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On Degenerations of the Projective Plane

Jürgen HAUSEN [a], *Katharina KIRÁLY* [a] *and Milena WROBEL* [b]

[a)] *Mathematisches Institut, Universität Tübingen,*
Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail: *juergen.hausen@uni-tuebingen.de*, *kaki@math.uni-tuebingen.de*

[b)] *Institut für Mathematik, Universität Oldenburg, 26111 Oldenburg, Germany*
E-mail: *milena.wrobel@uni-oldenburg.de*

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Abstract. Complementing results of Hacking and Prokhorov, we determine in an explicit manner all log terminal, rational, degenerations of the projective plane that allow a non-trivial torus action.

Key words: degenerations of the plane; Markov numbers; del Pezzo surfaces; torus action

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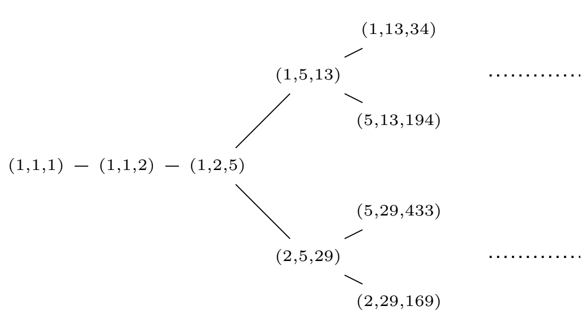
1 Introduction

The aim of this note is to determine explicitly all log terminal, rational degenerations of the projective plane $\mathbb{P}_2$ that admit a non-trivial torus action, see Theorem 5.5; note that log terminality for a surface merely means to have at most quotient singularities. Recall from [16] that a degeneration of $\mathbb{P}_2$ is the central fiber X_0 of a proper flat analytic family of surfaces over the unit disk such that $X_t \cong \mathbb{P}_2$ for all $t \neq 0$. Manetti [16] characterized the log terminal degenerations of $\mathbb{P}_2$ as the projective algebraic complex surfaces of Picard number one with vanishing plurigenera having at most singularities of the type $\frac{1}{n^2}(1, na-1)$ with coprime $a, n \in \mathbb{Z}_{>0}$, that means quotients of $0 \in \mathbb{C}^2$ by the linear action of a cyclic group of order n^2 with the weights $(1, na-1)$.

The description of all normal degenerations of the projective plane involves the *Markov numbers*. These are by definition the entries of the *Markov triples* which in turn are the triples (x, y, z) of positive integers satisfying the diophantic equation

$$x^2 + y^2 + z^2 = 3xyz.$$

From any Markov triple (x, y, z), one obtains new ones via mutations, that means by permuting its entries and passing to $(x, y, 3xy - z)$. The vertices of the *Markov tree* are the normalized (i.e., ascendingly ordered) Markov triples, representing all triples obtained from a given one by permuting its entries. In the Markov tree, two triple classes are *adjacent*, that means joined by an edge, if and only if they are distinct and arise from each other by a mutation.



The Markov tree admits an interpretation in terms of rational projective complex surfaces. With any vertex, represented by a Markov triple (k_1, k_2, l), Hacking and Prokhorov associate the weighted projective plane $\mathbb{P}_{(k_1^2,k_2^2,l^2)}$ and they show that these are in fact all toric normal degenerations of $\mathbb{P}_2$, see [9, Corollary 1.2 and Theorem 4.1]. The edges, joining adjacent Markov triples (k_1, k_2, l_1) and (k_1, k_2, l_2), are reflected combinatorially by mutations of the simplices associated with the two weighted projective planes [1, 2] and geometrically by a flat one parameter family having both of them as fibers [8, 14, 17]; moreover, [18] studies the birational geometry behind the Markov tree. For an explicit geometric realization of a given edge joining adjacent Markov triples (k_1, k_2, l_1) and (k_1, k_2, l_2), one associates with it the following surface, living in a weighted projective space, see also [8, Example 7.7]:

$$X(k_1, k_2, l_1, l_2) := V\left(T_1T_2 + T_3^{l_1} + T_4^{l_2}\right) \subseteq \mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)}.$$

By Theorem 4.5, the surface $X(k_1, k_2, l_1, l_2)$ is well defined, quasismooth (i.e., has at most cyclic quotient singularities), rational, del Pezzo, of Picard number one and it comes with an effective $\mathbb{C}^*$-action. Moreover, we obtain the following geometric connections between $X(k_1, k_2, l_1, l_2)$ and the weighted projective planes $\mathbb{P}_{(k_1^2,k_2^2,l_i^2)}$ given by the two adjacent triples, see also the (more general) Construction 3.4 and Propositions 3.2 and 3.6.

Theorem 1.1. *Let (k_1, k_2, l_1) and (k_1, k_2, l_2) be adjacent Markov triples and consider the surface $X(k_1, k_2, l_1, l_2)$. Then we have a commutative diagram*

$$\begin{tikzcd}
 & \mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)} \arrow[dl, dashed, "{[z_1,z_2,z_4^{l_1}] \mapsfrom [z_1,z_2,z_3,z_4]}"'] \arrow[dr, dashed, "{[z_1,z_2,z_3,z_4] \mapsto [z_1,z_2,z_3^{l_2}]}"] & \\
\mathbb{P}_{(k_1^2,k_2^2,l_1^2)} & X(k_1,k_2,l_1,l_2) \arrow[l, "1:l_1^2"'] \arrow[r, "l_2^2:1"'] \arrow[u] & \mathbb{P}_{(k_1^2,k_2^2,l_2^2)}
\end{tikzcd}$$

with finite coverings $X(k_1, k_2, l_1, l_2) \to \mathbb{P}_{(k_1^2,k_2^2,l_i^2)}$ of degree l_i^2, respectively. Moreover, there are flat families $\psi_i\colon \mathcal{X}_i \to \mathbb{C}$, where

$$\mathcal{X}_1 := V\left(T_1T_2 + ST_3^{l_1} + T_4^{l_2}\right) \subseteq \mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)} \times \mathbb{C},$$
$$\mathcal{X}_2 := V\left(T_1T_2 + T_3^{l_1} + ST_4^{l_2}\right) \subseteq \mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)} \times \mathbb{C}$$

and the ψ_i are obtained by restricting the projection $\mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)} \times \mathbb{C} \to \mathbb{C}$. For the fibers of these families, we have

$$\psi_i^{-1}(s) \cong X(k_1, k_2, l_1, l_2), \quad s \in \mathbb{C}^*, \qquad \psi_i^{-1}(0) \cong \mathbb{P}_{(k_1^2,k_2^2,l_i^2)}.$$

Whereas the degenerations $\mathcal{X}_i \to \mathbb{C}$ are, as mentioned, well known, the coverings haven't been observed so far to our knowledge. Note that there may occur adjacent Markov triples (k_1, k_2, l_i) with $l_i = 1$ for one of the i. This happens if and only if k_1 and k_2 are Fibonacci numbers of subsequent odd indices. In this case, the covering is in fact an isomorphism:

$$X(k_1, k_2, l_1, l_2) \cong \mathbb{P}_{(k_1^2,k_2^2,1)}.$$

If both l_i differ from one, $X(k_1, k_2, l_1, l_2)$ is non-toric. The *Fibonacci branch* of the Markov tree hosts the vertices $\mathbb{P}_{(1,k_2^2,l^2)}$ and the edges $X(1, k_2, l_1, l_2)$, where the latter surfaces also showed up in [15, Remark 6.6]. Our first main result characterizes the surfaces $X(k_1, k_2, l_1, l_2)$ by their geometric properties and shows in particular that they are uniquely determined by the underlying weighted projective planes.

Theorem 1.2. *Let X be a non-toric, log terminal, rational, projective $\mathbb{C}^*$-surface of Picard number $\rho(X) = 1$. Then the following statements are equivalent:*

(*i*) *The surface X is isomorphic to one of the surfaces $X(k_1, k_2, l_1, l_2)$.*

(*ii*) *The canonical self intersection number of X is given by $\mathcal{K}_X^2 = 9$.*

Moreover, if one of these statements holds, then X is determined up to isomorphy by the numbers k_1 and k_2.

