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1D Landau–Ginzburg Superpotential of Big Quantum Cohomology of $\mathbb{CP}^2$

Guilherme F. ALMEIDA [ab]

[a] *Mannheim University, Mannheim, Germany*

[b] *Max Planck Institute of Molecular Cell Biology and Genetics, Dresden, Germany*
E-mail: *feitosad@mpi-cbg.de*

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Abstract. Using the inverse period map of the Gauss–Manin connection associated with $QH^*\big(\mathbb{CP}^2\big)$ and the Dubrovin construction of Landau–Ginzburg superpotential for Dubrovin–Frobenius manifolds, we construct a one-dimensional Landau–Ginzburg superpotential for the quantum cohomology of $\mathbb{CP}^2$. In the case of small quantum cohomology, the Landau–Ginzburg superpotential is expressed in terms of the cubic root of the j-invariant function. For big quantum cohomology, the one-dimensional Landau–Ginzburg superpotential is given by Taylor series expansions whose coefficients are expressed in terms of quasi-modular forms. Furthermore, we express the Landau–Ginzburg superpotential for both small and big quantum cohomology of $QH^*\big(\mathbb{CP}^2\big)$ in closed form as the composition of the Weierstrass $\wp$-function and the universal coverings of $\mathbb{C}\setminus\big(\mathbb{Z}\oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}\big)$ and $\mathbb{C}\setminus(\mathbb{Z}\oplus z\mathbb{Z})$, respectively.

Key words: Dubrovin–Frobenius manifolds; big quantum cohomology; Landau–Ginzburg superpotential

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Contents

1 Introduction

The main goal of this paper is to investigate the geometric structure of quantum cohomology of $\mathbb{CP}^2$ by studying the correspondent 1D Landau–Ginzburg superpotential. We start by providing the necessary background and motivation to our goal.

1.1 Background

Definition 1.1 ([16, 18])**.** An analytic function $F(t)$, where $t = \left(t^1, t^2, \dots, t^n\right) \in U \subset \mathbb{C}^n$ defined in an open subset of $\mathbb{C}^n$, is considered a solution of the WDVV (Witten–Dijkgraaf–Verlinde–Verlinde) equation if its third derivatives $c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$ satisfy the following conditions:

(1) The coefficients $\eta_{\alpha\beta} = c_{1\alpha\beta}$ form elements of a constant nondegenerate matrix.

(2) The quantities $c_{\alpha\beta}^{\gamma} = \eta^{\gamma\delta} c_{\alpha\beta\delta}$ represent the structure constants of an associative algebra.

(3) The function $F(t)$ must be quasi-homogeneous.

In [16, Chapter 1], Dubrovin formulated a geometric interpretation of the WDVV equation which is given by the following.

Definition 1.2. A Frobenius algebra $\mathscr{A}$ is a finite-dimensional unital, commutative, associative algebra equipped with an invariant, non-degenerate bilinear pairing $\eta\colon \mathscr{A} \otimes \mathscr{A} \mapsto \mathbb{C}$, which is invariant in the following sense $\eta(A \bullet B, C) = \eta(A, B \bullet C)$, $\forall A, B, C \in \mathscr{A}$.

Definition 1.3 ([16, 18])**.** Let M be a complex manifold of dimension n. A Dubrovin–Frobenius structure over M consists of the following compatible objects:

(1) A family of Frobenius multiplications $\bullet_p\colon T_pM \times T_pM \mapsto T_pM$ that are analytically dependent on $p \in M$. This family induces a Frobenius multiplication on

$$\bullet\colon\ \Gamma(TM) \times \Gamma(TM) \mapsto \Gamma(TM).$$

(2) A flat pseudo-Riemannian metric η on $\Gamma(TM)$, also known as the Saito metric.

(3) A unity vector field e that is covariantly constant with respect to the Levi-Civita connection ∇ for the metric η, i.e., $\nabla e = 0$.

(4) Consider the tensor $c(X, Y, Z) := \eta(X \bullet Y, Z)$. We require the 4-tensor $(\nabla_W c)(X, Y, Z)$ to be symmetric with respect to $X, Y, Z, W \in \Gamma(TM)$.

(5) An Euler vector field E with the following properties

$$\nabla\nabla E = 0, \qquad \mathscr{L}_E \eta(X, Y) = (2 - d)\eta(X, Y), \qquad \mathscr{L}_E c(X, Y, Z) = c(X, Y, Z),$$

where $X, Y, Z \in \Gamma(TM)$. Moreover, we require ∇E to be diagonalizable. Let $\left(t^1, t^2, \dots, t^n\right)$ be the flat coordinates with respect to the metric η. These coordinates are denoted as Saito flat coordinates. The Euler vector E can be explicitly represented as

$$E = \sum_{i=1}^{n} ((1 - q_i)t_i + r_i)\partial_i.$$

Roughly speaking, Dubrovin–Frobenius manifold is the geometric structure that naturally arise in the domain of any WDVV solution, which is given by a family of Frobenius algebra on the sheaf of holomorphic vector fields, a flat structure and some suitable marked vector fields. An important example of WDVV solutions are the generating function of Gromov–Witten invariants called Gromov–Witten potential. Another source of Dubrovin–Frobenius manifolds comes from Landau–Ginzburg superpotential, which are unfolding of singularities or family of covering over $\mathbb{CP}^1$. In the analytic theory of Dubrovin–Frobenius manifold, there exist two flat connection. The 1st structure connection is called Dubrovin connection and it is defined below.

Definition 1.4 ([16, 18]). Consider the following deformation of the Levi-Civita connection defined on a Dubrovin–Frobenius manifold M, $\tilde{\nabla}_u v = \nabla_u v + zu \bullet v$, $u, v \in \Gamma(TM)$, where ∇ represents the Levi-Civita connection of the metric η, $\bullet$ denotes the Frobenius product, and $z \in \mathbb{CP}^1$. The Dubrovin connection defined in $M \times \mathbb{CP}^1$ is then given by

$$\begin{aligned} &\tilde{\nabla}_u v = \nabla_u v + zu \bullet v, \qquad \tilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}z}} \frac{\mathrm{d}}{\mathrm{d}z} = 0, \qquad \tilde{\nabla}_v \frac{\mathrm{d}}{\mathrm{d}z} = 0, \\ &\tilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}z}} v = \partial_z v + E \bullet v - \frac{1}{z}\mu(v). \end{aligned} \tag{1.1}$$

Here, μ is a diagonal matrix given by $\mu_{\alpha\beta} = \left(q_\alpha - \frac{d}{2}\right)\delta_{\alpha\beta}$.

The deformation of Levi-Civita connection (1.1) is again a flat connection. In Saito flat coordinates, the Dubrovin connection flat coordinate system, i.e., the solution of $\tilde{\nabla}\mathrm{d}\tilde{t} = 0$, can be written as

$$\begin{aligned} &\left(\tilde{\nabla}_\alpha \omega\right)_\beta = \partial_\alpha \omega_\beta - z c^\gamma_{\alpha\beta}\omega_\gamma = 0, \\ &\left(\tilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}z}} \omega\right)_\beta = \partial_z \omega_\beta - E^\sigma c^\gamma_{\sigma\beta}\omega_\gamma + \frac{\mu_\beta}{z}\omega_\beta = 0, \end{aligned}$$

where $\omega = \mathrm{d}\tilde{t} = \omega_\alpha \mathrm{d}t^\alpha$ and $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$. Alternatively, we could write the Dubrovin flat coordinate system in the matrix form as

$$\partial_\alpha \omega = z C_\alpha^{\mathsf{T}} \omega, \qquad \partial_z \omega = \left(\mathscr{U}^{\mathsf{T}} - \frac{\mu}{z}\right)\omega, \tag{1.2}$$

where

$$\omega = \sum_{\beta=1}^{n} \partial_\beta \tilde{t} \mathrm{d}t^\beta, \qquad C_\alpha = \left(c^\gamma_{\alpha\beta}\right), \qquad \mathscr{U} = \left(\mathscr{U}^\gamma_\beta\right) := \left(E^\epsilon c^\gamma_{\epsilon\beta}\right) \tag{1.3}$$

or by using the conjugate system

$$\partial_\alpha \xi = z C_\alpha \xi, \qquad \partial_z \xi = \left(\mathscr{U} + \frac{\mu}{z}\right)\xi, \qquad \text{where} \quad \xi = \eta^{-1}\omega. \tag{1.4}$$

The compatibility of the system (1.3) is guaranteed by the vanishing of the Riemann tensor associated with the connection (1.1).

The connection (1.1) is more suitable for presenting the Gromov–Witten potential. The second structure connection, also known as the extended Gauss–Manin connection, is more suitable for presenting the Landau–Ginzburg potential. In order to define this connection, we consider the multiplication by the Euler vector field,

$$E\bullet\colon\ \Gamma(TM) \mapsto \Gamma(TM), \qquad X \in \Gamma(TM) \mapsto E \bullet X \in \Gamma(TM). \tag{1.5}$$

Such an endomorphism gives rise to a bilinear form in the sections of the cotangent bundle of M as follows. Consider $x = \eta(X), y = \eta(Y) \in \Gamma(T^*M)$, where $X, Y \in \Gamma(TM)$. An induced Frobenius algebra is defined on $\Gamma(T^*M)$ by $x \bullet y = \eta(X \bullet Y)$.

Definition 1.5 ([16, 18])**.** The intersection form is a bilinear pairing in $\Gamma(T^*M)$ defined by $g^*(\omega_1, \omega_2) = \iota_E(\omega_1 \bullet \omega_2)$, where $\omega_1, \omega_2 \in \Gamma(T^*M)$, and $\bullet$ is the induced Frobenius algebra product in $\Gamma(T^*M)$.

In the flat coordinates of the Saito metric is given by

$$g^{\alpha\beta} = E^\epsilon \eta^{\alpha\mu} \eta^{\beta\lambda} c_{\epsilon\mu\lambda}. \tag{1.6}$$

The intersection form g^* of a Dubrovin–Frobenius manifold is a flat almost everywhere nondegenerate metric. The discriminant is the sub manifold such that the intersection form is degenerate

$$\Sigma = \{t \in M \mid \det(g) = 0\}. \tag{1.7}$$

Hence, the flat coordinate system of the intersection form in Saito flat coordinates

$$g^{\alpha\epsilon}(t) \frac{\partial^2 x}{\partial t^\beta \partial t^\epsilon} + \Gamma_\beta^{\alpha\epsilon}(t) \frac{\partial x}{\partial t^\epsilon} = 0 \tag{1.8}$$

has poles in (1.7), and consequently its solutions $x_a\big(t^1, \dots, t^n\big)$ are multivalued. The meromorphic connection (1.8) is called Gauss–Manin connection of the Dubrovin–Frobenius manifold. The analytical continuation of the solutions $x_a\big(t^1, \dots, t^n\big)$ has monodromy corresponding to loops around Σ. This gives rise to a monodromy representation of

$$\pi_1(M \setminus \Sigma) \mapsto \mathrm{GL}(\mathbb{C}^n), \tag{1.9}$$

which is called monodromy of the Dubrovin–Frobenius manifold. Moreover, we can extend the connection (1.8) to a connection on $M \times \mathbb{CP}^1$ as follows:

$$\left(g^{\alpha\epsilon}(t) - \lambda\eta^{\alpha\epsilon}\right) \frac{\partial^2 x}{\partial t^\beta \partial t^\epsilon} + \Gamma_\beta^{\alpha\epsilon}(t) \frac{\partial x}{\partial t^\epsilon} = 0. \tag{1.10}$$

The system (1.10) is isomonodromic, then its monodromy representation

$$\pi_1(M \times \mathbb{C} \setminus \Sigma_\lambda) \mapsto \mathrm{GL}(\mathbb{C}^n), \qquad \text{where} \quad \Sigma_\lambda = \det(\lambda - E_\bullet) \tag{1.11}$$

is isomorphic to (1.9). This fact is straightforward to check once we realise that if $x_a\big(t^1, t^2, \dots, t^n\big)$ is a solution of (1.9), then $x_a\big(t^1 - \lambda, t^2, \dots, t^n\big)$ is a solution of (1.10). The meromorphic connection (1.10) is called extended Gauss–Manin connection. Summarising,

Definition 1.6 ([16, 18])**.** Let $\tilde{\nabla}$ be the Levi-Civita connection of the intersection form (1.5). Then the extended Gauss–Manin connection is connection in $M \times \mathbb{CP}^1$ given by

$$\tilde{\nabla}_{\frac{\partial}{\partial t^\alpha}} = \frac{\partial}{\partial t^\alpha} + (\lambda - E_\bullet)^{-1} C_\alpha \left(\frac{1}{2} + \mu\right),$$
$$\tilde{\nabla}_{\frac{\partial}{\partial \lambda}} = \frac{\partial}{\partial \lambda} - (\lambda - E_\bullet)^{-1} \left(\frac{1}{2} + \mu\right).$$

The solutions of the flat coordinate system of (1.1) and (1.10) are related by a Fourier–Laplace transform, see Lemma 2.1.

Remark 1.7. The analytic continuation of solutions of the flat coordinates system (1.8) and (1.10) along any path of $M \setminus \Sigma$ and $M \times \mathbb{CP}^1 \setminus \Sigma_\lambda$, respectively,

$$t = \big(t^1, t^2, \dots, t^n\big) \mapsto (x_1(t), \dots, x_n(t)),$$
$$(\lambda, t) = \big(\lambda, t^1, t^2, \dots, t^n\big) \mapsto (x_1(\lambda, t), \dots, x_n(\lambda, t)) \tag{1.12}$$

are called period map and extended period map, respectively.

Remark 1.8. The solutions of flat coordinate system of (1.8) and (1.10) are quasi-homogeneous in the following sense:

$$E(x_a(t)) = \frac{1-d}{2}x_a(t), \qquad \left(\lambda\frac{\partial}{\partial\lambda} + E\right)(x_a(\lambda,t)) = \frac{1-d}{2}x_a(\lambda,t). \tag{1.13}$$

Moreover, let g^{ab} be the coefficients of the intersection form (1.6) in its flat coordinates. Then, there is a polynomial relation between Saito flat coordinates $\left(t^1, t^2, \dots, t^n\right)$ and intersection form flat coordinates $(x_1, x_2, \dots, x_n)$ given by

$$t_1 := \eta_{1\alpha}t^\alpha = g_{ab}x_a x_b, \qquad \text{where} \quad (g_{ab}) = \left(g^{ab}\right)^{-1}. \tag{1.14}$$

See in [16, Appendix G, Exercise G.1].

Recall that a point in a Dubrovin–Frobenius manifold is called semisimple if the Frobenius algebra in T_pM is semisimple. It is worth noting that semisimplicity constitutes an open condition. The Dubrovin–Frobenius structure around semisimple points becomes rather simple. Specifically, the Frobenius algebra becomes trivial. Moreover, both the Saito metric and the endomorphism resulting from the multiplication by the Euler vector field (1.5) are diagonal around such a point.

