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EULERIAN NUMBERS, FOULKES CHARACTERS AND LEFSCHETZ CHARACTERS OF S_n

BY

Adalbert KERBER and Karl-Joseph THÜRLINGS

Abstract. — The aim of this talk is to point out a connection between the characters which Foulkes introduced in order to give a representation theoretical generalization of Eulerian numbers and certain Lefschetz characters of S_n which A. Björner mentioned at his talk in Feuerstein and which were described in detail by R. Stanley in [1]. The missing link is a theorem on the irreducible constituents of Foulkes' characters.

1. Eulerian numbers. — Let $\pi = \big(\pi(1)\ldots\pi(n)\big)$ be an element of the symmetric group S_n, e.g. $(13248765) \in S_8$ (list notation!) with the *up-and-down-sequence* $A(\pi)$ (rises indicated by $+$, falls denoted by $-$), for example

$$A((13248765)) = \ + - + + - - - \, .$$

The number of permutations with a given number of rises, i.e. of entries $+$ in its up-down sequence, defines an *Eulerian number :*

$$A(n,k) := |\{\pi \in S_n \mid A(\pi) \text{ has } k \text{ rises}\}| \, .$$

According to Foulkes, $A(\pi)$ yields a skew diagram via the rule

$$\begin{array}{cc} + \leftarrow & \times \\ & \downarrow - \end{array}$$

This means, that to an entry $+$ of $A(\pi)$ there corresponds a node $\times$ that has to be added at the left of the last node added, and in the same row. Correspondingly to an entry $-$ there corresponds a node that has to be added just below the last node. For example the sequence $(+-++---)$ mentioned above gives

$$\begin{array}{cccc} & & \times & \times \\ \times & \times & \times & \\ \times & & & \\ \times & & & \\ \times & & & \end{array} \qquad \textit{according to} \qquad \begin{array}{cccc} & & + & \times \\ + & + & - & \\ - & & & \\ - & & & \\ - & & & \end{array} \; .$$

This resulting skew diagram is the rim part of $\lambda(A)$: $R_{11}^{\lambda(A)} = (431^3)/(2)$. Recall the definition of *skew representation* by the Littlewood-Richardson rule :

$$[\lambda/\mu] := \sum_{\nu}([\mu][\nu],[\lambda])[\nu].$$

THEOREM (FOULKES). — *The number of permutations with given up-down sequence A can be identified with a dimension of a skew representation :*

$$|\{\pi \mid A(\pi) = A\}| = dim[R_{11}^{\lambda(A)}].$$

For example, if we put $A := + - + + - - -$ then the number of permutations with this sequence isthe dimension of $[(431^3)/(2)]$ which has the decomposition $[421^2] + [41^4] + [3^2 1^2] + [321^3]$ and therefore the dimension 245. More generally we have the following generalization of Foulkes' result :

THEOREM (K/TH). — *The decomposition of $[R_{11}^{\lambda(A)}]$ is obtained from A by successive applications of the rule*

$$\begin{array}{ccc} & \times & \nearrow + \\ \swarrow - & & \end{array}$$

By this pictorial description we mean that to an entry $+$ of A there corresponds a node $\times$ which has to be added to the right of the last node, maybe in a higher row, while to an entry $-$ there corresponds a node added to the left of the last node or in a lower row. Consider once more the example $A = (+ - + + - - -)$. We start with a node $\times$, and the first entry of A is a $+$, so the corresponding node has to be added, according to the rule, to the right of the starting node, i.e. we obtain the diagram $\times\otimes$, where the last node added is encircled. Now the second entry of A is a minus sign, hence the corresponding addition of a node is again uniquely determined, and we get the diagram

$$\begin{array}{cc} \times & \times \\ \otimes & \end{array}.$$

The next entry of A is a plus sign, so that there are two places open for an additional node which are to the right of the node which was added last time :

$$\begin{array}{ccc} \times & \times & \otimes \\ \times & & \end{array} \quad and \quad \begin{array}{cc} \times & \times \\ \times & \otimes \end{array}.$$

The next steps yield the following cascade of diagrams :

$$\begin{array}{cccc} \times & \times & \times & \otimes \\ \times & & & \\ & \swarrow & \searrow & \end{array} \qquad \begin{array}{ccc} \times & \times & \otimes \\ \times & \times & \\ \swarrow & & \searrow \end{array}$$

$$
\begin{array}{cccc}
\begin{array}{llll} \times & \times & \times & \times \\ \times & & & \\ \otimes & & & \end{array}
&
\begin{array}{llll} \times & \times & \times & \times \\ \times & \otimes & & \end{array}
&
\begin{array}{lll} \times & \times & \times \\ \times & \times & \\ \otimes & & \end{array}
&
\begin{array}{lll} \times & \times & \times \\ \times & \times & \otimes \end{array}
\\
\downarrow & \downarrow & \downarrow & \downarrow
\\
\begin{array}{llll} \times & \times & \times & \times \\ \times & & & \\ \times & & & \\ \times & & & \\ \times & & & \end{array}
&
\begin{array}{llll} \times & \times & \times & \times \\ \times & \times & & \\ \times & & & \\ \times & & & \end{array}
&
\begin{array}{lll} \times & \times & \times \\ \times & \times & \\ \times & & \\ \times & & \\ \times & & \end{array}
&
\begin{array}{lll} \times & \times & \times \\ \times & \times & \times \\ \times & & \\ \times & & \end{array}
\end{array}
$$