Let us refer to the representatives $\mathbb{P}_{(k_1^2, k_2^2, l_i^2)}$ of the vertices of the Markov tree as the *toric Markov surfaces* and to the representatives $X(k_1, k_2, l_1, l_2)$ of the edges as the *Markov $\mathbb{C}^*$-surfaces*. Then the second main result of this note, Theorem 5.5, provides us with the following statement on degenerations of the projective plane $\mathbb{P}_2$ in the sense of Manetti [16, Definition 1].

Theorem 1.3. *Let X be a log terminal, rational, projective surface with a non-trivial torus action. Then the following statements are equivalent:*

(*i*) *X is a degeneration of the projective plane.*

(*ii*) *We have $\rho(X) = 1$ and $\mathcal{K}_X^2 = 9$.*

(*iii*) *X is a toric Markov surface or a Markov $\mathbb{C}^*$-surface.*

As a consequence of this characterization, given any log terminal, rational degeneration X of the projective plane with precisely one singularity $x \in X$, we can say the following: if the local Gorenstein index of $x \in X$ is a Markov but not a Fibonacci number, then X does not allow a non-trivial $\mathbb{C}^*$-action.

Moreover, let us relate the second main to *test configurations* of the projective plane. Roughly speaking, these are $\mathbb{C}^*$-equivariant flat families $\psi\colon \mathcal{X} \to \mathbb{C}$ with general fiber $\psi^{-1}(1) \cong \mathbb{P}_2$ such that the $\mathbb{C}^*$-action on $\mathbb{C}$ is given by the multiplication, see [4, Definition A.1]. According to [4, Theorem A.3], the surfaces which arise as the central fiber $\psi^{-1}(0)$ of a test configuration of $\mathbb{P}_2$ are given up to isomorphy by the vertices of the Fibonacci branch of the Markov tree and the edges touching one of the latter vertices. All the other vertices and edges from the Markov tree represent degenerations of the projective plane in the sense of [16, Definition 1] which do not occur as the central fiber of any test configuration of the projective plane.

2 Toric Markov surfaces

We provide the necessary facts on toric Markov surfaces, that means weighted projective planes given by a squared Markov triple. The main observation of the section, Proposition 2.7, characterizes the toric Markov surfaces as the toric surfaces of Picard number one and canonical self intersection nine. The reader is assumed to be familiar with the basics of toric geometry; we refer to [6] for the background.

Construction 2.1 (fake weighted projective spaces as toric varieties)**.** Consider an $n \times (n+1)$ *generator matrix*, that means an integral matrix

$$P = \begin{bmatrix} v_0 & \dots & v_n \end{bmatrix}$$

the columns $v_i \in \mathbb{Z}^n$ of which are parwise distinct, primitive and generate $\mathbb{Q}^n$ as a convex cone. For each $i = 0, \dots, n$, we obtain a convex, polyhedral cone

$$\sigma_i := \operatorname{cone}(v_j; j = 0, \dots, n, j \neq i).$$

These σ_i are the maximal cones of a fan $\Sigma = \Sigma(P)$ in $\mathbb{Z}^n$. The associated toric variety $Z = Z(P)$ is an n-dimensional *fake weighted projective space.*

The fake weighted projective spaces turn out to be precisely the $\mathbb{Q}$-factorial projective toric varieties of Picard number one. In particular, the fake weighted projective planes are exactly the projective toric surfaces of Picard number one. We will also benefit from the following alternative approach.

Remark 2.2. The fake weighted projective spaces $Z(P)$ are quotients of $\mathbb{C}^{n+1} \setminus \{0\}$. The matrix $P = (p_{ij})$ from Construction 2.1 defines a homomorphism

$$p\colon\ \mathbb{T}^{n+1} \to \mathbb{T}^n, \qquad (t_0, \ldots, t_n) \mapsto \left(t_0^{p_{10}} \cdots t_n^{p_{1n}}, \ldots, t_0^{p_{n0}} \cdots t_n^{p_{nn}}\right),$$

the subgroup $H := \ker(p) \subseteq \mathbb{T}^{n+1}$ is a direct product $\mathbb{C}^* \times G$ with a finite subgroup $G \subseteq \mathbb{T}^{n+1}$ and for the induced action of H on $\mathbb{C}^{n+1}$ we have

$$Z(P) = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/H.$$

This provides us with *homogeneous coordinates* as for the classical projective space: We write $[z_0, \ldots, z_n] \in Z(P)$ for the H-orbit through $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$.

Recall that for a point x of a normal variety X the *local class group* is the factor group $\mathrm{Cl}(X, x)$ of the group of all Weil divisors on X modulo those being principal on some open neighbourhood of x. We denote by $\mathrm{cl}(x)$ the order of $\mathrm{Cl}(X, x)$.

Remark 2.3. Consider $P = [v_0, \ldots, v_n]$ as in Construction 2.1 and the associated $Z = Z(P)$. The *fake weight vector* associated with P is

$$w = w(P) = (w_0, \ldots, w_n) \in \mathbb{Z}^{n+1}_{>0}, \qquad w_i := |\det(v_j; j = 0, \ldots, n, j \neq i)|.$$

For the divisor class group and the local class groups of the toric fixed points $z(i)$, having i-th homogeneous coordinate one and all others zero, we obtain

$$\mathrm{Cl}(Z) = \mathbb{Z}^n / \mathrm{im}(P^*) \cong \mathbb{Z} \oplus \Gamma, \qquad |\Gamma| = \gcd(w_0, \ldots, w_n), \qquad \mathrm{cl}(z(i)) = w_i.$$

Moreover, $\mathrm{Cl}(Z) \cong \mathbb{Z} \oplus \Gamma$ can be identified with the character group of $H \cong \mathbb{C}^* \times G$ from Remark 2.2 via the isomorphism $H \cong \mathrm{Spec}\, \mathbb{C}[\mathbb{Z}^n / \mathrm{im}(P^*)]$.

Remark 2.4. Let $Z = Z(P)$ arise from Construction 2.1 and $w = w(P)$ as in Remark 2.3. Then $\mathrm{Cl}(Z)$ is torsion free if and only if $w \in \mathbb{Z}^{n+1}$ is primitive. In the latter case, Z equals the *weighted projective space* $\mathbb{P}_{(w_0, \ldots, w_n)}$.

Proposition 2.5. *For a fake weighted projective plane $Z = Z(P)$ with fake weight vector $w = w(P) = (w_0, w_1, w_2)$, the canonical self intersection number is given by*

$$\mathcal{K}_Z^2 = \frac{(w_0 + w_1 + w_2)^2}{w_0 w_1 w_2}.$$

Proof. We may assume that the generator matrix P of our fake weighted projective plane Z is of the form

$$P = \begin{bmatrix} l_0 & l_1 & l_2 \\ d_0 & d_1 & d_2 \end{bmatrix}, \qquad l_i \neq 0, \quad i = 0, 1, 2.$$

Then we can use, for instance, [12, Remark 3.3] for the computation of the canonical self intersection number. ■

Definition 2.6. By a *toric Markov surface* we mean a surface isomorphic to a weighted projective plane $\mathbb{P}(k_0^2, k_1^2, k_2^2)$, where (k_0, k_1, k_2) is a Markov triple.

Proposition 2.7. *Let Z be a projective toric surface of Picard number one. Then the following statements are equivalent:*

(*i*) *Z is a toric Markov surface.*

(*ii*) *We have $\mathcal{K}_Z^2 = 9$.*

The proof relies on the following (known) elementary statement on the positive integer solutions of the "squared Markov identity"; see also [7] for recent, further going work in that direction.