Proposition 1.9 ([16, 18])**.** *Let $(u_1, u_2, \dots, u_n)$ be pairwise distinct roots of the characteristic equation*

$$\det\left(g^{\alpha\beta} - u\eta^{\alpha\beta}\right) = 0. \tag{1.15}$$

Then, the relation $(u_1(t), u_2(t), \dots, u_n(t))$ can serve as local coordinates, which are called canonical coordinates. In these coordinates, the Frobenius multiplication, Saito metric, unit vector field and Euler vector field can be written as

$$\frac{\partial}{\partial u_i} \bullet \frac{\partial}{\partial u_j} = \delta_{ij}\frac{\partial}{\partial u_i}, \qquad \eta = \sum_{i=1}^n \psi_{i1}^2(\mathrm{d}u_i)^2, \qquad e = \sum_{i=1}^n \frac{\partial}{\partial u_i}, \qquad E = \sum_{i=1}^n u_i\frac{\partial}{\partial u_i}, \tag{1.16}$$

where the matrix $\Psi = (\psi_{i\alpha})$ is given by $\psi_{i\alpha} = \psi_{i1}\frac{\partial u_i}{\partial t^\alpha}$.

At this stage, we can define a Landau–Ginzburg superpotental, which can be found in [18, Definition 5.7].

Definition 1.10 ([18])**.** Let D be an open domain of a Riemann surface. A Landau–Ginzburg superpotential associated with a Dubrovin–Frobenius manifold M of dimension n consists of a function $\lambda(p,u)$ on $D \times M$ and an Abelian differential ϕ in D satisfying

- The critical values of $\lambda(p,u)$ are the canonical coordinates $(u_1, \dots, u_n)$. In other words, the canonical coordinates $(u_1, u_2, \dots, u_n)$ are defined by the following system $\lambda(p_i) = u_i$, $\frac{\mathrm{d}\lambda}{\mathrm{d}p}(p_i) = 0$.
- For some cycles $Z_1, \dots, Z_n$ in D the integrals

  $$\tilde{t}_j(z,u) = \frac{1}{z^{\frac{3}{2}}}\int_{Z_j} \mathrm{e}^{z\lambda(p)}\phi, \qquad j = 1, \dots, n, \tag{1.17}$$

  converges and give a system of independent flat coordinates for the Dubrovin connection $\tilde{\nabla}$ in canonical coordinates, i.e., the matrix $Y = \Psi\eta^{-1}\omega := \left(\psi_{i\alpha}\eta^{\alpha\beta}\partial_\beta\tilde{t}\right)$ is a solution of the following system

  $$\frac{\partial Y}{\partial u_i} = (zE_i + V_i)Y, \qquad \frac{\mathrm{d}Y}{\partial z} = \left(U + \frac{V}{z}\right)Y, \tag{1.18}$$

where

$$U = \Psi \mathscr{U} \Psi^{-1}, \qquad V = \Psi \mu \Psi^{-1}, \qquad E_i = (\delta_{ij}\delta_{ik}), \qquad V_i := \frac{\partial \Psi}{\partial u_i}\Psi^{-1}. \tag{1.19}$$

- The following expressions for the coefficients of the tensors Saito metric η, intersection form g^* and the structure constants c, in any coordinate system $(x_1, x_2, \dots, x_n)$, holds true

$$\eta_{ij} = \sum \operatorname*{Res}_{\mathrm{d}\lambda=0} \frac{\partial_i\lambda\partial_j\lambda}{d_p\lambda}\phi, \qquad g_{ij} = \sum \operatorname*{Res}_{\mathrm{d}\lambda=0} \frac{\partial_i \log\lambda\partial_j \log\lambda}{d_p \log\lambda}\phi,$$
$$c_{ijk} = \sum \operatorname*{Res}_{\mathrm{d}\lambda=0} \frac{\partial_i\lambda\partial_j\lambda\partial_k\lambda}{d_p\lambda}\phi. \tag{1.20}$$

1.2 Problem setting

Let N_k be the number of rational curves $\mathbb{CP}^1 \to \mathbb{CP}^2$ of degree k passing through $3k-1$ generic points. Kontsevich [28] proved that the generating function

$$F\left(t^1, t^2, t^3\right) = \frac{\left(t^1\right)^2 t^3}{2} + \frac{\left(t^2\right)^2 t^1}{2} + \sum_{k=0}^{\infty} \frac{N_k}{(3k-1)!} \mathrm{e}^{kt^2}\left(t^3\right)^{3k-1} \tag{1.21}$$

satisfies a WDVV equation. Moreover, in [28] the Gromov–Witten invariants N_k were computed as follows. Note that the associativity equation for any 3d Dubrovin–Frobenius manifold is given by

$$c_{223}^2 = c_{333} + c_{222}c_{233}. \tag{1.22}$$

Defining the function

$$\Phi(X) = \sum_{k=0}^{\infty} \frac{N_k}{(3k-1)!}\mathrm{e}^{kX}, \qquad X := t^2 + 3\ln t^3, \tag{1.23}$$

we obtain the following third-order differential equation for the function $\Phi(X)$ by substituting (1.21) and (1.23) into (1.22),

$$-6\Phi + 33\Phi' - 54\Phi'' - \left(\Phi''\right)^2 + \Phi'''\left(27 + 2\Phi' - 3\Phi''\right) = 0. \tag{1.24}$$

Hence, using the Taylor series expansion of (1.23) in (1.24), we obtain the following recursion:

$$N_d = \sum_{m=1}^{d-1}\left[\binom{3d-4}{3m-2} m^2(d-m)^2 - \binom{3d-4}{3m-3} m(d-m)^3\right] N_m N_{d-m},$$

which explicitly provides all Gromov–Witten N_k by setting $N_1 = 1$. In principle, the Gromov–Witten potential (1.23) is only a formal power series. However, in [11] Di Francesco and Itzykson derived the following asymptotic behavior for N_k:

$$\frac{N_k}{(3k-1)!} = ba^k k^{-\frac{7}{2}}\left(1 + O\left(\frac{1}{k}\right)\right),$$

where a and b are numerically estimated as $a = 0.138$, $b = 6.1$, which implies that the radius of convergence of (1.23) is given by $\frac{1}{a}$. Then, the analytic domain of (1.21) is given by

$$D = \left\{\left(t^1, t^2, t^3\right) \in \mathbb{C}^3 \mid \left|\mathrm{e}^{t^2}\left(t^3\right)^3\right| < \frac{1}{a}\right\}. \tag{1.25}$$

Furthermore, the Euler vector field has the following form $E = t^1\partial_1 + 3\partial_2 - t^3\partial_3$.

Our aim is to investigate the analytic properties of (1.21) by constructing the corresponding 1D Landau–Ginzburg superpotential from the Gauss–Manin connection of $QH^*\left(\mathbb{CP}^2\right)$.

1.3 Main results

In order to simplify the problem, it is convenient to restrict to the affine small quantum cohomology locus, which is the sub manifold of (1.25) defined by

$$D = \left\{ \left(t^1, t^2, 0\right) \in \mathbb{C}^2 \times D\left(0, \frac{1}{a}\right) \right\} \cong \mathbb{C}^2.$$

Here the adjective affine means an affine extension of the standard small quantum cohomology locus

$$D^0 = \left\{ \left(0, t^2, 0\right) \in \mathbb{C}^2 \times D\left(0, \frac{1}{a}\right) \right\} \cong \mathbb{C}.$$

Moreover, we consider the local change of coordinates $Q\colon \mathbb{C} \mapsto \mathbb{C}^*$, $t^2 \mapsto Q = \mathrm{e}^{t^2}$. In this setting, we are able to state our first result.

Theorem 1.11.

(1) *The Landau–Ginzburg superpotential of small affine quantum cohomology of* $\mathbb{CP}^2$ *is a family of holomorphic functions* $\lambda\left(\tilde{\tau}, t^1, Q^{\frac{1}{3}}\right)\colon \mathbb{H} \mapsto \mathbb{C}$ *with holomorphic dependence in the parameter space*

$$\left(t^1, Q^{\frac{1}{3}}\right) \in \mathbb{C} \times \mathbb{C}^* \tag{1.26}$$

and given by

$$\lambda\left(\tilde{\tau}, t^1, Q\right) = t^1 + 3Q^{\frac{1}{3}} J^{\frac{1}{3}}(\tilde{\tau}), \tag{1.27}$$

where

$$J(\tilde{\tau}) = \frac{E_4^3(\tilde{\tau})}{E_4^3(\tilde{\tau}) - E_6^2(\tilde{\tau})}$$

is the j*-function and* $E_k(\tilde{\tau})$ *are the Eisenstein series of weight* k*. In addition, the correspondent Abelian differential* ϕ *is given by*

$$\phi = \frac{\Delta^{\frac{1}{6}}(\tilde{\tau})}{Q^{\frac{1}{6}}} \mathrm{d}\tilde{\tau},$$

where the function

$$\Delta(\tilde{\tau}) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \qquad q = \mathrm{e}^{2\pi \mathrm{i} \tilde{\tau}},$$

is the modular discriminant.

(2) *The Landau–Ginzburg superpotential of small affine quantum cohomology of* $\mathbb{CP}^2$ *is* $\Gamma^{(3)}$*-invariant, where*

$$\Gamma^{(3)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \text{ or } \begin{pmatrix} * & b \\ b & * \end{pmatrix} \mod 3 \right\} \tag{1.28}$$

is a free Coxeter group with 3 *generators.*

After an affine transformation the Landau–Ginzburg superpotential (1.27) is the Hauptmodul of the group (1.28). In particular, this group generate the monodromy of the Gauss–Manin connection associated to quantum cohomology of $\mathbb{CP}^2$. The Saito flat coordinates (t^1, Q) lives in the parameter space (1.26), which describes the deformations of Hauptmodul of $\Gamma^{(3)}$ by affine and rescaling transformations, respectively. See details in Section 3.2.

In addition, we consider a t^3-isomonodromic deformation of (1.27). In essence, the Landau–Ginzburg superpotential for big quantum cohomology of $\mathbb{CP}^2$ represents a power series expansion in the variable $Q^{\frac{1}{3}}t^3$, where its coefficients are quasi-modular forms. We present this result below without delving into specific details on obtaining the coefficients of the expansion. For further information, refer to Section 4.1.

Theorem 1.12.

(1) *The Landau–Ginzburg superpotential of big quantum cohomology of $\mathbb{CP}^2$ is a family of functions $\lambda(\tilde{\tau}_{12}, t^1, Q^{\frac{1}{3}}, Q^{\frac{1}{3}}t^3)\colon \mathbb{H} \mapsto \mathbb{C}$ with holomorphic dependence in the parameter space*

$$\left\{ (t^1, Q^{\frac{1}{3}}, Q^{\frac{1}{3}}t^3) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C} \mid \left|Q^{\frac{1}{3}}t^3\right| < \left(\frac{1}{a}\right)^{\frac{1}{3}} \right\},$$
$$\lambda(\tilde{\tau}_{12}, t^1, Q, t^3) = t^1 + 3Q^{\frac{1}{3}} J^{\frac{1}{3}}(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3),$$

where its pullback via the extended period map (1.12) coincide with the coordinate λ. Here $J^{\frac{1}{3}}(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3)$ is a t^3-deformation of $J^{\frac{1}{3}}(\tilde{\tau}_{12})$ in the following sense:

$$J^{\frac{1}{3}}(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3) = J^{\frac{1}{3}}(\tilde{\tau}_{12}) + \sum_{n=1}^{\infty} J_n^{\frac{1}{3}}(\tilde{\tau}_{12})(Q^{\frac{1}{3}}t^3)^n.$$

The coefficients $J_n^{\frac{1}{3}}(\tilde{\tau}_{12})$ belong to the following ring $J_n^{\frac{1}{3}}(\tilde{\tau}_{12}) \in \Delta^{\frac{-n}{3}}\mathbb{C}[E_2, E_4, E_6]$.

In addition, the correspondent Abelian differential ϕ is given by

$$\phi = \frac{\Delta^{\frac{1}{6}}(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3)}{Q^{\frac{1}{6}}} \mathrm{d}\tilde{\tau}_{12},$$

where $\Delta(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3)$ is a suitable t^3-deformation of $\Delta(\tilde{\tau}_{12})$ in the following sense:

$$\Delta(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3) = \Delta(\tilde{\tau}_{12}) + \sum_{n=1}^{\infty} \Delta_n(\tilde{\tau}_{12})(Q^{\frac{1}{3}}t^3)^n.$$

(2) *The Landau–Ginzburg superpotential of big quantum cohomology of $\mathbb{CP}^2$ is $\Gamma^{(3)}$-invariant, under a suitable action.*

In order to elucidate the geometric nature of the parameter space (1.26), we use an appropriate change of coordinates in the context of affine small quantum cohomology. More precisely, we utilize the following factorization of the Hauptmodul of $\Gamma^{(3)}$, which we denoted by Cohn identities [5, 10, 24]:

$$1 - J(\tau) = 4\wp^3(v(\tau), \mathrm{e}^{\frac{2\pi \mathrm{i}}{6}}) + 1, \qquad (v'(\tau))^6 = \Delta(\tau). \tag{1.29}$$

Here, $v(\tilde{\tau})$ represents the universal covering of $\mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}\}$, and $\wp(v, z)$ denotes the Weierstrass $\wp$ function. Further details can be found in Section 3.3. By employing the change of coordinates (1.29) in the small superpotential (1.27), we derive the following theorem.

Theorem 1.13. *The Landau–Ginzburg superpotential of small affine quantum cohomology of $\mathbb{CP}^2$ is a family of functions $\lambda(\tilde{\tau}, v_0, \omega)\colon \mathbb{H} \mapsto \mathbb{C}$ with holomorphic dependence in the parameter space $(v_0, \omega) \in \mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}\} \times \mathbb{C}^*$ and given by*

$$\lambda(\tilde{\tau}, v_0, \omega) = \frac{\wp\big(v(\tilde{\tau}), \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^2} - \frac{\wp\big(v_0, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^2},$$

where $v(\tilde{\tau})$ is the universal covering of $\mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}\}$. In addition, the correspondent Abelian differential ϕ is given by $\phi = 2\omega \mathrm{d}v(\tilde{\tau})$.

The elliptic curve associated with the lattice $\mathbb{Z} \oplus \mathrm{e}^{\frac{2\pi \mathrm{i}}{6}}\mathbb{Z}$, $y^2 = 4\big(x - t^1\big)^3 - 27Q$, is referred to as an equianharmonic elliptic curve. Subsequently, we establish a change of coordinates between $Q^{\frac{1}{3}}t^3 \in D\big(0, \frac{1}{a}\big)$ and $z \in U\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big) \subset \mathbb{H}$, where $U\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big)$ represents a suitable neighbourhood of $\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$. As a consequence of this, we can derive the following theorem.

Theorem 1.14. *The Landau–Ginzburg superpotential of big quantum cohomology of $\mathbb{CP}^2$ is a family of functions $\lambda(\tilde{\tau}_{12}, \tau_{12}, \tilde{\omega}, z)\colon \mathbb{H} \mapsto \mathbb{C}$ with holomorphic dependence in the parameter space*

$$\big\{(\tau_{12}, \tilde{\omega}, z) \in \mathbb{H} \times \mathbb{C}^* \times \mathbb{H} \mid z \in U\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big)\big\} \tag{1.30}$$

and given by

$$\lambda(v(\tilde{\tau}_{12}, z), \tau_{12}, \tilde{\omega}, z) = \frac{\wp(v(\tau_{12}, z), z)}{(2\tilde{\omega})^2} - \frac{\wp(v(\tilde{\tau}_{12}, z), z)}{(2\tilde{\omega})^2},$$

where, for $z \in \mathbb{H}$ close enough to $\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$, $v(\tilde{\tau}_{12}, z)$ is the universal covering of $\mathbb{C} \setminus \{\mathbb{Z} \oplus z\mathbb{Z}\}$. In addition, the correspondent Abelian differential ϕ is given by $\phi = 2\tilde{\omega}\mathrm{d}v(\tilde{\tau}_{12}, z)$, where $\Delta(\tilde{\tau}_{12}, z)$ is a suitable deformation of the modular discriminant.

