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COMPOSITIONAL INVERSION OF TRIANGULAR SETS OF SERIES

BY

Marilena BARNABEI

There are two possible definitions of formal Laurent series in infinitely many variables : they can be requested to be either series such that the set of their monomials has an infimum, or series for which the set of weights of their monomials has a minimum : in this second case, it is necessary to add a finiteness condition on monomials of the same weight, in order to perform products and, hence, compositions. So, we get two different algebras of Laurent series, say L^1, L^2, which are not comparable. L^1 has properties which are very similar to those of the algebra of Laurent series in one variable, and these properties have been extensively studied in [1]. On the contrary, in L^2 we find many pathological situations, including the fact that it seems impossible to characterize those sets of power series (namely, series whose monomials have only positive exponents) which can be composed with any Laurent series. Nevertheless, it is very natural to look for a Lagrange-type inversion formula which allows us to inverte such a "simple" set of series as $(x_i + x_{2i}^2)$, $i \in \mathbf{N}$. Here, we solve this problem for a special class of sets of power series in L^2, namely, triangular P-sets. Our result is completely characteristic-free.

Let $\mathbf{D}$ be the set of all maps $\mathbf{d} : \mathbf{N} \to \mathbf{Z}$ with finite support ; such maps will be called *degrees*. Sum and order relation between degrees are defined pointwise. The *weight* of a degree $\mathbf{d}$ is the integer $w(\mathbf{d}) := \sum \mathbf{d}(i)$. Set

$$\mathbf{D}^+ := \{\mathbf{d} \in \mathbf{D}; \mathbf{d} \geq \mathbf{0}\}$$

where $\mathbf{0}$ is the zero-degree.

Let now $\mathbf{A}$ be a commutative integral domain with unity, and let $\mathbf{U}$ denote the group of units of $\mathbf{A}$. A *formal series in infinitely many variables* over $\mathbf{A}$ will be a map $\alpha : \mathbf{D} \to \mathbf{A}$. We set $a_{\mathbf{d}} := \alpha(\mathbf{d}) =: \langle \mathbf{d} \mid \alpha \rangle$ and write $\alpha = \sum_{\mathbf{d}} a_{\mathbf{d}} \tau^{\mathbf{d}}$, where $\tau_1, \tau_2, \ldots$ are formal variables, and $\tau^{\mathbf{d}} := \prod_n \tau_n^{\mathbf{d}(n)}$.

A series α will be called a *Laurent series* whenever :

1) for every $k \in \mathbf{Z}$, the set $\{\mathbf{d} \in \mathbf{D}; \langle \mathbf{d} \mid \alpha \rangle \neq 0 \text{ and } w(\mathbf{d}) = k\}$ is finite, and

2) the set $\{w(\mathbf{d}); \mathbf{d} \in \mathbf{D}, \langle \mathbf{d} \mid \alpha \rangle \neq 0\}$ has a minimum, which is called the *weight* of α and will be denoted by $w(\alpha)$. The weight of the zero series is meant as $+\infty$.

Let α, β be two Laurent series; the *convolution product* $\alpha\beta$ is defined as

$$\alpha\beta := \sum_{\mathbf{d}} \left(\sum_{\mathbf{f}} \langle \mathbf{f} \mid \alpha \rangle \langle \mathbf{d} - \mathbf{f} \mid \beta \rangle \right) \tau^{\mathbf{d}};$$

this makes sense, since the inner sums are finite, and yields a Laurent series.

It is immediate that, under the usual pointwise sum and the convolution product, the set L of all Laurent series turns out to be a commutative $\mathbf{A}$-algebra. It can be also proved that L is an integral domain, and for every $\alpha, \beta \in L$, $w(\alpha, \beta) = w(\alpha) + w(\beta)$. The identity element with respect to the product will be denoted by υ. We explicitly note that the convolution product $\alpha\beta$ is defined even if α is a Laurent series and β is any series satisfying condition 2).

Straightforward computations show that a Laurent series α admits multiplicative inverse α^{-1} whenever it has exactly one monomial of minimum weight β. If this is the case, setting $\alpha = \beta + \gamma$, we have

$$\alpha^{-1} = \beta^{-1} \sum_{n \geq 0} (-1)^n \gamma^n \beta^{-n}.$$

A *power series* will be a Laurent series α such that $\langle \mathbf{d} \mid \alpha \rangle \neq 0$ only for $\mathbf{d} \in \mathbf{D}^+$. The set P of all power series is of course a subalgebra of L.

A collection $\alpha := (\alpha_i)$, $i \in \mathbf{N}$, of power series will be called a *P-set* whenever :

i) $w(\alpha_i) > 0$ for every i;

ii) for every degree $\mathbf{d}$, the set $\{i \in \mathbf{N}; \langle \mathbf{d} \mid \alpha_i \rangle \neq 0\}$ is finite.