Lemma 2.8. *The positive integer solutions of $(w_0 + w_1 + w_2)^2 = 9w_0w_1w_2$ are precisely the triples $\left(k_0^2, k_1^2, k_2^2\right)$, where (k_0, k_1, k_2) is a Markov triple.*

Proof. Clearly, every squared Markov triple solves the equation. For the converse, we build up the analogue of the Markov tree. Consider the involution

$$\lambda\colon\ (u_0, u_1, u_2) \mapsto \left(u_0, u_1, 9u_0u_1 - 6\sqrt{u_0u_1u_2} + u_2\right) = \left(u_0, u_1, \left(3\sqrt{u_0u_1} - \sqrt{u_2}\right)^2\right).$$

If u is a positive integer solution, then also $\lambda(u)$ is one. Moreover, if the entries of u are squares, then the entries of $\lambda(u)$ are so. Starting with $(1,1,1)$, we obtain

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & & (1,169,1156) & \\
 & & & (1,25,169) \arrow[ur, dash] \arrow[dr, dash] & & \cdots\cdots\cdots \\
 & & & & (25,169,37636) & \\
(1,1,1) \arrow[r, dash] & (1,1,4) \arrow[r, dash] & (1,4,25) \arrow[uur, dash] \arrow[ddr, dash] & & & \\
 & & & & (25,841,187489) & \\
 & & & (4,25,841) \arrow[ur, dash] \arrow[dr, dash] & & \cdots\cdots\cdots \\
 & & & & (4,841,28561) &
\end{tikzcd}$$

by successively applying λ to permutations of triples obtained so far. One directly checks that this yields the squared triples of the Markov tree. We claim

$$u_0 \le u_1 \le u_2,\ u_2 \ge 3 \quad \Longrightarrow \quad (3\sqrt{u_0u_1} - \sqrt{u_2})^2 < u_2$$

for any positive integer solution $u = (u_0, u_1, u_2)$. Suppose that we have "$\ge$" on the right hand side. Then $9u_0^2u_1^2 \ge 4u_0u_1u_2$ and we obtain

$$(u_0 + u_1 + u_2)^2 = 9u_0u_1u_2 \ge 4u_2^2.$$

Consequently $u_0 + u_1 \ge u_2$. This in turn gives us $2(u_0 + u_1) \ge u_0 + u_1 + u_2$ and the claim directly follows from the estimate

$$\frac{3}{u_2} + \frac{1}{u_0} \ge \frac{u_0}{u_1u_2} + \frac{2}{u_2} + \frac{u_1}{u_0u_2} = \frac{(u_0+u_1)^2}{u_0u_1u_2} \ge \frac{1}{4}\frac{(u_0+u_1+u_2)^2}{u_0u_1u_2} = \frac{9}{4}.$$

We conclude that every positive integer solution of $(w_0 + w_1 + w_2)^2 = 9w_0w_1w_2$ arises from $(1,1,1)$ by successively applying λ to permutations of triples. ∎

Proof of Proposition 2.7. Consider $Z = Z(P)$ as in Construction 2.1. Then, with $w = w(P)$, Proposition 2.5 tells us

$$\mathcal{K}_Z^2 = \frac{(w_0 + w_1 + w_2)^2}{w_0w_1w_2}.$$

The implication "(i) $\Rightarrow$ (ii)" is a direct consequence. For the reverse direction, we additionally use Lemma 2.8. ∎

We take a brief look at the singularities of the toric Markov surfaces; we refer to [9, Sections 2 and 4] for a comprehensive, more general treatment. Let k, p be coprime positive integers, denote by $C(k^2) \subseteq \mathbb{C}^*$ the group of k^2-th roots of unity and consider the action

$$C(k^2) \times \mathbb{C}^2 \to \mathbb{C}^2, \qquad \zeta \cdot z = (\zeta z_1, \zeta^{pk-1} z_2).$$

Then $U := \mathbb{C}^2 / C(k^2)$ is an affine toric surface and the image $u \in U$ of $0 \in \mathbb{C}^2$ is singular as soon as $k > 1$. A *singularity of type* $\frac{1}{k^2}(1, pk-1)$ is a surface singularity locally isomorphic to $u \in U$. The *local Gorenstein index* $\iota(x)$ of a point x in a normal variety X is the order of the canonical class in the local class group $\mathrm{Cl}(X, x)$.

Proposition 2.9. *The fixed points $z(i) \in Z$, $i = 0, 1, 2$, of a toric Markov surface $Z = \mathbb{P}(k_0^2, k_1^2, k_2^2)$ are of local Gorenstein index k_i and singularity type $\frac{1}{k_i^2}(1, p_i k_i - 1)$.*

Lemma 2.10. *Consider an affine toric surface U with fixed point $u \in U$. Then $u \in U$ is of type $\frac{1}{k^2}(1, pk-1)$ if and only if $\mathrm{cl}(u) = \iota(u)^2$.*

Proof. Let $u \in U$ be of type $\frac{1}{k^2}(1, pk-1)$. Choose $a, b \in \mathbb{Z}$ such that $ak - bp = 1$. Consider the affine toric surface U' given by the generator matrix

$$P' = \begin{bmatrix} k & k \\ k+b & b \end{bmatrix}.$$

The corresponding homomorphism $\mathbb{T}^2 \to \mathbb{T}^2$, $t \mapsto (t_1^k t_2^k, t_1^{k+b} t_2^b)$ extends to a toric morphism $\pi \colon \mathbb{C}^2 \to U'$. Moreover, we obtain an isomorphism

$$C(k^2) \to \ker(\pi), \qquad \zeta \mapsto (\zeta, \zeta^{pk-1}).$$

We conclude $U' \cong \mathbb{C}^2 / C(k^2)$ with $C(k^2)$ acting as needed for type $\frac{1}{k^2}(1, pk-1)$. Thus $U' \cong U$ and, using [12, Remark 3.7], we obtain

$$\mathrm{cl}(u) = |\det(P')| = k^2, \qquad \iota(u) = k.$$

Conversely, assume $\mathrm{cl}(u) = \iota(u)^2$. The affine toric surface U is given by a generator matrix P. With $k := \iota(u)$, a suitable unimodular transformation turns P into

$$P = \begin{bmatrix} k & k \\ c & b \end{bmatrix}, \qquad \gcd(c, k) = \gcd(b, k) = 1.$$

By assumption, $\mathrm{cl}(u) = |\det(P)|$ equals $\iota(u)^2 = k^2$. Thus, we may assume $c = k + b$. Take $a, p \in \mathbb{Z}$ with $ak - bp = 1$ and $p \geq 1$. Then we have an action

$$C(k^2) \times \mathbb{C}^2 \to \mathbb{C}^2, \qquad \zeta \cdot z = (\zeta z_1, \zeta^{pk-1} z_2).$$

With similar arguments as above, we verify that U is the quotient $\mathbb{C}^2 / C(k^2)$ for this action and thus see that $u \in U$ is of type $\frac{1}{k^2}(1, pk-1)$. ∎

Proof of Proposition 2.9. We have $\mathrm{Cl}(Z) = \mathbb{Z}$ and the anticanonical class of Z is given by $w_Z = k_0^2 + k_1^2 + k_2^2 = 3k_0k_1k_2 \in \mathbb{Z}$. Remark 2.3 tells us $\mathrm{Cl}(Z, z(i)) = \mathbb{Z}/k_i^2\mathbb{Z}$. As Markov numbers are pairwise coprime and not divisible by 3, we see that w_Z is of order k_i in $\mathrm{Cl}(Z, z(i))$. The assertion follows from Lemma 2.10. ∎

3 Rational projective $\mathbb{C}^*$-surfaces

We first recall the necessary theory of quasismooth, rational, projective $\mathbb{C}^*$-surfaces of Picard number one; see [3, Section 5.4] and the introductory part of [10] for the general background. Then, in Construction 3.4, we exhibit for each of our $\mathbb{C}^*$-surfaces two coverings onto fake weighted projective planes. Moreover, in Construction 3.5 and Proposition 3.6, we take an explicit look at the toric degenerations.

A point of a rational $\mathbb{C}^*$-surface X is called *quasismooth* if it is the image of a smooth point of the characteristic space $\hat{X}$ over X; see [10, Section 5]. It is a specific feature of a rational $\mathbb{C}^*$-surface that its singular quasismooth points are precisely its cyclic quotient singularities; see [13, Corollary 6.12].