From this perspective, the domain of the WDVV solution (1.21) could be identified as the parameter space of a family of elliptic functions following a change of coordinates. The parameters t^1 and Q are associated with v_0 and ω, representing the affine and rescaling freedom within the elliptic family (1.30). Additionally, the t^3 deformation corresponds to $z \in U\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big) \subset \mathbb{H}$, functioning as deformations of the equianharmonic lattice $\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}$. Further details can be found in Section 4.4.

Starting from a genus 0 Gromov–Witten theory applied to suitable Kähler manifolds, we observe that the Gromov–Witten potential satisfies a WDVV equation. Consequently, its domain exhibits a Dubrovin–Frobenius structure. This construction is recognized in the realm of the geometry of topological field theories as the A-model or big quantum cohomology. Alternatively, starting from a Landau–Ginzburg superpotential, one can establish a Dubrovin–Frobenius structure within the parameter space of the LG superpotential. This is accomplished through the utilization of Grothendieck residues (1.20). Such a construction is commonly referred to as LG models or the B model. In this setting, mirror symmetry is a Dubrovin–Frobenius manifold isomorphism between quantum cohomology and LG models.

Restricting to a specific sublocus of big quantum cohomology known as small quantum cohomology, Givental derived several mirror symmetry statements in [21, 22]. In particular, there exists a mirror symmetry between the small quantum cohomology of $\mathbb{CP}^n$ and the LG superpotential, represented by the Laurent polynomial

$$\lambda = \sum_{i=1}^{n} x_i + \frac{Q}{\prod_{i=1}^{n} x_i}.$$

This result is elucidated in Givental's notes [23]. Furthermore, Barannikov, in [4], developed an LG model corresponding to the big quantum cohomology of $\mathbb{CP}^n$. Additionally, Douai and Sabbah, in [14] and [15], systematically constructed a theory to derive LG models for a wide array of examples, notably including the case studied by Barannikov.

The primary objective of this manuscript is to establish a 1D Landau–Ginzburg (LG) superpotential for both small and big quantum cohomology of $\mathbb{CP}^2$. This signifies that the LG superpotential's domain is one-dimensional, differing from the two-dimensional domains of the Givental LG superpotential for small quantum cohomology and the Barannikov LG superpotential for big quantum cohomology. Our approach relies on the inversion of the period map (1.12), wherein the analytic properties of the associated LG superpotential are contingent on the analytical behavior of the period map governed by the given Gauss–Manin connection. This method circumvents the issue of non-isolated singularities for sufficiently large t^3. Despite encountering singular behavior for sufficiently large t^3 in our framework, it appears that a more lucid geometric interpretation of these singularities is feasible in terms of the degeneration of the universal covering of $\mathbb{C} \setminus \{\mathbb{Z} \oplus z\mathbb{Z}\}$. Further exploration of this problem will be the focus of subsequent publications.

Our primary motivation is to explore the Gromov–Witten potential with good analytic properties. In [16, Appendix A], Dubrovin conjectured that the Dubrovin–Frobenius structure of these Gromov–Witten potentials has a monodromy associated with certain reflection groups or their generalizations.

From the standard theory of Dubrovin–Frobenius manifolds, we know that the coefficients of the intersection form in Saito flat coordinates are essentially the Hessian of the corresponding WDVV solution. On the other hand, the intersection form is invariant with respect to the monodromy of the associated Gauss–Manin connection. Therefore, the WDVV solution and the inverse period map can be expressed in terms of monodromy-invariant functions.

In [26], Hertling proved that a particular class of polynomial WDVV solutions is in one-to-one correspondence with the orbit space of a finite Coxeter group. Examples of WDVV solutions associated with orbit spaces of natural extensions of finite Coxeter groups can be found in [1, 2, 6, 7, 17, 19, 32]. In particular, the extended affine Weyl groups and Jacobi groups are extensions of finite Coxeter groups, and the corresponding rings of invariant functions are trigonometric functions and Jacobi forms, respectively.

Motivated by these examples, we aim to study in detail the ring of invariant functions related to the quantum cohomology of $\mathbb{CP}^2$. The inverse period map constructed by Milanov in [29] serves as a good example of invariant functions for $QH^*(\mathbb{CP}^2)$. However, we need to understand how large the period domain found by Milanov is. We expect that the investigation of the 1D Landau–Ginzburg (LG) superpotential for the quantum cohomology of $\mathbb{CP}^2$ would be the first step in understanding the invariant functions of $QH^*(\mathbb{CP}^2)$.

The inverse period map of $QH^*(\mathbb{CP}^2)$ possesses intriguing arithmetic significance as it transforms like a Hilbert modular form under a diagonal action of $A_1 \times \mathrm{PSL}_2(\mathbb{Z})$. Furthermore, its Taylor expansion exhibits similarities to Jacobi forms. An interesting avenue of investigation is to explore whether these functions can be derived through methods akin to Cohen–Kuznetsov series.

This paper is organized in the following way. In Section 2, we revisit the Dubrovin construction of the Landau–Ginzburg superpotential associated with a Dubrovin–Frobenius manifold, as done in [16]. In Section 3, we recapitulate the inverse period map for affine small quantum cohomology as presented in [29] and apply this result to derive a Landau–Ginzburg superpotential for affine small quantum cohomology. Moreover, we reinterpret the Landau–Ginzburg superpotential for affine small quantum cohomology as a composition between the Weierstrass $\wp$ function and the universal covering of $\mathbb{C} \setminus \left\{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}\right\}$ using Cohn identities. In Section 4, we revise the Milanov deformations as discussed in [29] and apply these results to obtain a Landau–Ginzburg superpotential for the big quantum cohomology of $\mathbb{CP}^2$. Furthermore, we derive a change of coordinates using the results of [25] to rewrite the Landau–Ginzburg superpotential of big quantum cohomology of $\mathbb{CP}^2$ as a composition between the Weierstrass $\wp$ function and the universal covering of $\mathbb{C} \setminus \{\mathbb{Z} \oplus z\mathbb{Z}\}$ and derive an isomonodromic deformation of Cohn identities.

2 Construction of Dubrovin superpotential

The objective of this section is to review the Dubrovin construction of the superpotential associated with any WDVV solution in [16, Appendix J] and [18, Chapter 5]. Additionally, we will employ this approach to deduce the Landau–Ginzburg superpotential for the big quantum cohomology of $\mathbb{CP}^2$. The central element of this construction is the inverse period map of the extended Gauss–Manin connection (1.10).

2.1 Monodromy of Dubrovin–Frobenius manifolds

In this subsection, we describe the monodromy associated with the two flat meromorphic connections of a Dubrovin–Frobenius manifold and their relationship. The Dubrovin connection (1.1) is a meromorphic connection with a regular singularity at $z = 0$ and an irregular singularity at $z = \infty$. The fundamental solution of the flat coordinate system of (1.4) in Saito flat coordinates near $z = 0$ has the following form:

$$\xi_0(z,u) = \left(\eta^{\alpha\gamma}\partial_\gamma \tilde{t}_\beta\right) = \left(I + \sum_{n=1}^{\infty} H_n(t) z^n\right) z^\mu z^R, \qquad \tilde{t}_\beta = \eta_{\beta\epsilon}\tilde{t}^\epsilon, \tag{2.1}$$

or alternatively in canonical coordinates

$$Y_0(z,u) = \left(\Psi + \sum_{n=1}^{\infty} \Psi_n(u) z^n\right) z^\mu z^R, \tag{2.2}$$

where $R_{\alpha\beta} = 0$ if $\mu_\alpha - \mu_\beta \neq k > 0$, $k \in \mathbb{N}$.

The solutions (2.1) and (2.2) are related by the following Gauge transformation $Y_0 = \Psi\xi_0$. The coefficients $\Psi_k(u)$ in (2.2) are obtained by substituting (2.2) in (1.18), which concretely gives

$$\frac{\partial \Psi_k}{\partial u_i} = E_i \Psi_{k-1} + V_i \Psi_k.$$

The monodromy of the solution (2.1) is given by

$$Y_0\left(z e^{2\pi i}, u\right) = Y_0(z,u) M,$$

where $M = e^{2\pi i(\mu+R)}$.

To analyze the analytic behavior at ∞, consider a series $\sum_{n=0}^{\infty} \frac{a_n}{z^n}$ as an asymptotic expansion of the function $f(z)$ for $|z| \to \infty$ in the sector $\alpha < \arg(z) < \beta$, if for any n

$$z^n \left[f(z) - \sum_{n=0}^{n} \frac{a_n}{z^n}\right] \to 0, \qquad \text{as} \quad |z| \to \infty,$$

uniformly in the sector $\alpha + \epsilon < \arg(z) < \beta - \epsilon$. We denote this asymptotic expansion by

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \qquad \text{as} \quad |z| \to \infty,$$

uniformly in the sector $\alpha + \epsilon < \arg(z) < \beta - \epsilon$.

Moreover, a line l through the origin in the complex z-plane is called admissible for the system (1.1) if $\operatorname{Re}(z(u_i - u_j))|_{z\in l\setminus\{0\}} \neq 0$ for any $i \neq j$. Fixing an admissible line l with slope ϕ and its respective orientations l_+ and l_-, $l_+ = \{z \in \mathbb{C} \mid \arg(z) = \phi\}$, $l_- = \{z \in \mathbb{C} \mid \arg(z) = \phi - \pi\}$, and the sectors

$$\Pi_{\text{right}} = \phi - \pi - \epsilon < \arg(z) < \phi + \epsilon, \qquad \Pi_{\text{left}} = \phi - \epsilon < \arg(z) < \phi + \epsilon + \pi,$$

for small ϵ. Due to the irregular singularity at $z = \infty$, there is a formal solution (1.1) near $z = \infty$ of the form

$$Y_{\text{formal}}(z, u) = \left(I + \sum_{n=1}^{\infty} \frac{F_n(u)}{z^n}\right) \mathrm{e}^{zU}. \tag{2.3}$$

In addition, there exist analytic solutions of (1.1) Y_{right}, Y_{left} in the sectors Π_{right}, Π_{left} with the following asymptotic expansion

$$\begin{aligned} &Y_{\text{formal}}\mathrm{e}^{-zU} \sim Y_{\text{right}}\mathrm{e}^{-zU}, \qquad |z| \to \infty, \qquad z \in \Pi_{\text{right}}, \\ &Y_{\text{formal}}\mathrm{e}^{-zU} \sim Y_{\text{left}}\mathrm{e}^{-zU}, \qquad |z| \to \infty, \qquad z \in \Pi_{\text{left}}. \end{aligned}$$

Note that in the sector $\Pi_+ \colon \phi - \epsilon < \arg(z) < \phi + \epsilon$, the solutions Y_{right}, Y_{left} are defined. Hence, these solutions coincide up to a constant matrix, i.e.,

$$Y_{\text{left}}(z) = Y_{\text{right}}(z) S, \qquad z \in \Pi_+. \tag{2.4}$$

Similarly, in the opposite narrow sector Π_-

$$Y_{\text{left}}(z) = Y_{\text{right}}(z) S_-, \qquad z \in \Pi_-. \tag{2.5}$$

The constant matrices defined in (2.4) and (2.5) are called the Stokes matrices. From the standard theory of ODE in complex domain, the matrices S and S_- satisfy the following relation $S_- = S^{\mathsf{T}}$. The solution at $z = 0$ and the solution near $z = \infty$ are connected by a constant matrix C called the connection matrix, i.e., $Y_0(z) = Y_{\text{right}}(z) C$, $z \in \Pi_+$.

The set of parameters

$$(\mu, R, S, C), \tag{2.6}$$

represents the monodromy data associated with the Dubrovin connection. The flatness property of the Dubrovin connection implies that the monodromy data (2.6) remains constant with respect to the directions $\frac{\partial}{\partial t^\alpha}$, $\alpha = 1, \dots, n$. In other words, the system (1.2) is isomonodromic.

To provide a more detailed description of the monodromy associated with the second structure connection, it is advantageous to express the extended Gauss–Manin connection (1.10) in canonical coordinates. Consider a solution $x\big(t^1 - \lambda, t^2, \dots, t^n\big)$, of the flat coordinate system of the extended Gauss–Manin connection (1.10). Then, the gradient $\xi^\alpha = \eta^{\alpha\beta}\partial_\beta x\big(t^1 - \lambda, t^2, \dots, t^n\big)$ solves the following system:

$$(\mathscr{U} - \lambda)\partial_\beta \xi + C_\beta \left(\frac{1}{2} + \mu\right) \xi = 0, \qquad (\mathscr{U} - \lambda)\partial_\lambda \xi - \left(\frac{1}{2} + \mu\right) \xi = 0, \tag{2.7}$$

where $\mathscr{U}$ and C_β are defined in (1.3). In the semisimple case, a gauge transformation can be applied

$$\phi = \Psi \xi, \tag{2.8}$$

which is equivalent to

$$\phi_i = \sum_{\alpha,\beta} \psi_{i\alpha}\eta^{\alpha\beta}\partial_\beta x\big(t^1-\lambda, t^2,\dots,t^n\big). \tag{2.9}$$

Utilising the gauge (2.8), we express the system (2.7) in the form

$$\begin{aligned}
&(U-\lambda)\partial_\lambda\phi - \left(\frac{1}{2}+V\right)\phi = 0,\\
&(U-\lambda)\partial_i\phi + E_i\left(\frac{1}{2}+V\right)\phi = 0, \qquad i=1,\dots,n,
\end{aligned} \tag{2.10}$$

or alternatively

$$\partial_i\phi = -\frac{B_i}{\lambda-u_i}\phi + V_i\phi, \qquad i=1,\dots,n, \qquad \partial_\lambda\phi = \sum_{i=1}^n \frac{B_i}{\lambda-u_i}\phi, \tag{2.11}$$

where

$$B_i = -E_i\left(\frac{1}{2}+V\right) \tag{2.12}$$

and V, V_i, E_i are defined in (1.19).

The matrix B_i in (2.12) has one eigenvalue $-\frac{1}{2}$ and $n-1$ zero eigenvalues. Consequently, there exists a unique basis of solutions $\phi^{(1)},\phi^{(2)},\dots,\phi^{(n)}$ for (2.11) such that

$$\phi_a^{(j)}(\lambda) = \frac{\delta_{aj}}{\sqrt{u_j-\lambda}} + O\big(\sqrt{u_j-\lambda}\big), \qquad \lambda\to u_j. \tag{2.13}$$

Moreover, following the standard theory of Fuchsian systems,

$$\big(\phi_a^{(j)}(\lambda)\big) = \left(I + O\left(\frac{1}{\lambda}\right)\right)\lambda^{-\mu-\frac{1}{2}}\lambda^R, \qquad \lambda\to\infty.$$

Under the semi-simplicity condition, choosing a reference point $\big(u^0,\lambda^0\big)\in M\times\mathbb{C}$, the monodromy (1.11) splits on the short exact sequence

$$\pi_1\big(F^0,\lambda^0\big) \mapsto \pi_1\big(M\times\mathbb{C}\setminus\Sigma_\lambda, \big(u^0,\lambda^0\big)\big) \mapsto \pi_1\big(M,u^0\big), \tag{2.14}$$

where $p\colon M\times\mathbb{C}\setminus\Sigma_\lambda \mapsto M$, $(u,\lambda)\mapsto u$, and $F^0 := p^{-1}(u) = \mathbb{C}\setminus\big\{u_1^0,u_2^0,\dots,u_n^0\big\}$.