Hence from $A = (+-++---)$ we obtain the diagrams

$$[4, 1^4], [4, 2, 1^2], [3, 2, 1^3], [3^2, 1^2],$$

and each one of them exactly once.

Proof. — "By example"

$$
\begin{aligned}
R_{11}^{\lambda((+--++--+++))} &= \det \begin{pmatrix} [2] & [3] & [6] & [7] & [11] \\ 1 & [1] & [4] & [5] & [9] \\ 0 & 1 & [3] & [4] & [8] \\ 0 & 0 & 1 & [1] & [5] \\ 0 & 0 & 0 & 1 & [4] \end{pmatrix} \\
&= [4] \det \begin{pmatrix} [2] & [3] & [6] & [7] \\ 1 & [1] & [4] & [5] \\ 0 & 1 & [3] & [4] \\ 0 & 0 & 1 & [1] \end{pmatrix} - \det \begin{pmatrix} [2] & [3] & [6] & [11] \\ 1 & [1] & [4] & [9] \\ 0 & 1 & [3] & [8] \\ 0 & 0 & 1 & [5] \end{pmatrix} \\
&= [4][(7, 6, 6, 4)/(5, 5, 3, 3)] - [(8, 7, 7, 5)/(6, 6, 4)] \\
&= [4][R_{11}^{\lambda((+--++-))}] - [R_{11}^{\lambda((+--++-++++))}].
\end{aligned}
$$

Hence, the following lemma completes the proof.

LEMMA. — *Let A denote an up-and-down sequence. Then, for each $k \in \mathbf{N}$ we have*

$$[k+1][R_{11}^{\lambda(A)}] = [R_{11}^{\lambda((A++\cdots+))}] + [R_{11}^{\lambda((A-+\cdots+))}].$$

2. Foulkes characters. — Foulkes' result gives the following interpretation of Eulerian numbers as sums of dimensions of skew representations :

$$A(n, k) = \sum_{A,k \ ups} \dim[R_{11}^{\lambda(A)}],$$

while the above generalization gives a more general result in terms of characters :

$$\chi^{n,k} := \sum_{A,k \ ups} \chi^{R_{11}^{\lambda(A)}}.$$

We suggest to call these characters *Foulkes characters.* They have the following remarkable properties :

THEOREM (FOULKES). —

(i) no. of cycles of $\pi =$ *no. of cycles of* $\rho \Rightarrow \chi^{n,k}(\pi) = \chi^{n,k}(\rho)$;

(ii) $\chi^{n,0} = \zeta^{(1^n)}, \quad \chi^{n,n-1} = \zeta^{(n)}, \quad \chi^{n,k} = \zeta^{(1^n)} \otimes \chi^{n,n-1-k}$;

(iii) The Foulkes characters satisfy the following recursion :

$$\chi_\mu^{n,k} = \chi_{\mu^*}^{n-1,k-1} - \chi_{\mu^*}^{n-1,k}, \mu^* := (\mu_1, \ldots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \ldots).$$

FURTHER PROPERTIES.

(i) $(\chi^{n,k}, \zeta^\lambda) > 0 \Rightarrow \lambda_1 \leq k+1, \quad \lambda_1' \leq n-k$;

(ii) $(\chi^{n,k}, \zeta^{(j+1,1^{n-j-1})}) > 0 \Leftrightarrow j = k$;

(iii) *The* $\chi^{n,k}$ *are linearly independent;*

(iv) *If* $\chi : S_n \to \mathbf{C}$ *denotes a character, depending only on the number of cyclic factors, then we have*

$$\chi = \sum_i \frac{(\chi, \zeta^{(i+1,1^{n-i-1})})}{f^{(i+1,1^{n-i-1})}} \chi^{n,i}.$$

Using 5.8.30 in KERBER-THÜRLINGS we obtain

THEOREM. — *The "Pólya-character"* χ, *defined by*

$$\chi(\pi) := m^{\text{no. of cycles of } \pi}$$

has the following decomposition into irreducibles :

$$\chi = \sum_k \binom{m+k}{n} \chi^{n,k}.$$

3. Foulkes tables. — This section contains the Foulkes tables $F_i := (\chi_j^{n,k})$ of the symmetric groups S_n, for $n \leq 7$. We recall that the j-th column of the Foulkes table contains in its i-th row the value of the Foulkes characters $\chi^{n,i}$ on the classes of elements which consist of j cyclic factors.