Construction 3.1 (Quasismooth $\mathbb{C}^*$-surfaces of Picard number one)**.** Consider an integral 3×4 matrix of the following shape:

$$P = \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix},$$

$$1 \le d_1 \le l_1 \le l_2, \qquad \gcd(l_i, d_i) = 1, \qquad d_0 + \frac{d_1}{l_1} + \frac{d_2}{l_2} < 0 < \frac{d_1}{l_1} + \frac{d_2}{l_2}.$$

Let $Z(P)$ denote the fake weighted projective space having P as its generator matrix, see Construction 2.1. Then we obtain a surface

$$X(P) := \overline{V(1 + S_1 + S_2)} \subseteq Z(P),$$

where S_1, S_2, S_3 are the coordinates on the acting torus $\mathbb{T}^3 \subseteq Z$. The surface $X(P)$ inherits from $Z(P)$ the $\mathbb{C}^*$-action given on $\mathbb{T}^3 \subseteq Z(P)$ by

$$t \cdot s = (s_1, s_2, ts_3).$$

Proposition 3.2. *Consider $X = X(P)$ in $Z = Z(P)$ given by Construction 3.1. Then, in homogeneous coordinates on Z, we have the representation*

$$X = V\big(T_1T_2 + T_3^{l_1} + T_4^{l_2}\big) \subseteq Z.$$

The $\mathbb{C}^*$-surface X is projective, rational, quasismooth, del Pezzo and of Picard number one. With any l_1-th root ζ of -1, the $\mathbb{C}^*$-fixed points of X are

$$x_0 = [0, 0, \zeta, 1], \qquad x_1 = [0, 1, 0, 0], \qquad x_2 = [1, 0, 0, 0].$$

The fixed point x_0 is hyperbolic and x_1, x_2 are both elliptic. There are exactly two non-trivial orbits $\mathbb{C}^* \cdot z_1$ and $\mathbb{C}^* \cdot z_2$ with non-trivial isotropy groups:

$$z_1 = [-1, 1, 0, 1], \quad |\mathbb{C}^*_{z_1}| = l_1, \qquad z_2 = [-1, 1, 1, 0], \quad |\mathbb{C}^*_{z_2}| = l_2.$$

The fake weight vector $w(P) = (w_1, w_2, w_3, w_4)$ of the ambient fake weighted projective space $Z = Z(P)$ is given explicitly in terms of P as

$$w(P) = (-l_1 l_2 d_0 - l_2 d_1 - l_1 d_2, l_2 d_1 + l_1 d_2, -l_2 d_0, -l_1 d_0) \in \mathbb{Z}^4_{>0}.$$

Moreover, for the local class group orders of the three $\mathbb{C}^*$-fixed points $x_0, x_1, x_2 \in X$, we obtain

$$\mathrm{cl}(x_0) = -d_0, \qquad \mathrm{cl}(x_1) = w_2, \qquad \mathrm{cl}(x_2) = w_1.$$

Finally, the self intersection number of the canonical divisor $\mathcal{K}_X$ on X can be expressed as follows:

$$\mathcal{K}_X^2 = \left(\frac{1}{w_1} + \frac{1}{w_2}\right)\left(2 + \frac{l_1}{l_2} + \frac{l_2}{l_1}\right) = \frac{\mathrm{cl}(x_0)}{\mathrm{cl}(x_1)\,\mathrm{cl}(x_2)}(l_1 + l_2)^2.$$

Proof of Construction 3.1 and Proposition 3.2. The assumptions on l_i, d_i made in Construction 3.1 ensure that P fits into the setting of [10, Construction 4.2]. According to [10, Proposition 4.5], the output $X(P)$ is a normal, rational, projective $\mathbb{C}^*$-surface. Quasismoothness, $\rho(X) = 1$ and the statements on the fixed points are covered by [10, Propositions 4.9, 4.15 and 5.1].

We are left with the canonical self intersection number. Using the general formula [12, Proposition 7.9] for rational projective $\mathbb{C}^*$-surfaces, we directly compute

$$\mathcal{K}_X^2 = \frac{\left(\frac{1}{l_1}+\frac{1}{l_2}\right)^2}{\frac{d_1}{l_1}+\frac{d_2}{l_2}} - \frac{\left(\frac{1}{l_1}+\frac{1}{l_2}\right)^2}{d_0+\frac{d_1}{l_1}+\frac{d_2}{l_2}} = \left(\frac{1}{\mathrm{cl}(x_1)}+\frac{1}{\mathrm{cl}(x_2)}\right)\left(2+\frac{l_1}{l_2}+\frac{l_2}{l_1}\right),$$

where $\mathrm{cl}(x_i)$ are the local class group orders of the fixed points as just determined. The assertion then follows from $\mathrm{cl}(x_1) + \mathrm{cl}(x_2) = l_1 l_2 \,\mathrm{cl}(x_0)$. ∎

Proposition 3.3. *Let X be a non-toric, quasismooth, rational, projective $\mathbb{C}^*$-surface of Picard number one. Then $X \cong X(P)$ with P as in Construction 3.1.*

Proof. By [10, Theorem 4.18], we have $X = X(P)$ with a defining matrix P in the sense of [10, Construction 4.2]. Using [10, Propositions 4.9, 4.15 and 5.1], we see that P is as in Construction 3.1. ∎

Construction 3.4 (coverings onto fake weighted projective planes)**.** Let $X = X(P)$ in $Z = Z(P)$ arise via Construction 3.1 from a matrix

$$P = \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix}.$$

Let Σ denote the unique fan in $\mathbb{Z}^3$ having P as a generator matrix and define $\Sigma' \subseteq \Sigma$ to be the subfan with the maximal cones

$$\sigma_1 := \mathrm{cone}(v_1, v_3, v_4), \qquad \sigma_2 := \mathrm{cone}(v_2, v_3, v_4), \qquad \tau := \mathrm{cone}(v_1, v_2).$$

Then the open toric subvariety $Z' \subseteq Z$ given by the subfan $\Sigma' \subseteq \Sigma$ satisfies $X \subseteq Z'$. Set $\ell := \gcd(l_1, l_2)$ and $\ell_i := l_i/\ell$. Consider

$$P_1 := \begin{bmatrix} -1 & -1 & \ell_2 \\ 0 & d_0 l_1 & \ell_2 d_1 + \ell_1 d_2 \end{bmatrix}, \qquad P_2 := \begin{bmatrix} -1 & -1 & \ell_1 \\ 0 & d_0 l_2 & \ell_2 d_1 + \ell_1 d_2 \end{bmatrix}.$$

These are generator matrices for fake weighted projective planes Z_1 and Z_2. In terms of the fake weight vector $w(P) = (w_1, w_2, w_3, w_4)$ of $Z = Z(P)$, we have

$$w(P_1) = \left(\ell^{-1}w_1, \ell^{-1}w_2, w_3\right), \qquad w(P_2) = \left(\ell^{-1}w_1, \ell^{-1}w_2, w_4\right)$$

for the respective fake weight vectors. Let $\varphi_i\colon Z' \to Z_i$ be the toric morphisms defined by the linear maps $F_i\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ with the representing matrices

$$F_1 := \begin{bmatrix} 0 & 1 & 0 \\ -d_1 & d_1 & l_1 \end{bmatrix}, \qquad F_2 := \begin{bmatrix} 1 & 0 & 0 \\ d_2 & -d_2 & l_2 \end{bmatrix}.$$

Restricting to $X \subseteq Z'$ gives a finite covering $\varphi_1\colon X \to Z_1$ of degree l_1 and a finite covering $\varphi_2\colon X \to Z_2$ of degree l_2.