We refer [29, Section 1.2] or [30, Proposition 5.6.4] for this fact.

The image of $\pi_1\big(F^0,\lambda^0\big)$ under the monodromy representation, which we denote by W_R, is obtained as follows. Consider small loops $\gamma_1,\gamma_2,\dots,\gamma_n$ going around $u_1,u_2,\dots,u_n$ in the counterclockwise direction. Then, the generators $R_1,R_2,\dots,R_n$ of the monodromy W_R correspond to the image of the loops $\gamma_1,\gamma_2,\dots,\gamma_n$ under the monodromy representation. In particular, due to the monodromy of $\sqrt{u_i-\lambda}$ in (2.13), the R_i action on the solutions of (2.11) is given by $R_j\phi^{(j)} = -\phi^{(j)}$. The general action is given by

$$R_j\phi^{(i)} = \phi^{(i)} - \big(S+S^{\mathsf{T}}\big)_{ij}\phi^{(j)}, \tag{2.15}$$

where S is the Stokes matrices associated to the Dubrovin connection (1.1). See details in [18, Lemmas 5.3 and 5.4]. The compatibility of the system (2.10) implies that the system (2.10) is isomonodromic.

Choose an angle φ such that $\arg(u_i-u_j)\neq\frac{\pi}{2}+\varphi \mod 2\pi$, $i\neq j$.

Next, define the branch cuts

$$L_j = \{\lambda = u_j + \mathrm{i}\rho \mathrm{e}^{\mathrm{i}\varphi},\ \rho \geq 0\}, \qquad j = 1, 2, \dots, n,$$
$$L'_j = \{\lambda = u_j - \mathrm{i}\rho \mathrm{e}^{\mathrm{i}\varphi},\ \rho \geq 0\}, \qquad j = 1, 2, \dots, n,$$

and the positive and negative side of each L_j

$$L_j^+ = \left\{\lambda \mid \arg(u_j - \lambda) = \frac{-\pi}{2} - \varphi\right\}, \qquad L_j^- = \left\{\lambda \mid \arg(u_j - \lambda) = \frac{-\pi}{2} - \varphi + 2\pi\right\}.$$

Additionally, consider an infinite contour C_j coming from infinity along the positive side of L_j, encircling u_j, and returning to infinity along the negative side of the branchcut L_j. Also, consider the contour C'_j that encircles the branch cut L'_j. Then, we state here a lemma which can be derived from the proof of [18, Lemma 5.4].

Lemma 2.1 ([18])**.** *The Fourier–Laplace transform of the solution $\phi^{(j)}$ of* (2.11)

$$Y_{aj}^{\text{right}}(z) = -\frac{1}{2\sqrt{\pi}\sqrt{z}} \int_{C_j} \phi_a^{(j)}(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda, \qquad a, j = 1, 2, \dots, n,$$
$$Y_{aj}^{\text{left}}(z) = -\frac{1}{2\sqrt{\pi}\sqrt{z}} \int_{C'_j} \phi_a^{(j)}(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda, \qquad a, j = 1, 2, \dots, n, \tag{2.16}$$

converges for $z \in \Pi_{\text{right}}/\Pi_{\text{left}}$, respectively, and give n independent solutions of Dubrovin connection (1.18)*.*

Proof. Indeed,

$$\begin{aligned}
\left(U + \frac{V}{z}\right)_{ia} Y_{aj} = &- \frac{1}{2\sqrt{\pi}(z)^{\frac{3}{2}}} \int_{C_j} V_{ia} \phi_a^j(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda - \frac{1}{2\sqrt{\pi}\sqrt{z}} \int_{C_j} u_i \delta_{ia} \phi_a^j(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda \\
= &- \frac{1}{2\sqrt{\pi}(z)^{\frac{3}{2}}} \int_{C_j} \left(\frac{1}{2} + V_{ia}\right) \phi_a^j(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda \\
&- \frac{1}{2\sqrt{\pi}\sqrt{z}} \int_{C_j} (u_i - \lambda) \delta_{ia} \phi_a^j(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda \\
&+ \frac{1}{4\sqrt{\pi}(z)^{\frac{3}{2}}} \int_{C_j} \phi_a^j(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda - \frac{1}{2\sqrt{\pi}\sqrt{z}} \int_{C_j} \lambda \phi_a^j(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda \\
= &- \frac{1}{2\sqrt{\pi}(z)^{\frac{3}{2}}} \int_{C_j} \left(\frac{1}{2} + V_{ia}\right) \phi_a^j(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda \\
&- \frac{1}{2\sqrt{\pi}\sqrt{z}} \int_{C_j} (u_i - \lambda) \delta_{ia} \phi_a^j(\lambda) \mathrm{e}^{z\lambda} \mathrm{d}\lambda + \partial_z Y_{aj}.
\end{aligned} \tag{2.17}$$

Substituting (2.10) in (2.17) and using integration by parts, we obtain

$$\left(U + \frac{V}{z}\right)_{ia} Y_{aj} = \partial_z Y_{aj}.$$

Since $\lambda = \infty$ is a regular singularity, the solution $\phi_a^j(\lambda)$ does not grow faster than λ^k, for some k. Hence, the (2.16) converges absolutely for $z \in \Pi_{\text{right/left}}$. ∎

2.2 LG superpotential as family of isomonodromic covering maps

In this subsection, we aim to construct a family of covering maps by "inverting" a solution of the extended Gauss–Manin connection. To achieve this, we require several auxiliary lemmas.

Lemma 2.2 ([18])**.** *Consider the matrix G whose coefficients are given by $G^{ij} = \left(S + S^{\mathsf{T}}\right)_{ij}$, and its inverse G^{-1} with coefficients G_{ij}. Moreover, consider the solutions $\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(n)}$ of equation (2.11). Then, the solution*

$$\phi(\lambda, u) = (\phi_a(\lambda, u)) = \sum_{i,j=1}^{n} G_{ij} \phi^{(j)} \tag{2.18}$$

has the following asymptotic behavior:

$$\phi_a(\lambda) = \frac{\delta_{aj}}{\sqrt{u_j - \lambda}} + O\big(\sqrt{u_j - \lambda}\big), \qquad \lambda \to u_j. \tag{2.19}$$

Proof. The asymptotic expansion (2.19) follows from (2.13). See [18, Lemma 5.7] for details. ■

Lemma 2.3 ([18])**.** *Let $(x_1, x_2, \dots, x_n)$ be a solution of the extended Gauss–Manin connection (1.8) with charge $d \neq 1$, and let $(a_1, a_2, \dots, a_n) \in (\mathbb{C}^*)^n$. Then, the solution of the form*

$$p\big(\lambda, t^1, t^2, \dots, t^n\big) := \sum_{i=1}^{n} a_i x_i\big(t^1 - \lambda, t^2, \dots, t^n\big) \tag{2.20}$$

has the following behavior:

$$\begin{aligned} &p\big(\lambda, t^1, t^2, \dots, t^n\big) = p_j + \sqrt{2}\psi_{j1}\sqrt{u_j - \lambda} + O(u_j - \lambda), \\ &\lambda \to u_j, \qquad \forall j \in \{1, \dots, n\}, \end{aligned} \tag{2.21}$$

where $p_j = p(u_j, t)$.

Proof. According to [18, equation (3.20)], the transformation law between canonical vector field coordinates and Saito flat vector fields is given by

$$\frac{\partial}{\partial u_i} = \sum_{\alpha,\beta} \psi_{i\alpha} \eta^{\alpha\beta} \psi_{i1} \frac{\partial}{\partial t^\beta}. \tag{2.22}$$

Substituting (2.22) into (2.9), we obtain

$$\phi_i = \frac{1}{\psi_{i1}} \partial_i x, \tag{2.23}$$

where x in (2.23) is a generic solution of (2.7).

The solutions of the extended Gauss–Manin connection (2.7) are quasi-homogeneous (see [16, Appendix H, equation (H.19)]), i.e.,

$$\left(\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} + \mathrm{Lie}_E\right)(x(\lambda, t)) = \frac{1-d}{2} x(\lambda, t). \tag{2.24}$$

Moreover, recall the representation of the vector fields ∂_λ and Lie_E in canonical coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = -\frac{\partial}{\partial t^1} = \sum_{i=1}^{n} \frac{\partial}{\partial u_i}, \qquad E = \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i}. \tag{2.25}$$

Therefore, substituting (2.23) and (2.25) into (2.24), we obtain

$$x(\lambda,t)=\frac{2}{1-d}\sum_{i=1}^{n}(u_i-\lambda)\partial_i x(\lambda,t)=\frac{2}{1-d}\sum_{i=1}^{n}(u_i-\lambda)\psi_{i1}\phi_i(\lambda,t). \tag{2.26}$$

Here we choose ϕ to be given by (2.18). Using the asymptotic expansion (2.19) in (2.26), we can choose constants $a_1,\dots,a_n$ such that (2.21) holds. This completes the proof. ■

The solution $p(\lambda,t)$ (2.20) is analytic functions on $\mathbb{C}\setminus\bigcup_{j=1}^{n}L_j$. In particular, solutions of (2.7) are locally invertible outside the discriminant locus. Consider the set of the analytic continuations of $p(\lambda,t)$ $X:=\{p_\gamma(\lambda,u)\mid\gamma\in\pi_1(\mathbb{C}\setminus\{u_1^0,u_2^0,\dots,u_n^0\})\}/\sim$, here $(u_1^0,u_2^0,\dots,u_n^0)$ is a fixed reference point and the equivalence relation $\sim$ is given by

$$p_{\gamma_1}(\lambda,u)=p_{\gamma_2}(\lambda,u)\qquad\text{iff}\qquad\exists g\in\pi_1(\mathbb{C}\setminus\{u_1^0,u_2^0,\dots,u_n^0\}):\ g\gamma_1=\gamma_2,$$

and the following monodromy representation

$$\rho:\ \pi_1(\mathbb{C}\setminus\{u_1^0,u_2^0,\dots,u_n^0\})\mapsto\operatorname{Aut}(X) \tag{2.27}$$

given by $\rho(\gamma_1)(p_{\gamma_2}(\lambda,u))=p_{\gamma_1\bullet\gamma_2}(\lambda,u)$.

For fixed a reference point $(u_1^0,u_2^0,\dots,u_n^0)$, consider the universal covering of $\mathbb{C}\setminus\{u_1^0,u_2^0,\dots,u_n^0\}$ and denoted it by

$$\mathbb{C}\setminus\widetilde{\{u_1^0,u_2^0,\dots,u_n^0\}}.$$

Then, note that the kernel of the representation (2.27) acts properly discontinuously in

$$\mathbb{C}\setminus\widetilde{\{u_1^0,u_2^0,\dots,u_n^0\}},$$

because $\ker\rho$ is a normal subgroup of $\pi_1(\mathbb{C}\setminus\{u_1^0,u_2^0,\dots,u_n^0\})$. Then, we construct a Riemann surface given by

$$S_{(u_1^0,u_2^0,\dots,u_n^0)}:=\mathbb{C}\setminus\widetilde{\{u_1^0,u_2^0,\dots,u_n^0\}}/\operatorname{Ker}\rho.$$

The local inverses of the analytic continuation $p_\gamma(\lambda,u)$, $\lambda(p_\gamma(\lambda,u^0),u^0)=\lambda$ give rise to a covering map $\lambda\colon S_{(u_1^0,u_2^0,\dots,u_n^0)}\mapsto\mathbb{C}$, which group of deck transformations is image of the monodromy representation (2.27).

We can repeat this construction for any $(u_1,u_2,\dots,u_n)$ close enough to $(u_1^0,u_2^0,\dots,u_n^0)$. More specifically, it is sufficient that $(u_1,u_2,\dots,u_n)$ do not intersect the branch cut L_j^0. As a result, we build a family of Riemann surfaces $S_{(u_1,u_2,\dots,u_n)}$. Then, the family of covering maps $\lambda(p_\gamma(\lambda,u),u)=\lambda$ gives rise to an isomonodromic family of coverings

$$\lambda:\ S_{(u_1,u_2,\dots,u_n)}\mapsto\mathbb{C}. \tag{2.28}$$

Here, we state [16, Proposition I.1] as follows.

Lemma 2.4 ([16])**.** *Consider the family of coverings*

$$\lambda:\ S_{(u_1,u_2,\dots,u_n)}\mapsto\mathbb{C} \tag{2.29}$$

defined in (2.28)*. Then, the family* (2.29) *exhibits the following behavior:*

$$\lambda(p)=u_j+\frac{1}{2\psi_{j1}^2}(p-p_j)^2+O((p-p_j)^3),\qquad p\mapsto p_j. \tag{2.30}$$

Proof. The local behavior (2.30) is determined by the local inverse of equation (2.21). Lemma proved. ■

At this stage, we revisit the construction of the Landau–Ginzburg potential in the sense of Dubrovin as presented in [18, Theorem 5.3].

Theorem 2.5 ([18])**.** *Consider the family of coverings defined in* (2.28) $\lambda\colon S_{u_1,u_2,\dots,u_n} \mapsto \mathbb{C}$, $p \mapsto \lambda(p,u)$. *Then, the function* $\lambda(p,u)$*, together with the Abelian differential* $\phi = \mathrm{d}p$*, forms a Landau–Ginzburg superpotential.*

Proof. Due to Lemma 2.4, the critical values of $\lambda(p,u)$ are the canonical coordinates $(u_1,\dots,u_n)$, i.e., $(u_1,u_2,\dots,u_n)$ is defined by the following system $\lambda(p_i)=u_i$, $\frac{\mathrm{d}\lambda}{\mathrm{d}p}(p_i)=0$.

Consider the cycles $Z_j = p(C_j)$, which are the lifts of C_j in the correspondent family of Riemann surfaces $S_{u_1,u_2,\dots,u_n}$.

Integrating by parts and changing variables in (1.17),

$$\tilde{t}_j = \frac{1}{z^{\frac{3}{2}}}\int_{Z_j} \mathrm{e}^{z\lambda(p,t)}\mathrm{d}p = \frac{1}{z^{\frac{3}{2}}}\int_{C_j} \mathrm{e}^{z\lambda}\frac{\mathrm{d}p}{\mathrm{d}\lambda}\mathrm{d}\lambda = -\frac{1}{\sqrt{z}}\int_{C_j} p(\lambda,t)\mathrm{e}^{z\lambda}\mathrm{d}\lambda.$$

Hence, the gradient of (1.17)

$$Y_{aj}(z) = \frac{1}{\psi_{a1}}\frac{\partial\tilde{t}_j}{\partial u_a} = \frac{1}{\sqrt{z}}\int_{C_j}\phi_a(\lambda,t)\mathrm{e}^{z\lambda}\mathrm{d}\lambda$$

solves the Dubrovin connection flat coordinate system due to Lemma 2.1.