If $\alpha := (\alpha_i)$ is a P-set and $\mathbf{d} \in \mathbf{D}$, set $\alpha^{\mathbf{d}} := \sum_i \alpha_i^{\mathbf{d}(i)}$.

Let now $\alpha := (\alpha_i)$ be a P-set, and β a Laurent series. The *composition* $\beta \circ \alpha$ is defined as

$$\beta \circ \alpha := \sum_{\mathbf{d}} \langle \mathbf{d} \mid \beta \rangle \alpha^{\mathbf{d}}.$$

It is easily checked that the definition makes sense and, moreover :

1) if β is a power series, $\beta \circ \alpha$ is again a power series;

2) if β is a Laurent series, $\beta \circ \alpha$ is a series with minimum weight (namely, a series satisfying condition 2)), but not, in general, a Laurent series : as an example, take

$$\beta = \sum_{n \geq 2} \tau_n^{2n-2} / \tau_1^{n-1}$$

and $\alpha = (\alpha_i)$, with $\alpha_1 = \tau_1^2$ and $\alpha_n = \tau_n$ for $n \geq 2$; then

$$\beta \circ \alpha = \sum_{n \geq 2} (\tau_n / \tau_1)^{2(n-1)}$$

and this is a series consisting of infinite monomials of weight zero. In the sequel, we will consider only compositions between power series and P-sets.

The *composition of two P-sets* $\alpha := (\alpha_i)$ and β is defined as follows :

$$\alpha \circ \beta := (\alpha_i \circ \beta).$$

It is immediately seen that this gives an associative operation between P-sets, whose identity element is the P-set $\tau := (\tau_i)$.

We are now interested in those P-sets α which are invertible with respect to composition, namely, such that there exists another P-set β such that $\alpha \circ \beta = \tau = \beta \circ \alpha$. If this is the case, then β is of course unique, and we call it the *inverse set* of α, denoting it by $\tilde{\alpha}$. It is obvious that, if $\alpha := (\alpha_i)$ is an invertible P-set, then we must have $w(\alpha_i) = 1$ for every i. In order to study invertible P-sets, we begin with the following special case :

Proposition 2.1. — *Let $\alpha := (\alpha_i)$ be a P-set, with*

$$\alpha_i = a_i \tau_i + \hat{\alpha}_i, \quad a_i \in \mathbf{U}, \quad w(\hat{\alpha}_i) \geq 2, \quad i \in \mathbf{N}.$$

Then, α is invertible.

Obviously, the same result holds also for P-sets $\alpha := (\alpha_i)$ of the form

$$\alpha_i = a_i \tau_{\sigma(i)} + \hat{\alpha}_i \qquad (a_i \in \mathbf{U},\ w(\hat{\alpha}_i) \geq 2),$$

for some bijection $\sigma : \mathbf{N} \to \mathbf{N}$.

Let now $\alpha := (\alpha_i)$ be a P-set with $w(\alpha_i) = 1$ for every i; the *linear part* of α will be the P-set $\lambda := (\lambda_i)$, with

$$\lambda_i := \sum_{\mathbf{d} : w(\mathbf{d}) = 1} \langle \mathbf{d} \mid \alpha_i \rangle \tau^{\mathbf{d}} \qquad \text{for every } i.$$

A P-set which coincides with its linear part will be called a *linear P-set*.

There exist linear P-sets whose compositional inverse consists of series not satisfying condition 1) (so, they are not power series) : as an example, take $\alpha_i := \tau_i + \tau_{i+1}, \quad i \in \mathbf{N}$. Then, it is easily checked that the set of series

$$\beta_i := \sum_{k \geq i} (-1)^{k+i} \tau_k, \quad i \in \mathbf{N}$$

is the compositional inverse of (α_i), and each series β_i consists of infinitely many monomials of weight one. On the other hand, we have :

PROPOSITION 2. — *A P-set consisting of series of weight one is invertible if and only if its linear part is invertible.*

In the sequel, we will be concerned with sets of power series $\alpha := (\alpha_i)$, $i \in \mathbf{N}$, such that :

1) $\alpha_i = a_i\tau_i + \hat{\alpha}_i$, with $a_i \in \mathbf{U}$ and $w(\hat{\alpha}_i) \geq 2$;

2) $\partial\alpha_i/\partial\tau_j = 0$ for $i > j$, where $\partial/\partial\tau_j$ denotes the (formal) partial derivative with respect to τ_j.