Proof. Everything is basic toric geometry except the statement on $\varphi_i\colon X \to Z_i$. On the acting tori $\mathbb{T}^3 \subseteq Z'$ and $\mathbb{T}^2 \subseteq Z_i$, the map $\varphi_2\colon Z' \to Z_2$ is given by

$$\varphi_2(s_1, s_2, s_3) = \left(s_1, s_2^{-d_2} s_3^{l_2}\right).$$

The points of $X \cap \mathbb{T}^3$ are of the form $\xi = (\xi_1, -1-\xi_1, \xi_2)$ with $\xi_1, \xi_2 \in \mathbb{C}^*$ such that $\xi_1 \neq -1$. For the image and the fibers, we obtain

$$\varphi_2\left(X \cap \mathbb{T}^3\right) = \left\{\eta \in \mathbb{T}^2; \eta_1 \neq -1\right\}, \qquad \varphi_2^{-1}(\varphi_2(\xi)) = \left\{(\xi_1, -1-\xi_1, \zeta\xi_2); \zeta^{l_2} = 1\right\}.$$

Consequently, φ_2 is dominant, hence surjective and its general fiber contains precisely l_2 points. With the coordinate divisors $C_1, C_2, C_3 \subseteq Z_1$, we have

$$Z_2 \setminus \varphi_2\left(X \cap \mathbb{T}^3\right) = C_1 \cup C_2 \cup C_3 \cup C_4, \qquad C_4 := \overline{\left\{\eta \in \mathbb{T}^2;\ \eta_1 = -1\right\}} \subseteq Z_2.$$

Let $D_i \subseteq X$ be the prime divisors obtained by cutting down the coordinate divisors of Z; see [10, Proposition 4.9]. Using surjectivity of φ_2, we see

$$Z_2 \setminus \varphi_2\left(X \cap \mathbb{T}^3\right) = \varphi_2\left(X \setminus \mathbb{T}^3\right) = \varphi_2(D_1) \cup \cdots \cup \varphi_2(D_4).$$

Thus, $\varphi_2\colon X \to Z_2$ must have finite fibers, proving everything we need. The map $\varphi_1\colon X \to Z_1$ can be treated in an analogous manner. ■

Construction 3.5 (degenerations to fake weighted projective planes)**.** Consider $X = X(P)$ in $Z = Z(P)$ as provided by Construction 3.1 and set

$$\mathcal{X}_1 := V\left(T_1T_2 + ST_3^{l_1} + T_4^{l_2}\right) \subseteq Z \times \mathbb{C},$$
$$\mathcal{X}_2 := V\left(T_1T_2 + T_3^{l_1} + ST_4^{l_2}\right) \subseteq Z \times \mathbb{C},$$

where the T_i are the homogeneous coordinates on Z and S is the coordinate on $\mathbb{C}$. Then $\mathcal{X}_1$ and $\mathcal{X}_2$ are invariant under the respective $\mathbb{C}^*$-actions on $Z \times \mathbb{C}$ given by

$$\vartheta \cdot ([z_1, z_2, z_3, z_4], s) = \left(\left[z_1, z_2, \vartheta^{-1} z_3, z_4\right], \vartheta s\right),$$
$$\vartheta \cdot ([z_1, z_2, z_3, z_4], s) = \left(\left[z_1, z_2, z_3, \vartheta^{-1} z_4 r\right], \vartheta s\right).$$

Restricting the projection $Z \times \mathbb{C} \to \mathbb{C}$ yields flat families $\psi_i\colon \mathcal{X}_i \to \mathbb{C}$ being compatible with the above $\mathbb{C}^*$-actions and the scalar multiplication on $\mathbb{C}$. Set

$$\tilde{P}_1 := \begin{bmatrix} d_1 & d_1 + l_1 d_0 & d_2 \\ l_1 & l_1 & -l_2 \end{bmatrix}, \qquad \tilde{P}_2 := \begin{bmatrix} d_2 & d_2 + l_2 d_0 & d_1 \\ l_2 & l_2 & -l_1 \end{bmatrix},$$

and let $\tilde{Z}_1$, $\tilde{Z}_2$ denote the associated fake weighted projective planes. Then the central fiber $\psi_i^{-1}(0)$ equals $\tilde{Z}_i$ and any other fiber $\psi_i^{-1}(s)$ is isomorphic to X.

Proof. The families $\psi_i\colon \mathcal{X}_i \to \mathbb{C}$ are those provided by [11, Construction 4.1] for $\kappa = 1, 2$ and from [11, Proposition 4.6] we infer that the $\tilde{P}_i$ are the generator matrices of the central fibers. Thus, $\psi_i^{-1}(0) = \tilde{Z}_i$. ■

Proposition 3.6. *Consider $X = X(P)$ in $Z = Z(P)$ from Construction 3.1 and the families $\mathcal{X}_i \to \mathbb{C}$ from Construction 3.5. With $w(P) = (w_1, w_2, w_3, w_4)$, we have*

$$w\left(\tilde{P}_1\right) = \left(w_1, w_2, -l_1^2 d_0\right), \qquad w\left(\tilde{P}_2\right) = \left(w_1, w_2, -l_2^2 d_0\right)$$

for the fake weight vectors of the central fibers $\tilde{Z}_1$ and $\tilde{Z}_2$. Moreover, the canonical self intersection numbers of X, $\tilde{Z}_1$, $\tilde{Z}_2$ satisfy

$$\mathcal{K}_X^2 = \mathcal{K}_{\tilde{Z}_1}^2 = \mathcal{K}_{\tilde{Z}_2}^2.$$

Proof. The statement on the fake weight vectors is obtained by direct computation. Also the identity of the canonical self intersections can be directly verified, using Propositions 2.5 and 3.2. ■

Remark 3.7. The flat families $\psi_i\colon \mathcal{X}_i \to \mathbb{C}$ from Construction 3.5 are special equivariant test configurations in the sense of [11, Definition 5.2] for the del Pezzo $\mathbb{C}^*$-surface X. Moreover, according to [11, Proposition 5.4], any other special equivariant test configuration of X has limit $\tilde{Z}_1$ or $\tilde{Z}_2$.

4 Markov $\mathbb{C}^*$-surfaces

In Construction 4.2, we associate with each pair of adjacent Markov triples a rational, projective $\mathbb{C}^*$-surface. Theorem 4.5 gathers geometric properties of these *Markov $\mathbb{C}^*$-surfaces*, showing in particular that they naturally represent the edges of the Markov graph. Theorem 5.4 characterizes the Markov $\mathbb{C}^*$-surfaces as the rational, projective $\mathbb{C}^*$-surfaces of Picard number one of canonical self intersection nine. We begin with a couple of elementary observations around Markov triples.

Lemma 4.1. *For any $0 \le k_1 \le k_2 \in \mathbb{R}$ and $0 \le l_1 \le l_2 \in \mathbb{R}$, the following three conditions are equivalent:*

(i) $l_1 l_2 = k_1^2 + k_2^2$ *and* $l_1 + l_2 = 3k_1k_2$,

(ii) $l_1^2 + k_1^2 + k_2^2 = 3l_1k_1k_2$ *and* $l_2^2 + k_1^2 + k_2^2 = 3l_2k_1k_2$,

(iii) l_1, l_2 *are given in terms of* k_1, k_2 *as*

$$l_{1,2} = \frac{3k_1k_2 \pm \sqrt{9k_1^2k_2^2 - 4k_1^2 - 4k_2^2}}{2}.$$

In particular, given any two real numbers $0 \le k_1 \le k_2$, there exist unique real numbers $0 \le l_1 \le l_2$ satisfying Condition (i).

Proof. Suppose that (i) holds. Then the second equation gives $l_2 = 3k_1k_2 - l_1$. Plugging this into the first equation, we obtain the first equation of (ii). Similarly, considering $l_1 = 3k_1k_2 - l_2$, we arrive at the second equation of (ii). Solving the equations of (ii) for l_1 and l_2 gives (iii). If (iii) holds, then we directly compute $l_1l_2 = k_1^2 + k_2^2$ and $l_1 + l_2 = 3k_1k_2$. ■

Lemma 4.2. *Let $k_1 \le k_2$ and $l_1 \le l_2$ be positive integers. Then the following statements are equivalent:*

(i) $l_1 l_2 = k_1^2 + k_2^2$ *and* $l_1 + l_2 = 3k_1k_2$,

(ii) $l_1^2 + k_1^2 + k_2^2 = 3l_1k_1k_2$ *and* $l_2^2 + k_1^2 + k_2^2 = 3l_2k_1k_2$,

(iii) (l_1, k_1, k_2) *and* (k_1, k_2, l_2) *are Markov triples.*

If one of (i) to (iii) holds, then $\gcd(l_1, l_2) = 1$ and the triples (l_1, k_1, k_2), (k_1, k_2, l_2) are adjacent and normalized up to switching (l_1, k_1) in the first one.