Due to Lemma 2.4 and (1.16), we have that $\eta_{ii} = \frac{1}{\lambda''(p_i)}$ holds true. The Taylor expansion of λ, $\frac{\partial\lambda}{\partial p}$, $\frac{\partial\lambda}{\partial u_j}$ around p_i is given by

$$\begin{aligned}
&\lambda(p) = u_i + \lambda''(p_i)\frac{(p-p_i)^2}{2} + O\big((p-p_i)^3\big),\\
&\frac{\mathrm{d}\lambda}{\mathrm{d}p}(p) = \lambda''(p_i)(p-p_i) + O\big((p-p_i)^2\big), \qquad \frac{\mathrm{d}\lambda}{\partial u_j}(p) = \delta_{ij} + O((p-p_i)).
\end{aligned} \tag{2.31}$$

Then, substituting (2.31) in the right-hand side of (1.20) in canonical coordinates, we obtain

$$\begin{aligned}
&\sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\partial_i\lambda\partial_j\lambda}{\mathrm{d}_p\lambda}\mathrm{d}p = \frac{\delta_{kl}}{\lambda''(p_l)} = \eta_{kl}, \qquad \sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\partial_i\log\lambda\partial_j\log\lambda}{\mathrm{d}_p\log\lambda}\mathrm{d}p = \frac{\delta_{kl}}{u_l\lambda''(p_l)} = g_{kl},\\
&\sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\partial_i\lambda\partial_j\lambda\partial_k\lambda}{\mathrm{d}_p\lambda}\mathrm{d}p = \frac{\delta_{kl}\delta_{km}}{\lambda''(p_l)} = c_{klm}.
\end{aligned}$$

Hence, the third statement of 1.10 is proved for canonical coordinates. In order to compute the coefficients of the tensors (1.20) in a generic coordinate system, it is enough to use the change of coordinates in λ. Indeed,

$$\begin{aligned}
&\sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\frac{\partial\lambda}{\partial v_i}\frac{\partial\lambda}{\partial v_j}}{\frac{\mathrm{d}\lambda}{\mathrm{d}p}}\mathrm{d}p = \frac{\partial u_k}{\partial v_i}\frac{\partial u_l}{\partial v_j}\left(\sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\frac{\partial\lambda}{\partial u_k}\frac{\partial\lambda}{\partial u_l}}{\frac{\mathrm{d}\lambda}{\mathrm{d}p}}\mathrm{d}p\right) = \eta_{ij},\\
&\sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\frac{\partial\lambda}{\partial v_i}\frac{\partial\lambda}{\partial v_j}}{\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}p}}\mathrm{d}p = \frac{\partial u_k}{\partial v_i}\frac{\partial u_l}{\partial v_j}\left(\sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\frac{\partial\lambda}{\partial u_k}\frac{\partial\lambda}{\partial u_l}}{\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}p}}\mathrm{d}p\right) = g_{ij},\\
&\sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\frac{\partial\lambda}{\partial v_i}\frac{\partial\lambda}{\partial v_j}\frac{\partial\lambda}{\partial v_k}}{\frac{\mathrm{d}\lambda}{\mathrm{d}p}}\mathrm{d}p = \frac{\partial u_k}{\partial v_i}\frac{\partial u_l}{\partial v_j}\frac{\partial u_l}{\partial v_m}\left(\sum\operatorname*{Res}_{\mathrm{d}\lambda=0}\frac{\frac{\partial\lambda}{\partial u_k}\frac{\partial\lambda}{\partial u_l}\frac{\partial\lambda}{\partial u_m}}{\frac{\mathrm{d}\lambda}{\mathrm{d}p}}\mathrm{d}p\right) = c_{ijk}.
\end{aligned}$$ ■

Remark 2.6. From the data of covering over $\lambda\colon S_{u_1,\dots,u_n} \mapsto \mathbb{CP}^1$ and Abelian differential $\mathrm{d}p$, one can construct a Dubrovin–Frobenius manifold according Theorem 2.5. The Abelian differential $\mathrm{d}p$ depend on a choice of a point in $(a_1,a_2,\dots,a_n) \in (\mathbb{C}^*)^n$. If one choose a different point $(b_1,b_2,\dots,b_n) \in (\mathbb{C}^*)^n$, the Dubrovin–Frobenius structures of $(a_1,a_2,\dots,a_n)$ and $(b_1,b_2,\dots,b_n)$ are related by a Legendre transform. See [16, Appendix B, Chapter 5] for details.

3 Small quantum cohomology of $\mathbb{CP}^2$

3.1 Monodromy of $QH^*(\mathbb{CP}^2)$

In this subsection, we will review the main steps for computing the monodromy of $QH^*(\mathbb{CP}^2)$. For the convenience of the reader, we summarize the main steps of [18, Examples 4.4 and 5.5]. Consider the Dubrovin connection (1.2) associated to the WDVV solution (1.21) at the locus of small quantum cohomoology, i.e., we consider the sublocus $t^1 = t^3 = 0$ in $QH^*(\mathbb{CP}^2)$. Explicitly such system can be reduced to

$$\partial_2^3 \phi = z^3 Q\phi, \qquad (z\partial_z)^3\phi = 27z^3 Q\phi, \tag{3.1}$$

where the relation of ϕ and $\xi = \eta^{-1}\omega = (\xi_1, \xi_2, \xi_3)$ in (1.2) is represented as follows:

$$(\xi_1, \xi_2, \xi_3) = \left(z\phi, \frac{1}{3}z\partial_z\phi, \frac{1}{9}\partial_z(z\partial_z\phi)\right).$$

Using the quasi homogeneous condition $\phi(z, t^2) = \phi(Q^{\frac{1}{3}}z)$, the system (3.1) reduces to

$$(z\partial_z)^3\phi = 27z^3 Q\phi. \tag{3.2}$$

Under this setting, the multiplication by the Euler vector field becomes

$$\left(g^{\alpha\beta}\right) = \begin{pmatrix} 3Q & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \qquad \text{or alternatively,} \qquad (g^{\alpha}_{\beta}) = \begin{pmatrix} 0 & 0 & 3Q \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}. \tag{3.3}$$

The canonical coordinates, which are the roots of (1.15) with respect (3.3), are

$$u_1 = 3Q^{\frac{1}{3}}, \qquad u_2 = 3Q^{\frac{1}{3}}\zeta_3^2, \qquad u_3 = 3Q^{\frac{1}{3}}\zeta_3, \qquad \text{where} \quad \zeta_3 = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}.$$

Because the transition function $\psi_{i\alpha}$ diagonalizes the multiplication by the Euler vector field, it can be concluded that

$$\Psi = \frac{1}{\sqrt{3}}\begin{pmatrix} Q^{\frac{-1}{3}} & 1 & Q^{\frac{1}{3}} \\ \bar{\zeta}_6 Q^{\frac{-1}{3}} & -1 & \zeta_6 Q^{\frac{1}{3}} \\ \zeta_6 Q^{\frac{-1}{3}} & -1 & \bar{\zeta}_6 Q^{\frac{1}{3}} \end{pmatrix}, \qquad \text{where} \quad \zeta_6 = \mathrm{e}^{\frac{\pi\mathrm{i}}{3}}.$$

The monodromy of the Dubrovin flat coordinate system at $z = 0$ is determined by its matrix μ and its first Chern class $c_1(\mathbb{CP}^2)$, see [18, Examples 1.3 and 2.2],

$$\mu = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}.$$

To compute the monodromy at $z = \infty$, we calculate the asymptotic behaviour of the solutions of (3.2). Then, it is necessary to choose basis $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{t}_3$ such that the matrix $Y_{ij} = \frac{\partial_i \tilde{t}_j}{\psi_{i1}}$ has the asymptotic behaviour of the form (2.3), i.e.,

$$Y_{ij} \sim \left(\delta_{ij} + O\left(\frac{1}{z}\right)\right)\mathrm{e}^{zu_j}, \qquad i, j = 1, 2, 3.$$

Then, we obtain three solutions ϕ_1, ϕ_2, ϕ_3 of (3.2) such that

$$\phi_j = \frac{\partial_1 \tilde{t}_j}{z} = \frac{1}{z}\sum_{i=1}^{3} \partial_i \tilde{t}_j = \frac{1}{z}\sum_{i=1}^{3} \psi_{i1} Y_{ij}.$$

Furthermore, the functions ϕ_1, ϕ_2, ϕ_3 have the following asymptotic behaviour:

$$\phi_1 \sim \frac{1}{\sqrt{3}}\frac{\mathrm{e}^{3Q^{\frac{1}{3}}z}}{Q^{\frac{1}{3}}z}\left(1+O\left(\frac{1}{z}\right)\right), \qquad \phi_2 \sim \frac{\bar{\zeta}_6}{\sqrt{3}}\frac{\mathrm{e}^{3Q^{\frac{1}{3}}\bar{\zeta}_6^2 z}}{Q^{\frac{1}{3}}z}\left(1+O\left(\frac{1}{z}\right)\right),$$
$$\phi_3 \sim \frac{\zeta_6}{\sqrt{3}}\frac{\mathrm{e}^{3Q^{\frac{1}{3}}\zeta_6^2 z}}{Q^{\frac{1}{3}}z}\left(1+O\left(\frac{1}{z}\right)\right).$$

At this stage, we can compute the correspondent Stokes matrices of $QH^*(\mathbb{CP}^2)$ by using [18, Lemma 4.9]. Here, we state as follows.

Lemma 3.1 ([18])**.** *The Meijer function*

$$g(z,Q) = \frac{1}{(2\pi)^2\mathrm{i}}\int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty}\Gamma^3(-s)\mathrm{e}^{\pi\mathrm{i}s}Q^s z^{3s}\mathrm{d}s, \qquad -\frac{5\pi}{6} < \arg(z) < \frac{\pi}{6},$$

defined for $z \neq 0$, where c is any positive number, satisfies the equation (3.2). The analytic continuation of this function has the asymptotic development

$$g(z,Q) \sim \frac{1}{\sqrt{3}}\bar{\zeta}_6\frac{\mathrm{e}^{3Q^{\frac{1}{3}}\bar{\zeta}_6^2 z}}{Q^{\frac{1}{3}}z}, \qquad |z| \mapsto \infty, \qquad \textit{in the sector} \qquad \frac{-5\pi}{3} < \arg(z) < \pi.$$

Moreover, it satisfies the following identity

$$g\big(z\mathrm{e}^{2\pi\mathrm{i}}\big) - 3g\big(z\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\big) + 3g\big(z\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big) - g(z) = 0. \tag{3.4}$$

To construct $\phi_{\text{left}} = \big(\phi^1_{\text{left}}, \phi^2_{\text{left}}, \phi^3_{\text{left}}\big)$ and $\phi_{\text{right}} = \big(\phi^1_{\text{right}}, \phi^2_{\text{right}}, \phi^3_{\text{right}}\big)$ such that

$$\phi^j_{\text{right}} \cong \phi^j, \qquad |z| \mapsto \infty, \qquad -\pi < \arg(z) < \frac{\pi}{3}, \qquad j=1,2,3,$$
$$\phi^j_{\text{left}} \cong \phi^j, \qquad |z| \mapsto \infty, \qquad 0 < \arg(z) < \frac{4\pi}{3}, \qquad j=1,2,3,$$

we take

$$\phi_{\text{right}} = \big(-g\big(z\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big), g(z), g\big(z\mathrm{e}^{\frac{-2\pi\mathrm{i}}{3}}\big)\big),$$
$$\phi_{\text{left}} = \big(-g\big(z\mathrm{e}^{\frac{-4\pi\mathrm{i}}{3}}\big), g\big(z\mathrm{e}^{-2\pi\mathrm{i}}\big) - 3g\big(z\mathrm{e}^{\frac{-4\pi\mathrm{i}}{3}}\big), g\big(z\mathrm{e}^{\frac{-2\pi\mathrm{i}}{3}}\big)\big).$$

Using the identity (3.4) in the sector $0 < \arg(z) < \frac{\pi}{3}$, we deduce that $\phi_{\text{left}} = \phi_{\text{right}}S$, where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & -3 & 1 \end{pmatrix}$$

is the Stokes matrices of $QH^*(\mathbb{CP}^2)$. Here, we used the fact that the Dubrovin flat coordinate system (1.2) is isomonodromic. Then, the monodromy computed in a fixed point of $QH^*(\mathbb{CP}^2)$ is the same as the monodromy computed in a neighbourhood of this reference point.

The monodromy group of the Gauss–Manin connection (1.10) splits in short exact sequence (2.14). The monodromy representation of $\pi_1\big(F^0,\lambda^0\big)$, denoted by W_R, is characterized by the reflections given in (2.15). Hence the generators of W_R associated with $QH^*(\mathbb{CP}^2)$ are given by

$$R_1 = \begin{pmatrix} -1 & -3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & -1 \end{pmatrix}. \tag{3.5}$$

The monodromy $\pi_1(D,t)$, where D is defined in (1.25), is given by loops around $Q=0$. The image under the monodromy representation of loops around $Q=0$ correspond to loops $z \mapsto z\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$. This loop acts on the solutions of Dubrovin flat coordinate system as follows:

$$\Phi_{\text{left}}\left(z\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right) = \Phi_{\text{left}}(z)T, \qquad \text{where} \quad T = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{pmatrix}.$$

Defining

$$T_0 := TR_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

we have the following relations $T_0^3 = -I$, $R_2 = T_0 R_1 T_0^{-1}$, $R_3 = T_0^2 R_1 T_0^{-2}$. Consequently, the monodromy group W of $QH^*(\mathbb{CP}^2)$ is generated by

$$W = \langle R_1, T_0^4, T_0^3 \rangle. \tag{3.6}$$

To better understand the groups W_R, W, we make a digression on modular forms and elliptic functions. For comprehensive details, we refer the reader to [12] and [31]. Consider the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$.

The special linear group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \qquad \tau \in \mathbb{H}. \tag{3.7}$$

The standard generators of $\mathrm{SL}_2(\mathbb{Z})$ are $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which satisfy only the relations $S^2 = (ST)^3 = I$.

The action (3.7) acts properly discontinuously in $\mathbb{H}$ having

$$F_0 = \left\{ \tau \in \mathbb{H} \mid -\frac{1}{2} \leq \operatorname{Re}(\tau) \leq \frac{1}{2},\ |\tau| > 1 \right\}$$

as the fundamental domain. The quotient map $\mathbb{H} \mapsto \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ is a branched covering map with branches of order 2 and 3 in the $\mathrm{SL}_2(\mathbb{Z})$ orbit of i and $\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$, respectively.

The quotient space $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ represents the moduli space of isomorphic classes of complex tori. Indeed, consider $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ and the lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The quotient space $\mathbb{C}/\Lambda$ is a torus, and two tori $\mathbb{C}/\Lambda$, $\mathbb{C}/\Lambda'$ are biholomorphic iff the correspondents lattices Λ, Λ' are homothetics, i.e., there exist $\alpha \in \mathbb{C}^*$ such that $\alpha\Lambda = \Lambda'$. Therefore, if we multiply the lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ by $\alpha = \omega_1^{-1}$, assuming that $\tau := \frac{\omega_2}{\omega_1}$ has positive imaginary part, we have that $\mathbb{C}/\Lambda \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. Moreover,

$$\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}/\mathbb{Z} + \tau'\mathbb{Z} \qquad \text{iff} \qquad \begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

A modular form of weight k is a holomorphic function $f\colon \mathbb{H} \mapsto \mathbb{C}$ such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Furthermore, we have some controlled growth condition at $i\infty$. More precisely, its Fourier series around $i\infty$ has the following form: $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$, $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$.