Such sets of series are automatically P-sets, and we will call them *triangular P-sets*. By PROPOSITION 2.1, every triangular P-set is invertible. If $\alpha := (\alpha_i)$ is a triangular P-set, set

$$P(\alpha_i) := \frac{\tau_i}{\alpha_i}\frac{\partial\alpha_i}{\partial\tau_i} \qquad \text{for every } i\ ;$$

$P(\alpha_i)$ is a Laurent series of weight zero, $\langle 0 \mid P(\alpha_i)\rangle = 1$, and in its monomials τ_i is the only variable which can appear with negative exponent. Moreover, by definition, $\langle \mathbf{d} \mid P(\alpha_i)\rangle \neq 0$ implies $i \leq \min\operatorname{supp}(\mathbf{d})$. Hence, setting $P(\alpha_i) = \upsilon + \gamma_i$, we have $\langle \mathbf{0} \mid \gamma_i\rangle = 0$ and, for every $J \subseteq \mathbf{N}$, J finite, $\langle \mathbf{d} \mid \prod_{j\in J}\gamma_i\rangle \neq 0$ only if $j \leq \max\operatorname{supp}(\mathbf{d})$ for every $j \in J$. This implies that the infinite product

$$\prod_{i\in\mathbf{N}} P(\alpha_i) := \upsilon + \sum_{\emptyset\neq J\subseteq\mathbf{N}}\prod_{j\in J}\gamma_j \qquad (J \text{ finite})$$

is well-defined. We set

$$P(\alpha) := \prod_{i\in\mathbf{N}} P(\alpha_i).$$

By previous remarks, for every $\mathbf{d} \in \mathbf{D}$, we have :

$$\langle \mathbf{d} \mid P(\alpha)\rangle = \langle \mathbf{d} \mid \prod_{i\leq a} P(\alpha_i)\rangle$$

with $a := \max\operatorname{supp}(\mathbf{d})$. We note explicitly that $P(\alpha)$ is a series, but not, in general, a Laurent series : for example, if $\alpha_i := \tau_i(\upsilon + \tau_j)^{-1}$, we have $P(\alpha_i) = (\upsilon + \tau_i)^{-1}$, and $P(\alpha)$ has infinitely many monomials of given weight. Nevertheless, $P(\alpha)$ can be multiplied by any Laurent series, since it has a minimum weight.

LEMMA. — *Let $\alpha := (\alpha_i)$ be a triangular P-set ; then, for every $\mathbf{d} \in \mathbf{D}$,*

$$\langle \mathbf{0} \mid \alpha^{\mathbf{d}} P(\alpha)\rangle = \delta_{\mathbf{0},\mathbf{d}}.$$

THEOREM 2 (Lagrange inversion formula). — *Let* $\alpha : (\alpha_i)$ *be a triangular* P*-set, and* $\tilde{\alpha} := (\tilde{\alpha}_i)$ *its compositional inverse; then, for every* $\mathbf{d} \in \mathbf{D}$,

$$\langle \mathbf{d} \mid \tilde{\alpha}_i \rangle = \langle -\mathbf{e}_i \mid \alpha^{-\mathbf{d}} P(\alpha) \rangle.$$

In particular, $\tilde{\alpha}$ *is again a triangular* P*-set.*

In [2], A. JOYAL, defines a composition between two power series in infinitely many variables, associating to any power series a set of series in a canonical way. As an example, we will translate this definition in our terminology, and derive an inversion formula for such sets of series.

For any fixed $n \in \mathbf{N}$ let $\tau(n)$ be the P-set

$$\tau(n) := (\tau_n, \tau_{2n}, \dots).$$

It is evident that $\tau(k) \circ \tau(n) = \tau(kn)$ for every $k, n \in \mathbf{N}$. Let now α be any power series; the *canonical set* associated to α will be the set $(\alpha \circ \tau(n))$, $n \in \mathbf{N}$. It is immediately seen that the canonical set associated to a series α is a P-set if and only if $w(\alpha)\rangle 0$. Moreover, if a canonical set α is invertible, its inverse is again a canonical set :

PROPOSITION 5.1. — *Let* α *be a power series whose canonical set* α *admits the inverse* $\beta := (\beta_n)$*; then,* β *is the canonical set associated to* β_1.

THEOREM 3. — *Let* $\alpha = a\tau_1 + \hat{\alpha}$, $a \in \mathbf{U}$ *and* $w(\hat{\alpha}) \geq 2$, *be a power series with canonical set* α, *and let* β *be the power series whose canonical set is the compositional inverse of* α*; then*

$$\langle \mathbf{d} \mid \beta \rangle = \langle -\mathbf{e}_1 \mid \prod_{k \geq 1} \big(\alpha^{-\mathbf{d}(k)} P(\alpha)\big) \circ \tau(k) \rangle$$

where

$$P(\alpha) := \frac{\tau_1}{\alpha} \frac{\partial \alpha}{\partial \tau_1}.$$

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Marilena Barnabei,
Dipartimento di matematica,
Piazza di Porta San Donato 5,
Università di Bologna,
I-40127 Bologna, Italie