Proof. The equivalence of (i) and (ii) holds by Lemma 4.1 and the equivalence of (ii) and (iii) is valid by the definition of a Markov triple. Now assume that one of the three conditions holds. Then k_1, k_2, l_2 form a Markov triple, hence are pairwise coprime and, using $l_2 = 3k_1k_2 - l_1$, we see that l_1 and l_2 must be coprime as well. Moreover, $k_1 \le k_2$ and $l_1 \le l_2$ together with $l_1 + l_2 = 3k_1k_2$ imply that the triples are normalized up to switching (l_1, k_1) in the first one. Finally, $l_2 = 3k_1k_2 - l_1$ merely means that the triples are adjacent. ■

Lemma 4.3. *Let (l_1, k_1, k_2) and (k_1, k_2, l_2) be Markov triples. Then there exist integers d_1, d_2 such that*

$$k_2^2 = l_2 d_1 + l_1 d_2, \qquad 1 \le d_1 \le l_1.$$

The integers d_1 and d_2 are uniquely determined by these properties. Moreover, they satisfy $\gcd(l_i, d_i) = 1$.

Proof. Consider the factor ring $\mathbb{Z}/l_1\mathbb{Z}$. Lemma 4.2 says that l_2 and l_1 are coprime. Consequently, there is a multiplicative inverse $\bar{c}_1 \in \mathbb{Z}/l_1\mathbb{Z}$ of $\bar{l}_2 \in \mathbb{Z}/l_1\mathbb{Z}$. We claim that there is a unique $d_1 \in \mathbb{Z}$ with

$$1 \le d_1 \le l_1, \qquad \bar{d}_1 = \bar{c}_1 \cdot \bar{k}_2^2 \in \mathbb{Z}/l_1\mathbb{Z}.$$

Indeed, since (l_1, k_1, k_2) is a Markov triple, the numbers l_1 and k_2 are coprime. Thus, being a product of units, $\bar{d}_1 := \bar{c}_1 \cdot \bar{k}_2^2$ is a unit in $\mathbb{Z}/l_1\mathbb{Z}$. We take the unique representative $1 \le d_1 \le l_1$. Now, there is a unique $d_2 \in \mathbb{Z}$ with

$$l_1 d_2 + l_2 d_1 = k_2^2,$$

because of $\bar{l}_2 \cdot \bar{d}_1 = \bar{l}_2 \cdot \bar{c}_1 \cdot \bar{k}_2^2 = \bar{k}_2^2$ in $\mathbb{Z}/l_1\mathbb{Z}$. To obtain $\gcd(l_2, d_2) = 1$, look at the factor ring $\mathbb{Z}/l_2\mathbb{Z}$. There $\bar{l}_1$ admits a multiplicative inverse $\bar{c}_2$. This gives us $\bar{d}_2 = \bar{c}_2 \cdot \bar{k}_2^2$ in $\mathbb{Z}/l_2\mathbb{Z}$. Hence $\bar{d}_2$ is a unit in $\mathbb{Z}/l_2\mathbb{Z}$. ■

Construction 4.4. Let $\mu = ((l_1, k_1, k_2), (k_1, k_2, l_2))$ be a pair of adjacent Markov triples, the second triple normalized and the first one up to switching (l_1, k_1). Let $d_1, d_2 \in \mathbb{Z}$ be as provided by Lemma 4.3 and set

$$P(\mu) := \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & -1 & d_1 & d_2 \end{bmatrix}.$$

As in Construction 3.1, let $Z(\mu)$ be the toric threefold defined by the complete fan in $\mathbb{Z}^3$ having $P(\mu)$ as its generator matrix, let S_1, S_2, S_3 be the coordinates of the acting torus $\mathbb{T}^3 \subseteq Z(\mu)$ and set

$$X(\mu) := \overline{V(1 + S_1 + S_2)} \ \subseteq \ Z(\mu).$$

Theorem 4.5. *Let $X = X(\mu)$ be as in Construction 4.4. Then X is a quasismooth, rational $\mathbb{C}^*$-surface with $\mathrm{Cl}(X) = \mathbb{Z}$ and Cox ring $\mathcal{R}(X)$ given by*

$$\mathcal{R}(X) = \mathbb{C}[T_1, T_2, T_3, T_4]/\langle T_1T_2 + T_3^{l_1} + T_4^{l_2} \rangle,$$
$$\deg(T_1) = k_1^2, \qquad \deg(T_2) = k_2^2, \qquad \deg(T_3) = l_2, \qquad \deg(T_4) = l_1.$$

The ambient toric threefold $Z = Z(\mu)$ is the weighted projective space $\mathbb{P}_{(k_1^2, k_2^2, l_2, l_1)}$ and X is the zero set of a homogeneous equation of degree $k_1^2 + k_2^2 = l_1 l_2$:

$$X = V\big(T_1T_2 + T_3^{l_1} + T_4^{l_2}\big) \subseteq \mathbb{P}_{(k_1^2, k_2^2, l_2, l_1)} = Z.$$

The surface X is of Picard number one, it is non-toric if and only if $l_1 > 1$ and the $\mathbb{C}^*$-action on X is given in homogeneous coordinates by

$$t \cdot [z] = \big[t \cdot z_1, t^{-1} \cdot z_2, z_3, z_4\big].$$

The only possible singularities of X are the elliptic fixed points, given together with their local class group order, local Gorenstein index and singularity type by

$$x_1 = [0,1,0,0], \qquad \mathrm{cl}(x_1) = k_2^2, \qquad \iota(x_1) = k_2, \qquad \frac{1}{k_2^2}(1, p_2k_2 - 1),$$
$$x_2 = [1,0,0,0], \qquad \mathrm{cl}(x_2) = k_1^2, \qquad \iota(x_2) = k_1, \qquad \frac{1}{k_1^2}(1, p_1k_1 - 1),$$

with $p_i \in \mathbb{Z}_{\geq 1}$ such that $\gcd(p_i, k_i) = 1$. For the canonical self intersection of X, we have $\mathcal{K}_X^2 = 9$. Moreover, there is a commutative diagram

$$\begin{tikzcd}
& \mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)} \arrow[dl, dashed, "{[z_1,z_2,z_4^{l_1}] \mapsfrom [z_1,z_2,z_3,z_4]}"'] \arrow[dr, dashed, "{[z_1,z_2,z_3,z_4] \mapsto [z_1,z_2,z_3^{l_2}]}"] & \\
\mathbb{P}_{(k_1^2,k_2^2,l_1^2)} & X \arrow[l, "1:l_1^2"] \arrow[r, "l_2^2:1"'] \arrow[u] & \mathbb{P}_{(k_1^2,k_2^2,l_2^2)}
\end{tikzcd}$$

with finite coverings $X(k_1,k_2,l_1,l_2) \to \mathbb{P}_{(k_1^2,k_2^2,l_i^2)}$ of degree l_i^2, respectively. Finally, we obtain flat families $\psi_i \colon \mathcal{X}_i \to \mathbb{C}$, where

$$\mathcal{X}_1 := V\big(T_1T_2 + ST_3^{l_1} + T_4^{l_2}\big) \subseteq \mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)} \times \mathbb{C},$$
$$\mathcal{X}_2 := V\big(T_1T_2 + T_3^{l_1} + ST_4^{l_2}\big) \subseteq \mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)} \times \mathbb{C}$$

and the ψ_i are given by restricting the projection $\mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)} \times \mathbb{C} \to \mathbb{C}$. For the fibers of these families, we have

$$\psi_i^{-1}(s) \cong X(k_1,k_2,l_1,l_2), \quad s \in \mathbb{C}^*, \qquad \psi_i^{-1}(0) \cong \mathbb{P}_{(k_1^2,k_2^2,l_i^2)}.$$