The ring of meromorphic functions on a torus $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ provides examples of modular forms. A function defined on the $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ must be double periodic, i.e., $f(z+m+n\tau)=f(z)$. Due to maximum modulus principle and Liouville's theorem, a function defined on the torus must be meromorphic. An example of such function is the Weierstrass $\wp$ function given by

$$\wp(v,\tau)=\frac{1}{v^2}+\sum_{m^2+n^2\neq 0}\left(\frac{1}{(v+m+n\tau)^2}-\frac{1}{(m+n\tau)^2}\right),\qquad m,n\in\mathbb{Z}. \tag{3.8}$$

The series (3.8) is absolutely convergent, holomorphic in $\mathbb{C}\setminus\mathbb{Z}+\tau\mathbb{Z}$ and its poles are in the lattice $\mathbb{Z}+\tau\mathbb{Z}$. Hence, the function (3.8) has the following transformation laws:

$$\begin{gathered}\wp(v+m+n\tau,\tau)=\wp(v,\tau),\qquad m,n\in\mathbb{Z},\\ \wp\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2\wp(v,\tau),\qquad \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}).\end{gathered} \tag{3.9}$$

In addition, the function (3.8) satisfies the non-linear differential equation

$$(\wp'(v,\tau))^2=4\wp^3(v,\tau)-g_2(\tau)\wp(v,\tau)-g_3(\tau), \tag{3.10}$$

where

$$g_2(\tau)=60\sum_{m^2+n^2\neq 0}\frac{1}{(m+n\tau)^4},\qquad g_3(\tau)=140\sum_{m^2+n^2\neq 0}\frac{1}{(m+n\tau)^6}, \tag{3.11}$$

which are absolutely convergent series. The functions (3.11) are modular forms of weight 4 and 6, respectively, i.e.,

$$\begin{gathered}g_2\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^4g_2(\tau),\qquad \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}),\\ g_3\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^6g_3(\tau),\qquad \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}).\end{gathered} \tag{3.12}$$

The functions (3.11) are called Eisenstein series of weight 4 and 6, respectively. Another normalizations of (3.11) are given by

$$G_k(\tau)=\frac{1}{2}\sum_{m^2+n^2\neq 0}\frac{1}{(m+n\tau)^k},\qquad E_k(\tau)=\frac{1}{\zeta(k)}G_k(\tau), \tag{3.13}$$

where $\zeta(k)=\sum_{n=0}^{\infty}\frac{1}{n^k}$.

The differential equation (3.10) provides a biholomorphism between the torus $\mathbb{C}\setminus\mathbb{Z}+\tau\mathbb{Z}$ and the compactification of the algebraic curve

$$y^2=4x^3-g_2x-g_3. \tag{3.14}$$

Consider the image of the half periods under (3.8)

$$e_1(\tau)=\wp\left(\frac{1}{2},\tau\right),\qquad e_2(\tau)=\wp\left(\frac{1+\tau}{2},\tau\right),\qquad e_3(\tau)=\wp\left(\frac{\tau}{2},\tau\right), \tag{3.15}$$

we can show that e_1, e_2, e_3 are the roots of (3.14)

$$4\wp^3-g_2\wp-g_3=4\left(\wp-e_1\right)\left(\wp-e_2\right)\left(\wp-e_3\right). \tag{3.16}$$

Therefore, the discriminant of the cubic (3.16) is given by

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2,$$

or alternatively

$$\Delta(\tau) = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_3 - e_2)^2. \tag{3.17}$$

The modular discriminant (3.17) is a modular form of weight 12, which only vanishes at the cusp, i.e., at $\mathbb{Q} \cup \{\mathrm{i}\infty\}$. The Dedeking η function is modular form of weight $\frac{1}{2}$ given by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \qquad \text{where} \quad q = \mathrm{e}^{2\pi\mathrm{i}\tau}. \tag{3.18}$$

It obeys the following transformation laws:

$$\eta(\tau + 1) = \mathrm{e}^{\frac{\pi\mathrm{i}}{12}}\eta(\tau), \qquad \eta\left(\frac{-1}{\tau}\right) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau). \tag{3.19}$$

The 24-th power of (3.18) is the modular discriminant up to a constant, i.e.,

$$\Delta(\tau) = (2\pi)^{12}\eta^{24}(\tau). \tag{3.20}$$

Now define the following group homomorphism:

$$\begin{aligned} &\chi_2\colon\ \mathrm{SL}_2(\mathbb{Z}) \mapsto Z_2, \qquad \chi_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\eta^{12}\left(\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^6\eta^{12}(\tau)}, \\ &\chi_3\colon\ \mathrm{SL}_2(\mathbb{Z}) \mapsto Z_3, \qquad \chi_3\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\eta^{8}\left(\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^4\eta^{8}(\tau)}. \end{aligned} \tag{3.21}$$

More explicitly, substituting (3.19) in (3.21), we obtain

$$\begin{aligned} &\chi_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \zeta_2, \qquad \chi_2\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \zeta_2, \qquad \zeta_2 = \mathrm{e}^{\pi\mathrm{i}}, \\ &\chi_3\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1, \qquad \chi_3\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \zeta_3^2, \qquad \zeta_3 = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}. \end{aligned}$$

We are interested in the subgroups of $\mathrm{SL}_2(\mathbb{Z})$ provided by $\Gamma^{(n)} := \operatorname{Ker}\chi_n$.

For our current purposes, we will consider holomorphic functions $f\colon \mathbb{H} \mapsto \mathbb{C}$ that are invariant under $\mathrm{SL}_2(\mathbb{Z})$, $\Gamma^{(2)}$, $\Gamma^{(3)}$. Consider the function $j\colon \mathbb{H} \mapsto \mathbb{C}$ given by

$$j(\tau) = 1728\frac{g_2^3(\tau)}{\Delta(\tau)}, \qquad \text{or alternatively} \qquad j(\tau) = 1728\frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)}. \tag{3.22}$$

The function (3.22) is holomorphic, surjective, and $\mathrm{SL}_2(\mathbb{Z})$-invariant. Furthermore, it is a local biholomorphism in the subspace $\mathbb{H} \setminus \left\{\mathrm{SL}_2(\mathbb{Z})\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\right)\right\} \cup \{\mathrm{SL}_2(\mathbb{Z})(\mathrm{i})\}$, providing the following branched covering map $j\colon \mathbb{H} \mapsto \mathbb{C} \cong \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. The Fourier expansion of (3.22) $j(q) = \frac{1}{q} + 744 + O(q)$ give a compactification of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ as $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})^C \cong \mathbb{CP}^1$. In this way, up affine shift and rescaling constant, by Liouville's theorem, the j-function (3.22) is the unique modular $\mathrm{SL}_2(\mathbb{Z})$ invariant function, which it is called by Hauptmodul of $\mathrm{SL}_2(\mathbb{Z})$. A univalent automorphic function f for a genus zero Fuchsian group is called a Hauptmodul. The Hauptmodul of the group $\Gamma^{(2)}$, $\Gamma^{(3)}$ are given by the following Weber function, see [3, Section 3.4],

$$\gamma_2(\tau) = j^{\frac{1}{3}}(\tau), \qquad \text{or alternatively,} \qquad \gamma_2(\tau) = \frac{12}{(2\pi)^4}\frac{g_2(\tau)}{\eta^8(\tau)},$$

$$\gamma_3(\tau) = \sqrt{1728 - j(\tau)}, \qquad \text{or alternatively,} \qquad \gamma_3(\tau) = i\frac{216}{(2\pi)^6}\frac{g_3(\tau)}{\eta^{12}(\tau)}, \tag{3.23}$$

the cube root being chosen in such a way that $j^{\frac{1}{3}}(\tilde{\tau})$ is positive on the imaginary axis.

Indeed, consider $\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in \Gamma^{(3)}$, $\left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right) \in \Gamma^{(2)}$. Then, using that $\Gamma^{(n)} := \operatorname{Ker}\chi_n$ and transformation law of $\eta(\tau)$, $g_2(\tau)$, $g_3(\tau)$ defined in (3.12) and (3.19), we have the following:

$$\gamma_2\left(\frac{a_2\tau + b_2}{c_2\tau + d_2}\right) = \frac{12}{(2\pi)^4}\frac{g_2\left(\frac{a_2\tau+b_2}{c_2\tau+d_2}\right)}{\eta^8\left(\frac{a_2\tau+b_2}{c_2\tau+d_2}\right)} = \frac{1}{\chi_3\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}}\frac{12}{(2\pi)^4}\frac{(c_2\tau + d_2)^4 g_2(\tau)}{(c_2\tau + d_2)^4\eta^8(\tau)} = \gamma_2(\tau),$$

$$\gamma_3\left(\frac{a_3\tau + b_3}{c_3\tau + d_3}\right) = i\frac{216}{(2\pi)^6}\frac{g_3\left(\frac{a_3\tau+b_3}{c_3\tau+d_3}\right)}{\eta^{12}\left(\frac{a_3\tau+b_3}{c_3\tau+d_3}\right)} = \frac{1}{\chi_2\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}} i\frac{216}{(2\pi)^6}\frac{(c_3\tau + d_3)^6 g_3(\tau)}{(c_3\tau + d_3)^6\eta^{12}(\tau)} = \gamma_3(\tau).$$

From [8, Section 3.1], the Weber function $\gamma_2(\tau)$ has the following transformation law under $\mathrm{SL}_2(\mathbb{Z})$:

$$\gamma_2\left(\frac{a\tau + b}{c\tau + d}\right) = \zeta_3^{ac-ab+a^2cd-cd}\gamma_2(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{3.24}$$

and the group $\Gamma^{(3)}$ is represented as follows:

$$\Gamma^{(3)} := \operatorname{Ker}\chi_3 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \ \textit{or} \ \begin{pmatrix} * & b \\ b & * \end{pmatrix} \mod 3\right\}. \tag{3.25}$$

Due to the group homomorphism (3.21), the group $\Gamma^{(3)}$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with index 3 and is generated by the following three matrices

$$r_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}. \tag{3.26}$$

Consider the linear map $B\colon \mathrm{M}_{3\times 3}(\mathbb{R}) \mapsto \mathrm{M}_{3\times 3}(\mathbb{R})$,

$$B\left(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right) = \frac{1}{4}\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 & -6 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & -2 \end{pmatrix}. \tag{3.27}$$

The change of basis (3.27) sends the generators of (3.6) to

$$B(R_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B\left(T_0^4\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix},$$
$$B\left(T_0^3\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.28}$$

Moreover, the generators (3.5) are mapped to

$$B(R_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B(R_2) = \begin{pmatrix} -1 & 4 & -4 \\ -1 & 3 & -2 \\ -1 & 2 & -1 \end{pmatrix},$$

$$B(R_3) = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ -4 & 4 & -1 \end{pmatrix}. \tag{3.29}$$

Hence, the matrices (3.28) and (3.29) provide another generators for W and W_R, respectively.

To define a group action of W_R and W on the upper half-plane, consider the spaces

$$\left\{(\omega_1, \omega_1', \omega_2, \omega_2') \in (\mathbb{C}^*)^4 \mid \operatorname{Im}\left(\frac{\omega_i'}{\omega_i}\right) > 0,\ i = 1, 2\right\} \cong \mathbb{C}^* \times \mathbb{H} \times \mathbb{C}^* \times \mathbb{H},$$

where the above isomorphism is given explicitly by

$$(\omega_1, \omega_1', \omega_2, \omega_2') \mapsto (\omega_1, \tau_1, \omega_2, \tau_2) = \left(\omega_1, \frac{\omega_1'}{\omega_1}, \omega_2, \frac{\omega_2'}{\omega_2}\right).$$

Note that there exists a natural $A_1 \ltimes (\mathrm{PSL}_2(\mathbb{Z}) \times \mathrm{PSL}_2(\mathbb{Z}))$ action on the space $\mathbb{C}^* \times \mathbb{H} \times \mathbb{C}^* \times \mathbb{H}$ given by

$$\begin{aligned}(\gamma_1 \times \gamma_2)(\omega_1, \tau_1, \omega_2, \tau_2) &= \left((c_1\tau_1 + d_1)\omega_1, \frac{a_1\tau_1 + b_1}{c_1\tau_1 + d_1}, (c_2\tau_2 + d_2)\omega_2, \frac{a_2\tau_2 + b_2}{c_2\tau_2 + d_2}\right), \\ \sigma(\omega_1, \tau_1, \omega_2, \tau_2) &= (\omega_2, \tau_2, \omega_1, \tau_1),\end{aligned}$$

where

$$\sigma \in A_1, \qquad \gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \qquad i = 1.2.$$

In addition, consider the symmetric square polynomials with respect the periods $\omega_1, \omega_1', \omega_2, \omega_2'$

$$G\colon\ \mathbb{C}^* \times \mathbb{H} \times \mathbb{C}^* \times \mathbb{H} \mapsto \mathbb{C}^4, \qquad \left(\omega_1, \frac{\omega_1'}{\omega_1}, \omega_2, \frac{\omega_2'}{\omega_2}\right) \mapsto (w_1, w_2, w_3, w_4), \tag{3.30}$$

where

$$w_1 = \omega_1'\omega_2', \qquad w_2 = \frac{\omega_1\omega_2' + \omega_1'\omega_2}{2}, \qquad w_3 = \omega_1\omega_2, \qquad w_4 = \frac{\omega_1\omega_2' - \omega_1'\omega_2}{2}.$$

Consider the diagonal action of $A_1 \times \mathrm{PSL}_2(\mathbb{Z})$ given by

$$\begin{aligned}\gamma(\omega_1, \tau_1, \omega_2, \tau_2) &= \left((c\tau_1 + d)\omega_1, \frac{a\tau_1 + b}{c\tau_1 + d}, (c\tau_2 + d)\omega_2, \frac{a\tau_2 + b}{c\tau_2 + d}\right), \\ \sigma(\omega_1, \tau_1, \omega_2, \tau_2) &= (\omega_2, \tau_2, \omega_1, \tau_1),\end{aligned}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \qquad \sigma \in A_1.$$

Hence, the image of this diagonal action under the map (3.30) preserves the space generated by w_1, w_2, w_3 and the space generated by w_4^2. This action restricted to the space generated by w_1, w_2, w_3, gives rise to the following group homomorphism:

$$\rho\colon\ \mathrm{PSL}_2(\mathbb{Z}) \mapsto \mathrm{SL}_3(\mathbb{Z}), \qquad \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix},$$

Observe that the image of the generators of $\mathrm{PSL}_2(\mathbb{Z})$ are given by

$$\rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$$

Furthermore, the image of the generators of $\Gamma^{(3)}$ are given by

$$\rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 & -4 \\ -1 & 3 & -2 \\ -1 & 2 & -1 \end{pmatrix},$$
$$\rho\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ -4 & 4 & -1 \end{pmatrix}. \tag{3.31}$$

Hence, we have group isomorphism given by

$$A_1 \times \mathrm{PSL}_2(\mathbb{Z}) \mapsto W, \qquad \left(\sigma, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto \sigma\chi_2\begin{pmatrix} a & b \\ c & d \end{pmatrix}\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.32}$$

Moreover, the restriction of the group homomorphism (3.32) to $\Gamma^{(3)}$ provides an isomorphism from $\Gamma^{(3)}$ to W_R due to the relations (3.31) and (3.26). Summarising, we demonstrated the results of [29, Section 4.1] and [18, Theorem 5.5].

Lemma 3.2 ([18, 29])**.** *The monodromy groups W_R and W associated to $QH^*(\mathbb{CP}^2)$ are*

$$W_R = \Gamma^{(3)}, \qquad W = A_1 \times \mathrm{PSL}_2(\mathbb{Z}).$$

Remark 3.3. In [27, Theorem 5.12 (g)], Hertling proved that the group W_R is a free Coxeter group with 3 generators.