The 0-th row indicates the column numbers j, while the 0-th column shows the row numbers i.

$$F_1 = \begin{array}{cc} i\backslash j & 1 \\ 0 & 1 \end{array},\ F_2 = \begin{array}{ccc} i\backslash j & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{array},\ F_3 = \begin{array}{cccc} i\backslash j & 3 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \\ 2 & 1 & 1 & 1 \end{array},$$

$$F_4 = \begin{array}{ccccc} i\backslash j & 4 & 3 & 2 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 1 & 11 & -3 & -1 & 3 \\ 2 & 11 & 3 & -1 & -3 \\ 3 & 1 & 1 & 1 & 1 \end{array},\ F_5 = \begin{array}{cccccc} i\backslash j & 5 & 4 & 3 & 2 & 1 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & 26 & -10 & 2 & 2 & -4 \\ 2 & 66 & 0 & -6 & 0 & 6 \\ 3 & 26 & 10 & 2 & -2 & -4 \\ 4 & 1 & 1 & 1 & 1 & 1 \end{array},$$

$$F_6 = \begin{array}{ccccccc} i\backslash j & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 57 & -25 & 9 & -1 & -3 & 5 \\ 2 & 302 & -40 & -10 & 8 & 2 & -10 \\ 3 & 302 & 40 & -10 & -8 & 2 & 10 \\ 4 & 57 & 25 & 9 & 1 & -3 & -5 \\ 5 & 1 & 1 & 1 & 1 & 1 & 1 \end{array},$$

$$F_7 = \begin{array}{cccccccc} i\backslash j & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 120 & -56 & 24 & -8 & 0 & 4 & -6 \\ 2 & 1191 & -245 & 15 & 19 & -9 & -5 & 15 \\ 3 & 2416 & 0 & -80 & 0 & 16 & 0 & -20 \\ 4 & 1191 & 245 & 15 & -19 & -9 & 5 & 15 \\ 5 & 120 & 56 & 24 & 8 & 0 & -4 & -6 \\ 6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}.$$

4. The connection with a result of Stanley. — We consider groups acting on posets M such that $x \leq y \Leftrightarrow gx \leq gy$. An important example is the action of S_n on 2^n, the power set of n, with the inclusion as partial order.

Denote by R a subset of the set of ranks, and by $K_R(M)$ a set of rank selected chains. Put $K_R(M, \mathbf{C}) := \mathbf{C}^{K_R(M)}$ and denote by $H_i(M_R, \mathbf{C})$ the *homology group*. Using these notions we can introduce

$$\kappa_R(g) := \text{trace of } g \text{ on } K_R(M, \mathbf{C}), \gamma_{R,i}(g) := \text{trace of } g \text{ on } H_i(M_R, \mathbf{C}),$$

and

$$\nu_R(g) := \sum_{i=0}^{r} (-1)^{|R|-i} \gamma_{R,i}(g),$$

the *Lefschetz character*. Then, to begin with, we have the following well known facts :

$$\kappa_R = \sum_{T \subseteq R} \nu_T, \quad \textit{or, equivalently,} \quad \nu_R = \sum_{T \subseteq R} (-1)^{|R \setminus T|} T.$$

THEOREM (STANLEY). — *If* $R := (n_1, \dots, n_k)_<$, $\rho := (n_1, n_2 - n_1, \dots, n_k - n_{k-1}, n - n_k)$ *;* ρ^* *:= partition obtained by reordering, then*

(i) $\kappa_R = \xi^{\rho^*}$*, the Young character,* $= \sum_{\lambda \vdash n} \left| ST^{\lambda'}(\rho^*) \right| \zeta^\lambda$ *(standard tableaux, shape* λ'*, content* ρ^**).*

(ii) $\nu_R = \sum_{\lambda \vdash n} \left| ST_R^{\lambda'}(1^n) \right| \zeta^\lambda$ *(standard Young tableaux with* R *as set of ascents).*

Hence we obtain from the above discussion of Foulkes characters :

THEOREM.

$$\chi^{n,n-k-1} = \sum_R \nu_R \qquad (|R| = k).$$

This shows the connection between Foulkes characters and the Lefschetz characters of S_n on 2^n.

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Adalbert KERBER and Karl-Josef THÜRLINGS,
Lehrstuhl II für Mathematik,
Universität Bayreuth,
D-95440 Bayreuth, Deutschland
email : kerber@btm2x6.mat.uni-bayreuth.de