Proof of Construction 4.4 and Theorem 4.5. First, we check that $P(\mu)$ fits into Construction 3.1 with $d_0 = -1$. By the normalizedness assumption, we have $l_1 \leq l_2$. Lemma 4.3 delivers $1 \leq d_1 \leq l_1$ and $\gcd(l_i, d_i) = 1$. Moreover,

$$-1 + \frac{d_1}{l_1} + \frac{d_2}{l_2} = \frac{-l_1l_2 + k_2^2}{l_1l_2} = -\frac{k_1^2}{l_1l_2} < 0 < \frac{k_2^2}{l_1l_2} = \frac{d_1}{l_1} + \frac{d_2}{l_2}.$$

Thus, $P = P(\mu)$ is as wanted and Proposition 3.2 says that $X = X(P)$ is a quasismooth, rational, projective $\mathbb{C}^*$-surface. For the divisor class group and the Cox ring of X, we need to have an exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\quad P^* \quad} \mathbb{Z}^4 \xrightarrow{\quad Q \quad} \mathbb{Z} \longrightarrow 0.$$

The transpose matrix P^* is injective and, as l_1 and l_2 are coprime, the columns of P^* generate a primitive sublattice of $\mathbb{Z}^4$ and thus we have a torsion free cokernel. Moreover, we directly check that $Q \cdot P^* = 0$ holds with

$$Q = \begin{bmatrix} k_1^2 & k_2^2 & l_2 & l_1 \end{bmatrix}.$$

Since k_1^2 and k_2^2 are coprime, Q maps onto $\mathbb{Z}$. Consequently, [10, Propositions 4.13 and 4.16] yield $\mathrm{Cl}(X) = \mathbb{Z}$ and the desired presentation of $\mathcal{R}(X)$. Moreover, we can identify the ambient toric variety as

$$Z(\mu) = \mathbb{P}_{(k_1^2,k_2^2,l_2,l_1)}.$$

The hyperbolic fixed point x_0 is smooth by [10, Proposition 5.1]. Proposition 3.2 and Lemmas 4.2 and 4.3 give us $\mathrm{cl}(x_1)$ and $\mathrm{cl}(x_2)$. For the local Gorenstein indices, consider the following linear forms

$$u_1 = \left[\frac{d_2-d_1}{k_2^2}, \frac{d_1-d_2}{k_2^2}, \frac{3k_1}{k_2}\right], \qquad u_2 = \left[\frac{d_1-d_2+l_2}{k_1^2}, \frac{d_2-d_1+l_1}{k_1^2}, -\frac{3k_2}{k_1}\right].$$

Using the identities $k_1^2 = l_1l_2 - l_2d_1 - l_1d_2$ and $k_2^2 = l_2d_1 + l_1d_2$ just established, one directly checks that the linear forms u_1, u_2 evaluate on the columns v_1, v_2, v_3, v_4 of P as follows:

$$\begin{array}{lll} \langle u_1, v_1\rangle = 0, & \langle u_1, v_3\rangle = 1, & \langle u_1, v_4\rangle = 1, \\ \langle u_2, v_2\rangle = 0, & \langle u_2, v_3\rangle = 1, & \langle u_2, v_4\rangle = 1. \end{array}$$

We can conclude that k_2u_1 and k_1u_2 are primitive integral vectors and [12, Proposition 8.9] yields $\iota(x_1) = k_2$ and $\iota(x_2) = k_1$. As X is quasismooth, x_1, x_2 are toric singularities and Lemma 2.10 gives us their singularity type. We obtain

$$\mathcal{K}_X^2 = \frac{\mathrm{cl}(x_0)}{\mathrm{cl}(x_1)\,\mathrm{cl}(x_2)}(l_1+l_2)^2 = \left(\frac{l_1+l_2}{k_1k_2}\right)^2 = 9$$

for the canonical self intersection number, using Proposition 3.2 and Lemma 4.2 for the last equation. The desired coverings from X onto the weighted projective planes are those from Construction 3.4 followed by the obvious ones:

$$\mathbb{P}_{(k_1^2,k_2^2,l_i)} \to \mathbb{P}_{(k_1^2,k_2^2,l_i^2)}, \qquad [z_1, z_2, z_3] \mapsto \left[z_1, z_2, z_3^{l_i}\right].$$

Finally, the families $\mathcal{X}_i \to \mathbb{C}$ are provided by Construction 3.5 and their properties claimed in the assertion are guaranteed by Proposition 3.6. ■

5 Proof of the main results

Here we prove the main results of this note, Theorems 5.4 and 5.5. A first observation is that log terminal $\mathbb{C}^*$-surfaces of Picard number one and canonical self intersection nine are quasismooth.

Proposition 5.1. *Let X be a log terminal, rational, projective $\mathbb{C}^*$-surface of Picard number one with $\mathcal{K}_X^2 = 9$. Then X is quasismooth.*

Proof. We use the description of rational, projective $\mathbb{C}^*$-surfaces of Picard number one via defining matrices P provided by [10, Construction 4.2, Proposition 4.5]. Log terminality of X imposes strong conditions on the upper rows of P, see [10, Proposition 5.9].

The non-quasismooth log terminal surface singularities are those of the types D or E, where "type D" refers to the platonic triples $(2, 2, n)$ and "type E" gathers the platonic triples $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ in Brieskorn's result [5, Satz 2.10]; see also [12, Proposition 8.14]. The first tuple of possible upper entries from [10, Proposition 5.9] is $(1, y, 2, 2)$ and the associated defining matrix is of the form

$$P = \begin{bmatrix} -1 & -l_0 & 2 & 0 \\ -1 & -l_0 & 0 & 2 \\ 0 & d_0 & d_1 & 1 \end{bmatrix}, \qquad \gcd(l_0, d_0) = \gcd(2, d_1) = 1,$$

$$d_0 < 0, \qquad l_0 \geq 2, \qquad \frac{d_1}{2} + \frac{1}{2} > 0, \qquad \frac{d_0}{l_0} + \frac{d_1}{2} + \frac{1}{2} < 0.$$

The resulting $\mathbb{C}^*$-surface X, built as in Construction 3.1, has $[1,0,0,0] \in X$ as singular point of type D. From [12, Proposition 7.9], we infer

$$\begin{aligned}\mathcal{K}_X^2 &= \frac{1}{\frac{d_1}{2}+\frac{1}{2}} - \frac{2-l_0-\frac{1}{l_0}}{d_0} - \frac{1}{l_0^2\left(\frac{d_0}{l_0}+\frac{d_1}{2}+\frac{1}{2}\right)} \\ &= \frac{2}{d_1+1} + \frac{(l_0-1)^2}{d_0 l_0} - \frac{2}{l_0^2(d_1 l_0 + 2d_0 + l_0)}.\end{aligned}$$

Since the second right-hand side term is negative, we have $\mathcal{K}_X^2 < 4$ in this case. The next tuple of upper entries is $(1,2,y,2)$, which leads to the defining matrix

$$P = \begin{bmatrix} -1 & -2 & l_1 & 0 \\ -1 & -2 & 0 & 2 \\ 0 & d_0 & d_1 & 1 \end{bmatrix}, \qquad \gcd(2,d_0) = \gcd(l_1,d_1) = 1,$$

$$d_0 < 0, \qquad l_1 \geq 2, \qquad \frac{d_1}{l_1} + \frac{1}{2} > 0, \qquad \frac{d_0}{2} + \frac{d_1}{l_1} + \frac{1}{2} < 0.$$

As before, the resulting $\mathbb{C}^*$-surface X has $[1,0,0,0] \in X$ as a singular point of type D. This time, the canonical self intersection is the following:

$$\mathcal{K}_X^2 = \frac{\left(\frac{1}{l_1}+\frac{1}{2}\right)^2}{\frac{d_1}{l_1}+\frac{1}{2}} + \frac{1}{2d_0} - \frac{1}{l_1^2\left(d_0+\frac{d_1}{l_1}+\frac{1}{2}\right)}.$$

Note that the first term is not bounded from above, so we can't estimate $\mathcal{K}_X^2$. Instead we observe that $\mathcal{K}_X^2 = 9$ is a quadratic equation in d_0 with discriminant

$$\Delta = 36d_1^2 + 36d_1 l_1 + 9l_1^2 - 8d_1 - 4l_1.$$

The key observations are that Δ factors as $\Delta = a(9a-4)$ for $a = 2d_1 + l_1 > 0$ and that $a(9a-4)$ never is a square. Thus we can conclude $\mathcal{K}_X^2 \neq 9$.