3.2 LG superpotential for small quantum cohomology of $\mathbb{CP}^2$

In this subsection, we construct a LG superpotential for the small quantum cohomology of $\mathbb{CP}^2$ using the Dubrovin superpotential construction discussed in Section 2. To proceed with this approach, we require some results from Milanov [29, Section 4].

Recall that the Dubrovin–Frobenius manifold associated with quantum cohomology of $\mathbb{CP}^2$ is the domain of the WDVV solution (1.21), which is given by

$$D = \left\{ \left(t^1, t^2, t^3\right) \in \mathbb{C}^3 \mid \left|\mathrm{e}^{t^2}\left(t^3\right)^3\right| < \frac{1}{a} \right\}. \tag{3.33}$$

In this context, the period map is the set of three independent solutions of the system (1.8)

$$w_1\left(t^1, t^2, t^3\right), \qquad w_2\left(t^1, t^2, t^3\right), \qquad w_3\left(t^1, t^2, t^3\right). \tag{3.34}$$

By introducing the change of coordinates

$$t^1 = t^1, \qquad Q = \mathrm{e}^{t^2}, \qquad t^3 = t^3, \tag{3.35}$$

we can express the domain (3.33) in an alternative way

$$D = \left\{ \left(t^1, t^2, Q\left(t^3\right)^3\right) \in \mathbb{C}^3 \mid \left|Q\left(t^3\right)^3\right| < \frac{1}{a} \right\} \cong \mathbb{C}^2 \times \mathbb{H}.$$

In order to simplify the problem, in [29, Section 4], Milanov considered the sublocus

$$D_{\text{asmall}} = \{(t^1, t^2, 0) \in \mathbb{C}^3\} \cong \mathbb{C}^2,$$

which is now denoted as affine small quantum cohomology. In addition, we use the change of coordinates (3.35) to conveniently write the domain D_{asmall} as

$$\mathscr{D}_{\text{asmall}} = \{(t^1, Q, 0) \in \mathbb{C}^3\} \cong \mathbb{C} \times \mathbb{C}^*. \tag{3.36}$$

The discriminant locus (1.7) of $QH^*(\mathbb{CP}^2)$ in the sublocus (3.36) is explicitly written as

$$\det \begin{pmatrix} 3Q & 0 & t^1 \\ 0 & t^1 & 3 \\ t^1 & 3 & 0 \end{pmatrix} = 0.$$

Hence, the period domain for the affine small quantum cohomology of $\mathbb{CP}^2$ is given by

$$\mathscr{D}_{\text{asmall}} \setminus \Sigma_{\text{asmall}} = \{(t^1, Q) \in \mathbb{C} \times \mathbb{C}^* \mid (t^1)^3 + 27Q \neq 0\}. \tag{3.37}$$

At this stage, we follow the Milanov notation in [29, Sections 1.3 and 4]. Recall that the solution $w_E(t^1, t^2, \dots, t^n, \lambda)$ of the extended Gauss–Manin connection flat coordinate system (1.10) is obtained from the solution $w(t^1, t^2, \dots, t^n)$ of the Gauss–Manin connection flat coordinate system (1.8) by shifting with $-\lambda$. In other words, we have the symmetry relation

$$w_E(t^1, t^2, \dots, t^n, \lambda) = w(t^1 - \lambda, t^2, \dots, t^n). \tag{3.38}$$

Due to the symmetry (3.38), Milanov considered the following restriction to the extended Gauss–Manin connection of $QH^*(\mathbb{CP}^2)$ in the small quantum cohomology locus

$$w_E(0, Q, 0, \lambda) = w(-\lambda, Q, 0). \tag{3.39}$$

Note that the period domain of (3.39) is given by

$$\mathscr{D}_{\text{asmall},\lambda} \setminus \Sigma_{\text{asmall},\lambda} = \{(\lambda, Q) \in \mathbb{C} \times \mathbb{C}^* \mid (\lambda)^3 - 27Q \neq 0\}. \tag{3.40}$$

There is no structural difference by considering the solutions (1.8) in the affine small quantum cohomology locus $w(t^1, Q)$ or the restricted solutions $w_E(0, Q, 0, \lambda)$ of (1.10). The only issue to which we need to pay attention is the sign change from t^1 to $-\lambda$. Currently, we follow [29, Section 4.2] and use λ instead of t^1 and later we change back to t^1.

The system (1.10) in the affine small quantum cohomology locus (3.40) is reduced to

$$\big((Q\partial_Q)^3 - Q\partial_\lambda^3\big) w(\lambda, Q) = 0, \qquad (\lambda\partial_\lambda + 3Q\partial_Q) w(\lambda, Q) = -\frac{w(\lambda, Q)}{2}, \tag{3.41}$$

where

$$w(\lambda, Q) = (w_1(\lambda, Q), w_2(\lambda, Q), w_3(\lambda, Q)). \tag{3.42}$$

The second equation of (3.41) implies that $w(\lambda, Q)$ is of the form $w(\lambda, Q) = Q^{-\frac{1}{6}} f(x)$, $x := \frac{(\lambda)^3}{27Q}$. Moreover, due the first equation of (3.41), the vector valued function $f(x)$ solves some generalized hypergeometric equation. This equation has a special basis of solution of the symmetric square form. More precisely, writing a basis of solutions in the form $f_1 = u^2$, $f_2 = uv$, $f_3 = v^2$, we have that u, v are solutions of a differential equation, which is equivalent to the classic hypergeometric equation

$$\big((1 - x)x\partial_x^2 + (c - (a + b + 1)x) - ab\big) z = 0, \qquad a = b = \frac{1}{12}, \qquad c = \frac{2}{3}. \tag{3.43}$$

Moreover, due to the [29, Lemma 4.1], the solutions (3.34) in the affine small quantum cohomology locus (3.37) satisfies a quadratic relation

$$w_2^2 = 4w_1w_3. \tag{3.44}$$

Therefore, there exist a basis of solution of (3.41) of the form

$$w(\lambda, Q) = \left(\tau^2(x)f_3(x)Q^{\frac{-1}{6}}, -2\tau(x)f_3(x)Q^{\frac{-1}{6}}, f_3(x)Q^{\frac{-1}{6}}\right), \qquad x = \frac{(\lambda)^3}{27Q},$$

where

$$\tau(x) = -\frac{w_2\left(t^1, Q\right)}{2w_3\left(t^1, Q\right)} = -\frac{f_2(x)}{2f_3(x)}. \tag{3.45}$$

The hypergeometric equation (3.43) has singularities in 0, 1, ∞. As a consequence, the solutions of (3.41) are multivalued function with branch points at 0, 1, ∞. Denote l_0, l_1, l_∞ the loops around 0, 1, ∞, in this setting, we write the [29, Proposition 4.2] as follows.

Proposition 3.4 ([29])**.**

(1) *The analytic continuation of the Schwarz map* (3.45) *gives a representation* $\pi_1\left(\mathbb{CP}^1 \setminus \{0,1,\infty\}\right) \mapsto \operatorname{Aut}(\mathbb{H})$ *such that the generators* l_1, l_∞ *are mapped to the transformations* $\tau \mapsto \frac{-1}{\tau}$, $\tau \mapsto \tau + 1$, *respectively.*

(2) *The image of the lower and upper half plane* $\operatorname{Im}(x) \le 0$, $\operatorname{Im}(x) \ge 0$ *under the Schwarz map are the hyperbolic triangles with vertices* ∞, $\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}$, i *and* ∞, $\mathrm{e}^{2\frac{\pi \mathrm{i}}{3}}$, i, *respectively.*

Recall that the $\mathrm{SL}_2(\mathbb{Z})$ orbits of the vertices $\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}$ and i can be written as zeros of the Eisenstein series E_4, E_6 defined in (3.13). Indeed

$$E_4\left(\frac{1}{1-\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}}\right) = E_4\left(\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\right) = \left(\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\right)^4 E_4\left(\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\right) = 0, \qquad E_6(\mathrm{i}) = (-\mathrm{i})^6 E_6(\mathrm{i}) = -E_6(\mathrm{i}) = 0.$$

Hence, the covering map associated to analytic continuation of Schwarz map (3.45) is given by

$$J\colon\ \mathbb{H} \setminus \{E_4(\tau) = 0\} \cup \{E_6(\tau) = 0\} \mapsto \mathbb{C} \setminus \{0, 1\}. \tag{3.46}$$

Remarkably, the monodromy and the Fourier expansion around 0, 1, ∞ of the hypergeometric equation (3.43) are well know. Milanov used these results to prove the [29, Lemma 4.3], which we state here.

Lemma 3.5 ([29])**.**

(1) *The function* J *defined in* (3.46) *extends to a holomorphic function on* $\mathbb{H}$ *and it coincides with the unique* $\mathrm{SL}_2(\mathbb{Z})$ *holomorphic invariant function such that* $J(\mathrm{i}) = 1$, *i.e.,*

$$J(\tau) = \frac{1}{1728} j(\tau), \tag{3.47}$$

where $j(\tau)$ *is defined in* (3.22)*.*

(2) *The pullback of any branch of* $\frac{8}{27}(2\pi)^6 f_3(x)^6$ *to* $\mathbb{H}$ *via the map* J *extends to a holomorphic function on* $\mathbb{H}$ *and its coincides with the modular form* $E_4^3(\tau) - E_6^2(\tau)$*.*

From now, we return to the variable t^1 by replacing λ to $-t^1$.

Using Lemma 3.5, we can write the discriminant locus $\left(t^1\right)^3 + 27Q = 0$ as the zeros of E_6. Indeed,

$$J(\tau) = -\frac{\left(t^1\right)^3}{27Q} \implies J(i) = -\frac{\left(t^1\right)^3}{27Q} = 1 \qquad \text{iff} \qquad \left(t^1\right)^3 + 27Q = 0.$$

Due to Proposition 3.4, Lemma 3.5 and the quadratic relation (3.44), the period map (3.41)

$$w\colon\ \mathscr{D}_{\text{asmall}} \setminus \Sigma_{\text{asmall}} \mapsto \Omega_{\text{small}} \setminus \left\{ E_6\left(\frac{-w_2}{2w_3}\right) = 0 \right\}$$

is given explicitly by

$$\begin{aligned} \tau &= \frac{w_2}{2w_3}\left(t^1, Q\right) = J^{-1}\left(-\frac{\left(t^1\right)^3}{27Q}\right), \\ r &:= w_3\left(t^1, Q\right) = 2\pi\sqrt{\frac{2}{3}}\left(\frac{\left(E_4^3(\tau) - E_6^2(\tau)\right)}{Q}\right)^{\frac{1}{6}}, \end{aligned} \tag{3.48}$$

where

$$\Omega_{\text{small}} = \left\{ (w_1, w_2, w_3) \in \mathbb{C}^3 \mid w_2^2 - 4w_1w_3 = 0,\ \operatorname{Im}\left(\frac{-w_2}{2w_3}\right) > 0 \right\}.$$

In [29, Section 2.3], it was introduced the following bijection:

$$\phi\colon\ \mathbb{H} \times \mathbb{C}^* \mapsto \Omega_{\text{small}}, \qquad (\tau, r) \mapsto \left(\tau^2 r, 2\tau r, r\right).$$

At this stage, we can state the [29, Theorem 2.3] as follows.

Lemma 3.6 ([29]).

(1) *The inverse period map of quantum cohomology of $\mathbb{CP}^2$ for $t^3 = 0$ is the map*

$$t\colon\ (\mathbb{H} \setminus \{E_6(\tau) = 0\}) \times \mathbb{C}^* \mapsto \left\{ \left(t^1, Q\right) \in \mathbb{C} \times \mathbb{C}^* \mid \left(t^1\right)^3 + 27Q \neq 0 \right\}$$

given by

$$\begin{aligned} & t^1(\tau, r) = -2\frac{(2\pi)^2}{r^2}E_4(\tau), \qquad Q(\tau, r) = \frac{8}{27}\frac{(2\pi)^6}{r^6}\left(E_4^3(\tau) - E_6^2(\tau)\right), \\ & t^3 = 0, \end{aligned} \tag{3.49}$$

where $E_k(\tau)$ are defined in (3.13). Moreover, the map (3.49) extends to the holomorphic function $t\colon \mathbb{H} \times \mathbb{C}^* \mapsto \mathbb{C} \times \mathbb{C}^*$.

(2) *The fibers of the map (3.49) are $A_1 \times \mathrm{PSL}_2(\mathbb{Z})$-invariant and its concrete action in the domain of (3.49) is generated by*

$$\begin{aligned} & A(\tau, r) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^2\right) r, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \\ & B(\tau, r) = (\tau, -r). \end{aligned} \tag{3.50}$$

Using the Lemma 3.6, we can prove the main result of this section.

Theorem 3.7.

(1) *The Landau–Ginzburg superpotential of small affine quantum cohomology of $\mathbb{CP}^2$ is a family of functions $\lambda\big(\tilde{\tau}, t^1, Q\big)\colon \mathbb{H} \mapsto \mathbb{C}$ with holomorphic dependence in the parameter space $\big(t^1, Q^{\frac{1}{3}}\big) \in \mathbb{C} \times \mathbb{C}^*$ and given by*

$$\lambda\big(\tilde{\tau}, t^1, Q\big) = t^1 + 3Q^{\frac{1}{3}} J^{\frac{1}{3}}(\tilde{\tau}) \tag{3.51}$$

In addition, the correspondent Abelian differential ϕ is given by

$$\phi = -\frac{2^{\frac{5}{2}}}{2\pi} \frac{\Delta^{\frac{1}{6}}(\tilde{\tau})}{Q^{\frac{1}{6}}} \mathrm{d}\tilde{\tau}, \tag{3.52}$$

where $\Delta(\tilde{\tau})$ is defined in (3.20).

(2) *The Landau–Ginzburg superpotential (3.51) is invariant with respect the group $\Gamma^{(3)}$, i.e., $\lambda\big(\frac{a\tilde{\tau}+b}{c\tilde{\tau}+d}\big) = \lambda(\tilde{\tau})$.*

Proof. The Dubrovin construction of Landau–Ginzburg superpotential applies to any Dubrovin–Frobenius manifold with $d \neq 1$, in order to construct concretely the Landau–Ginzburg superpotentialone should only choice a suitable solution of the Gauss–Manin connection $p(\lambda, u)$ and invert it in λ. Consider a basic solution $(x_1(\lambda, t), x_2(\lambda, t), x_3(\lambda, t))$ of the extended Gauss–Manin connection of $QH^*\big(\mathbb{CP}^2\big)$, which is given by (2.13), (2.15) and (2.26). Then, consider the change of basis

$$\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

sends the basic solution $(x_1(\lambda, t), x_2(\lambda, t), x_3(\lambda, t))$ to (3.42). In order to obtain the Landau–Ginzburg superpotential of affine small quantum cohomology of $\mathbb{CP}^2$, we choice $p\big(\lambda, t^1, Q, 0\big) := \tilde{w}_2\big(\lambda, t^1, Q, 0\big)$ and apply the Lemma 2.3 and Theorem 2.5 to $p\big(\lambda, t^1, Q, 0\big)$. On another words, we invert $p\big(\lambda, t^1, Q, 0\big) := \tilde{w}_2\big(\lambda, t^1, Q, 0\big)$ in λ.

Inverting the map provided by (3.49), we obtain the functions defined in (3.48).

