi_1 + n - 1, \phi_2 + n - 2, \ldots, \phi_{n-1} + 1, \phi_n)$ are distinct, then there is a unique permutation $\sigma_{\phi} \in \mathcal{S}_n$ that orders the parts of $(\phi + \delta_n)$ in decreasing order. Denote the resulting partition (with distinct parts) by $\sigma_{\phi}(\phi + \delta_n)$. In addition, let $(\phi)_{\sigma_{\phi}}$ be the partition

$$(\phi)_{\sigma_{\phi}} \equiv \sigma_{\phi}(\phi + \delta_n) - \delta_n. \tag{21}$$

Now, given any n-tuple $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$, we determine $S_{\phi}(E-F)$ by means of

$$S_{\phi}(E-F), \quad \text{if the parts of } (\phi + \delta_n) \text{ are } \textit{not} \text{ distinct.} \tag{22a}$$

Otherwise,

$$S_\phi(E-F)=\epsilon(\sigma_\phi)\cdot S_{(\phi)_{\sigma\phi}}(E-F), \tag{22b}$$

where $(\phi)_{\sigma_\phi}$ is the partition given by (21) and $\epsilon(\sigma_\phi)$ is the sign of the permutation σ_ϕ. (Note that if ϕ is a partition, then the parts of $(\phi+\delta_n)$ are distinct, δ_ϕ is the identity permutation, and the right-side of (22b) is simply $(S_\phi(E-F)$.)

THEOREM 23. — *One has*

$$S_\lambda(X+Y)=\sum_{\pi,\,\alpha\subset\lambda} c^\lambda_{\pi\alpha}S_\alpha(Y)S_\pi(X), \tag{24}$$

where $c^\lambda_{\pi\alpha}$ are the L–R *coefficients in* (13), *and X and Y are two (signed) sums of monomials.*

THEOREM 25 (A. Lascoux). — *One has*

$$S_\pi(-1+\gamma_1,\ldots,-1+\gamma_{\mu+1})=\sum_{\nu\subset\pi} d_{\pi\nu}S_\nu(A), \tag{26}$$

where

$$d_{\pi\nu}=(-1)^{|\pi|-|\nu|}\det\left[\begin{pmatrix}\pi_i+\mu+1-i\\ \nu_j+\mu+1-j\end{pmatrix}\right]_{1\le i,\,j\le\mu+1}, \tag{27}$$

and

$$A=\{\gamma_1,\gamma_2,\ldots,\gamma_{\mu+1}\}.$$

THEOREM 23 appears in [9; see eq. (5.9) on p. 41] and THEOREM 25 is due to A. LASCOUX [8] and can be found in [9; see ex. 10, p. 30].

The first step of the umbral calculus is to compute

$$ {}^{m}_{\mu}G^{(n)}_{q-1}(\gamma_1-1,\gamma_2-1,\ldots,\gamma_{\mu+1}-1,B;F). \tag{28}$$

This is accomplished by means of the $q-1$ case of THEOREM 16, and THEOREMS 23 and 25. The second (and deepest) part of the umbral calculus is to replace $S_\nu(A)\cdot S_\alpha(B-F)$ in the sum giving (28) by

$$S_{(((\mu+1+m-n)^{\mu+1}+\nu)\oplus\alpha)}(E-F). \tag{29}$$

where (29) is determined by the $\phi=\big(((\mu+1+m-n)^{\mu+1}+\nu)\oplus\alpha\big)$ case of *Definition* 20. The resulting sum is equal to ${}^{m}_{\mu}G^{(n)}_{q}(E;F)$. That is, we have

THEOREM 30 (umbral calculus). — *Let* ${}^{m}_{\mu}G^{(n)}_{q-1}(E;F)$ *be given by the* $q-1$ *case of* (17). *For technical reasons assume that the following inequalities hold :* $(\mu+1)q \le n$, $m \ge \mu+1$, $m-n=k$ *is constant,* $l(\nu) \le \mu+1$, *and* $l(\alpha) \le n-(\mu+1)$. *We then have*

$$(31)\quad {}^{m}_{\mu}G^{(n)}_{q}(E;F) = (-1)^{\binom{\mu+1}{2}} \sum_{\lambda} a_{\lambda,q-1} \cdot \left\{ \sum_{\substack{\pi,\,\alpha\subset\lambda \\ \nu\subset\pi}} c^{\lambda}_{\pi\alpha} \cdot d_{\pi\nu} \cdot S_{\left(((\mu+1+m-n)^{\mu+1}+\nu)\oplus\alpha\right)}(E-F) \right\},$$

where $|\lambda| \le (\mu+1)(\mu+1+k)(q-1)$ *and* $\lambda \subset [(\mu+1+k)(q-1)]^{(\mu+1)(q-1)}$ *and where* $S_{\left(((\mu+1+m-n)^{\mu+1}+\nu)\oplus\alpha\right)}(E-F)$ *is determined by the* $\phi = (((\mu+1+m-n)^{\mu+1}+\nu)\oplus\alpha)$ *case of Definition* 20, $c^{\lambda}_{\pi\alpha}$ *are the* L–R *coefficients in* (13), *and the* $d_{\pi\nu}$ *are defined by* (27).

THEOREM 1.22 of [5] expressed ${}^{m}_{\mu}G^{(n)}_{1}(E;F)$ as a sum of products of Schur functions $S_\alpha(F)\cdot S_\beta(F)$. Using (15b) to rewrite this sum immediately gives

$$(32)\qquad {}^{m}_{\mu}G^{(n)}_{1}(E;F) = (-1)^{\binom{\mu+1}{2}} S_{(\mu+1+m-n)^{\mu+1}}(E-F),$$

where $(\mu+1+m-n)^{\mu+1}$ denotes the partition consisting of $(\mu+1)$ parts equal to $(\mu+1+m-n)$.

In [5] it took three pages to write ${}^{2}_{1}G^{(n)}_{2}(E;F)$ as a sum of products of Schur functions. Starting with (32) and making use of THEOREM 30 it was shown in [1] that

$$\begin{aligned}(33)\quad {}^{2}_{1}G^{(n)}_{2}(E;F) = \{S_{2^2}(D)\}^2 + \{&-[2S_{4,3}(D) + 2S_{2^3,1}(D) \\ &+ 3S_{3,2^2}(D) + 3S_{3^2,1}(D) + S_{4,3,1}(D) + S_{3,2,1^2}(D)] \\ &+ [S_{4,2}(D) + S_{2^2,1^2}(D) + 3S_{2^3}(D) + 3S_{3^2}(D) + 3S_{3,2,1}(D)] \\ &+ [2S_{3,2}(D) + 2S_{2^2,1}(D)] + S_{2^2}(D)\},\end{aligned}$$

where D denotes the set difference $E-F$.

Further applications of THEOREMS 16 and 30, as well as detailed proofs, can be found in [1].

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