These two sample cases show all the arguments for the remaining ones: either we directly estimate $\mathcal{K}_X^2 < 9$ or we can show that $\mathcal{K}_X^2 = 9$ admits no integral solution. Concerning the latter, $(1,2,y,2)$ is in fact the most tricky case. ∎

Remark 5.2. Log terminality is essential in Proposition 5.1. For instance, consider the rational, projective $\mathbb{C}^*$-surfaces X built from the matrices

$$P = \begin{bmatrix} -1 & -l^2+4l-1 & l^2-l+1 & 0 \\ -1 & -l^2+4l-1 & 0 & 2 \\ 0 & -1 & \frac{l-l^2}{2} & 1 \end{bmatrix}, \qquad l \geq 5,$$

exactly as in Construction 3.1. Then $\rho(X) = 1$ and $\mathcal{K}_X^2 = 9$ by [12, Proposition 6.9]. From [12, Corollary 8.12], we infer that X is not log terminal, thus not quasismooth.

Definition 5.3. By a *Markov* $\mathbb{C}^*$*-surface* we mean a $\mathbb{C}^*$-surface isomorphic to some $X(\mu)$ arising from Construction 4.4.

Theorem 5.4. *Let X be a non-toric, log terminal, rational $\mathbb{C}^*$-surface of Picard number one. Then the following statements are equivalent:*

(i) *X is a non-toric Markov $\mathbb{C}^*$-surface.*

(ii) *We have $\mathcal{K}_X^2 = 9$.*

Moreover, if X satisfies (i) or (ii), then it is determined up to isomorphy by the local class group orders $\mathrm{cl}(x_1)$, $\mathrm{cl}(x_2)$ *of its elliptic fixed points.*

Proof. The implication "(i) $\Rightarrow$ (ii)" holds by Theorem 4.5. Let (ii) be valid. Then Proposition 5.1 shows that X is quasismooth. Thus, Proposition 3.3 allows us to assume $X = X(P)$ with

$$P = \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix},$$

$$2 \le l_1 \le l_2, \qquad 1 \le d_1 < l_1, \qquad \gcd(l_i, d_i) = 1, \qquad d_0 + \frac{d_1}{l_1} + \frac{d_2}{l_2} < 0 < \frac{d_1}{l_1} + \frac{d_2}{l_2}.$$

Recall from Proposition 3.2 that the fake weight vector $w(P) = (w_1, w_2, w_3, w_4)$ of the matrix P is given explicitly by

$$w(P) = (-l_1 l_2 d_0 - l_2 d_1 - l_1 d_2, l_2 d_1 + l_1 d_2, -d_0 l_2, -d_0 l_1) \in \mathbb{Z}^4_{\ge 1}.$$

We first show that $d_0 = -1$ holds and that $\left(w_1, w_2, l_i^2\right)$ is a squared Markov triple $\left(k_1^2, k_2^2, l_i^2\right)$. Consider the toric degenerations $\tilde{Z}_1$ and $\tilde{Z}_2$ of X as provided by Construction 3.5. Due to Proposition 3.6, their fake weight vectors are $w\left(\tilde{P}_1\right) = \left(w_1, w_2, -l_1^2 d_0\right)$, $w\left(\tilde{P}_2\right) = \left(w_1, w_2, -l_2^2 d_0\right)$. Moreover, this Proposition gives us $\mathcal{K}^2_{\tilde{Z}_i} = \mathcal{K}^2_X = 9$. Consequently, Proposition 2.7 provides us with Markov triples $(k_1, k_2, l_i \delta_0)$ such that $w\left(\tilde{P}_1\right) = \left(k_1^2, k_2^2, l_1^2 \delta_0^2\right)$, $w\left(\tilde{P}_2\right) = \left(k_1^2, k_2^2, l_2^2 \delta_0^2\right)$, where $d_0 = -\delta_0^2$. To conclude the first step, we have to show $\delta_0 = 1$. Computing the anticanonical self intersection of $\tilde{Z}_i$ according to Proposition 2.5, we obtain

$$9 = \frac{\left(w_1 + w_2 + l_i^2 \delta_0^2\right)^2}{w_1 w_2 l_i^2 \delta_0^2} = \frac{\left(l_1 l_2 \delta_0^2 + l_i^2 \delta_0^2\right)^2}{\left(l_i^2 \delta_0^2 - w_2\right) w_2 l_i^2 \delta_0^2}.$$

Consequently,

$$9\left(l_i^4 \delta_0^2 w_2 - l_i^2 w_2^2\right) = \delta_0^2 \left(l_1 l_2 + l_i^2\right)^2.$$

Thus, δ_0 divides $3 l_i w_2$. Since the entries of the squared Markov triple $\left(w_1, w_2, l_i^2 \delta_0^2\right)$ are pairwise coprime, we have $\gcd(l_1, l_2) = 1$ and $\delta_0 \mid 3 l_i$. Thus, $\delta_0 = 1$ or $\delta_0 = 3$. The latter is excluded, because no Markov number is divisible by 3.

Having two members in common, the Markov triples (k_1, k_2, l_i), where $i = 1, 2$, are adjacent. As seen above, we have $k_2^2 = l_2 d_1 + l_1 d_2$. Thus, the uniqueness part of Lemma 4.3 shows that we are in the setting of Construction 4.4. The equivalence of (i) and (ii) is proven. The supplement follows directly from Theorem 4.5 and, again, the uniqueness statement of Lemma 4.3. ∎

Theorem 5.5. *Let X be a log terminal, rational, projective surface with a non-trivial torus action. Then the following statements are equivalent:*

(i) *X is a degeneration of the projective plane.*

(ii) *We have $\rho(X) = 1$ and $\mathcal{K}_X^2 = 9$.*

(iii) *X is a toric Markov surface or a Markov $\mathbb{C}^*$-surface.*

Proof. Assume that (i) holds. From [16, Theorem 4, Corollary 5], we infer $\rho(X) = 1$ and $K_X^2 = 9$. If (ii) holds, then Proposition 2.7 and Theorem 5.4 yield (iii). Now, let (iii) be valid. Then Proposition 2.9 and Theorem 4.5 ensure property (d) of Manetti's Main Theorem [16, p. 90], telling us that then X is a degeneration of the plane. ∎

Remark 5.6. Note that also non-rational normal $\mathbb{C}^*$-surfaces show up as degenerations of the projective plane. To obtain a classical example, fix $h \in \mathbb{C}[T_1, T_2, T_3]$ of degree d with $V(h) \subseteq \mathbb{P}_2$ smooth and consider $\mathcal{X} := V(T_4S - h) \subseteq \mathbb{P}(1,1,1,d) \times \mathbb{C}$, where S is the coordinate on $\mathbb{C}$. The projection $\mathcal{X} \to \mathbb{C}$ to the last coordinate has fibers $\mathcal{X}_s \cong \mathbb{P}_2$ over $s \in \mathbb{C}^*$ and the central fiber $\mathcal{X}_0$ is the cone over the smooth curve $V(h) \subseteq \mathbb{P}_2$, acted on by $\mathbb{C}^*$ via $t \cdot [z] = [z_1, z_2, z_3, tz_4]$.

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