Next, consider the affine extended $\tilde{w}_2$, $\tilde{w}_3$,

$$\begin{aligned} \tilde{\tau} &:= \frac{-\tilde{w}_2}{2\tilde{w}_3}\big(t^1 - \lambda, Q\big) = J^{-1}\left(\frac{\big(\lambda - t^1\big)^3}{27Q}\right), \\ \tilde{r} &:= \tilde{w}_3\big(t^1 - \lambda, Q\big) = 2\pi\sqrt{\frac{2}{3}}\left(\frac{\big(E_4^3(\tilde{\tau}) - E_6^2(\tilde{\tau})\big)}{Q}\right)^{\frac{1}{6}}. \end{aligned} \tag{3.53}$$

Inverting the first equation of (3.53) in λ, we obtain

$$\lambda\big(\tilde{\tau}, t^1, Q\big) = t^1 + 3Q^{\frac{1}{3}} J^{\frac{1}{3}}(\tilde{\tau}). \tag{3.54}$$

Hence, by Theorem 2.5 the data $\big(\lambda\big(\tilde{\tau}, t^1, Q\big) = t^1 + 3Q^{\frac{1}{3}} J^{\frac{1}{3}}(\tilde{\tau}), \mathrm{d}\tilde{w}_2\big)$, where $\tilde{w}_2$ is the pullback of the map $\tilde{w}_2 := w_2\big(t^1 - \lambda, Q, 0\big)$ via the map (3.51), gives a Landau–Ginzburg superpotential for the affine small quantum cohomology of $\mathbb{CP}^2$.

The domain of (3.51) is a family of $\mathbb{H}$ parametrized by $(\tau, r) \in (\mathbb{H} \setminus \{E_6(\tau) = 0\}) \times \mathbb{C}^*$, because of the parametrized LG superpotential (3.54) and the change of coordinates (3.49).

In order to write $\mathrm{d}\tilde{w}_2$ in terms the coordinate $\tilde{\tau}$, it is convenient to consider the following spaces.

Here consider this family of $\mathbb{H}$ defined by

$$\mathbb{H}_{\tau,r} = \left\{ (\tilde{\tau}, \tilde{r}, \tau, r) \in \mathbb{H} \times \mathbb{C}^* \times \mathbb{H} \times \mathbb{C}^* \mid \tilde{r} = 2\pi \sqrt{\frac{2}{3}} \left(\frac{\left(E_4^3(\tilde{\tau}) - E_6^2(\tilde{\tau})\right)}{Q} \right)^{\frac{1}{6}}, \, E_6(\tau) \neq 0 \right\}.$$

Let the space $\Omega_{w_2,r}$ be defined by

$$\Omega_{w_2,r} = \left\{ (w_2, r) \in \mathbb{C}^* \times \mathbb{C}^* \mid \mathrm{Im}\left(\frac{-w_2}{2r} \right) > 0 \right\}.$$

Consider the following isomorphism $S\colon \mathbb{H} \times \mathbb{C}^* \times \mathbb{H} \times \mathbb{C}^* \mapsto \Omega_{\tilde{w}_2,\tilde{r}} \times \Omega_{w_2,r}$ $S(\tilde{\tau}, \tilde{r}, \tau, r) = (\tilde{w}_2, \tilde{r}, w_2, r)$, where $\tilde{w}_2 = -2\tilde{r}\tilde{\tau}$, $w_2 = -2r\tau$. Note that the section $\mathrm{d}\tilde{w}_2 \in \Gamma(T^*(\mathbb{H}_{\tau,r}))$ is the pullback of the section $\mathrm{d}\tilde{w}_2 \in \Gamma(T^*(\Omega_{\tilde{w}_2,\tilde{r}} \times \Omega_{w_2,r}))$ and it is given by

$$\mathrm{d}\tilde{w}_2 = -2\tilde{r}\mathrm{d}\tilde{\tau} = -\frac{2^{\frac{5}{2}}}{2\pi} \frac{\Delta^{\frac{1}{6}}(\tilde{\tau})}{Q^{\frac{1}{6}}} \mathrm{d}\tilde{\tau}.$$

Here, it was used the following relation $\Delta(\tau) = (2\pi)^{12} \frac{E_4^3(\tau) - E_6^2(\tau)}{1728}$. The Landau–Ginzburg potential (3.51) is $\Gamma^{(3)}$ invariant, because the function $J^{\frac{1}{3}}(\tilde{\tau})$ is $\Gamma^{(3)}$ invariant due to (3.24) and (3.25), see [8, Section 3.1] for details. Theorem proved. ■

Even though, the construction given by Dubrovin guarantee that the function (3.51) works as superpotential for the affine small quantum cohomology of $\mathbb{CP}^2$. It would be instructive to directly verify the first condition.

Note that the LG variable p in the notation of (1.10) is the variable $\tilde{w}_2$ and not $\tilde{\tau}$. Due to (3.53) and the relation $\Delta(\tau) = (2\pi)^{12} \frac{E_4^3(\tau) - E_6^2(\tau)}{1728}$, $\tilde{\tau}$ and $\tilde{w}_2$ are related by the following:

$$\tilde{\tau} = -\frac{\tilde{w}_2}{2\tilde{r}}, \qquad \frac{\partial}{\partial \tilde{w}_2} = -\frac{2\pi}{2^{\frac{5}{2}}} \frac{Q^{\frac{1}{6}}}{\Delta^{\frac{1}{6}}(\tilde{\tau})} \frac{\partial}{\partial \tilde{\tau}}.$$

Then,

$$\begin{aligned} \frac{\partial \lambda}{\partial \tilde{w}_2} &= -\frac{(2\pi) \times 3}{2^{\frac{5}{2}}} \frac{Q^{\frac{1}{2}}}{\Delta^{\frac{1}{6}}(\tilde{\tau})} \frac{\partial J^{\frac{1}{3}}(\tilde{\tau})}{\partial \tilde{\tau}} = -\frac{(2\pi) \times 3}{2^{\frac{5}{2}}} \frac{Q^{\frac{1}{2}}}{\Delta^{\frac{1}{6}}(\tilde{\tau})} \frac{E_4'(\tilde{\tau}) - \frac{E_2(\tilde{\tau}) E_4(\tilde{\tau})}{3}}{\Delta^{\frac{1}{3}}(\tilde{\tau})} \\ &= \frac{(2\pi)}{2^{\frac{5}{2}}} Q^{\frac{1}{2}} \frac{E_6(\tilde{\tau})}{\Delta^{\frac{1}{2}}(\tilde{\tau})}, \end{aligned} \tag{3.55}$$

where Ramanujan identities [31] were used from the second to the third line of (3.55)

$$E_2' = \frac{E_2^2 - E_4}{12}, \qquad E_4' = \frac{E_2 E_4 - E_6}{3}, \qquad E_6' = \frac{E_2 E_6 - E_4^2}{2}. \tag{3.56}$$

Then, $\frac{\partial \lambda}{\partial \tilde{w}_2} = 0$ iff $E_6(\tilde{\tau}) = 0$. The zeros of E_6 are the elements of orbit of the imaginary number i under the $\mathrm{SL}_2(\mathbb{Z})$ action [31]. Recall that the function $\gamma_2(\tilde{\tau}) = j^{\frac{1}{3}}(\tilde{\tau})$ is the Weber function (3.23), which is known to be the Hauptmodul of the group $\Gamma^{(3)}$ defined in (3.25). In addition, we have that $J^{\frac{1}{3}}(\tau) = \frac{j^{\frac{1}{3}}(\tau)}{12} = \frac{\gamma_2(\tau)}{12}$.

In this context, the zeros of E_6 are $i-1$, i, $i+1$ mod $\Gamma^{(3)}$. Moreover, using the relation (3.24), we derive $\frac{\gamma_2(i)}{12} = 1$, $\frac{\gamma_2(i+1)}{12} = \mathrm{e}^{\frac{-2\pi \mathrm{i}}{3}}$, $\frac{\gamma_2(i-1)}{12} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$.

As a consequence,

$$u_k = t^1 + 3Q^{\frac{1}{3}} \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right)^k, \qquad k = 1, 2, 3. \tag{3.57}$$

Remark 3.8. The zeros of the LG superpotential (3.51) is given by $\tilde{\tau} = \tau$. Indeed,

$$\begin{aligned}\lambda\big(\tau, t^1, Q\big) &= t^1 + 3Q^{\frac{1}{3}} J^{\frac{1}{3}}(\tau) = -2\frac{(2\pi)^2}{r^2} E_4(\tau) + 2\frac{(2\pi)^2}{r^2}\big(E_4^3(\tau) - E_6^2(\tau)\big)^{\frac{1}{3}} J^{\frac{1}{3}}(\tau)\\ &= 2\frac{(2\pi)^2}{r^2}[-E_4(\tau) + E_4(\tau)] = 0.\end{aligned}$$

Remark 3.9. The LG superpotential $\lambda = t^1 + \frac{1}{4}Q^{\frac{1}{3}}\gamma_2(\tilde{\tau})$ (3.51) can be thought of as being a family of the covering maps over $\mathbb{CP}^1$ which share the same monodromy. This family is an isomonodromic deformation of the Hauptmodul of the group $\Gamma^{(3)}$, the parameter Q, t^1 are a rescaling and affine isomonodromic deformation parameter, respectively.

3.3 Cohn interpretation

The aim of this subsection is to give a geometric interpretation of LG superpotential (3.51) in terms of elliptic curves with respect to the lattice $\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}$.

Consider a family of hyperelliptic curves whose roots are given by the small canonical coordinates (3.57)

$$y^2 = 4(\lambda - u_1)(\lambda - u_2)(\lambda - u_3) = 4\big[\big(\lambda - t^1\big)^3 - 27Q\big]. \tag{3.58}$$

With respect to the notation of the equation (3.14), the cubic (3.58) do not have the linear term, i.e., $g_2 = 0$, which means that the correspondent lattice of the elliptic curve (3.58) is equiharmonic, i.e., $\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}$. Because, $g_2(\tau)$ is proportional to $E_4(\tau)$, whose zeros are the $\mathrm{SL}_2(\mathbb{Z})$ orbit of $\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}$. The uniformization of the cubic (3.58) is given by

$$\lambda - t^1 = \frac{\wp\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^2}, \qquad y = \frac{\wp'\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^3}, \qquad 27Q = -\frac{1}{4(2\omega)^6},$$

which gives rise to a natural change of coordinates $C(v_0, \omega) = \big(t^1, Q\big)$

$$C\colon\ \big(\mathbb{C} \setminus \big\{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\mathbb{Z}\big\} \times \mathbb{C}^*\big) \setminus \left\{\frac{\mathbb{Z}}{2} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\frac{\mathbb{Z}}{2} \times \mathbb{C}^*\right\} \mapsto \mathbb{C} \times \mathbb{C}^* \setminus \big\{\big(t^1\big)^3 + 27Q = 0\big\},$$

given by

$$t^1(v_0, \omega) = -\frac{\wp\big(v_0, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^2}, \qquad 27Q(v_0, \omega) = -\frac{1}{4(2\omega)^6}. \tag{3.59}$$

Indeed, the functions (3.59) are holomorphic and locally invertible in its domain. In addition, the discriminant $\big(t^1\big)^3 + 27Q = 0$, in this coordinates is written as

$$\begin{aligned}\big(t^1\big)^3 + 27Q &= -\frac{\wp^3\big(v_0, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^6} - \frac{1}{4(2\omega)^6} = -\frac{1}{4(2\omega)^6}\big(4\wp^3\big(v_0, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big) + 1\big)\\ &= -\frac{1}{4(2\omega)^6}\big(\wp'\big(v_0, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)\big)^2 = 0,\end{aligned}$$

which is equivalent to the locus $\frac{\mathbb{Z}}{2} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\frac{\mathbb{Z}}{2} \times \mathbb{C}^*$ due to (3.15).

Then, the uniformization of the cubic (3.58) is the following family of $\wp$ functions:

$$\lambda(v, v_0, \omega) = \frac{\wp\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^2} - \frac{\wp\big(v_0, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^2}, \tag{3.60}$$

parametrized by $(v_0, \omega) \in \left(\mathbb{C} \backslash \left\{\mathbb{Z} \oplus e^{\frac{\pi i}{3}} \mathbb{Z}\right\} \times \mathbb{C}^*\right) \backslash \left\{\frac{\mathbb{Z}}{2} \oplus e^{\frac{\pi i}{3}} \frac{\mathbb{Z}}{2}\right\} \times \mathbb{C}^*$. One might initially assume that the family (3.60) serves as an LG superpotential for the small quantum cohomology of $\mathbb{CP}^2$ since it produces the correct canonical coordinates by construction. Nevertheless, it fails to meet the remaining conditions outlined in (1.10). The good news is that we can enrich the family (3.60) to a truly LG superpotential. For this purpose, we have to consider a suitable Abelian differential ϕ which would satisfy the other conditions of (1.10). This Abelian differential ϕ is constructed explicitly by considering the universal covering of $\mathbb{C} \setminus \left\{\mathbb{Z} \oplus e^{\frac{\pi i}{3}} \mathbb{Z}\right\}$. Let us make a digression on factorisation of $\mathrm{SL}_2(\mathbb{Z})$ group. Recall that there exist group homomorphisms from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}_n)$, i.e.,

$$\pi_n \colon\ \mathrm{SL}_2(\mathbb{Z}) \mapsto \mathrm{SL}_2(\mathbb{Z}_n). \tag{3.61}$$

The kernel of (3.61) provide the congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$

$$\Gamma(n) := \mathrm{Ker}\, \pi_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}.$$

We are particularly interested in the case $n = 2$

$$\pi_2 \colon\ \mathrm{SL}_2(\mathbb{Z}) \mapsto S_3, \qquad \mathrm{SL}_2(\mathbb{Z})/\Gamma(2) \cong S_3. \tag{3.62}$$

The factorisation of the $\mathrm{SL}_2(\mathbb{Z})$ for π_2 in (3.62) induces a factorisation of the j-function (3.47) in the following sense:

$$J \colon\ \mathbb{H} \mapsto \mathbb{C} \setminus \{0,1\} \mapsto \mathbb{C}, \qquad J(\tau) = \frac{4}{27} \frac{\left(1 - x(\tau) + x^2(\tau)\right)^3}{(x(\tau))^2 (x(\tau) - 1)^2},$$

where

$$x \colon\ \mathbb{H} \mapsto \mathbb{C} \setminus \{0,1\}, \qquad x(\tau) = \frac{\wp\left(\frac{1+\tau}{2}, \tau\right) - \wp\left(\frac{\tau}{2}, \tau\right)}{\wp\left(\frac{1}{2}, \tau\right) - \wp\left(\frac{\tau}{2}, \tau\right)}, \tag{3.63}$$

is the universal covering $\mathbb{C} \setminus \{0,1\}$, which is a $\Gamma(2)$- invariant function, i.e.,

$$x(\tau + 2) = x(\tau), \qquad x\left(\frac{\tau}{2\tau + 1}\right) = x(\tau),$$

and have the following transformation law from under the $\mathrm{Aut}(\mathbb{C} \setminus \{0,1\}) \cong S_3$ action:

$$x(\tau + 1) = \frac{x(\tau)}{x(\tau) - 1}, \qquad x\left(\frac{-1}{\tau}\right) = 1 - x(\tau), \qquad x\left(\frac{\tau}{1 - \tau}\right) = \frac{1}{x(\tau)},$$
$$x\left(\frac{1}{1 - \tau}\right) = \frac{1}{1 - x(\tau)}, \qquad x\left(\frac{\tau - 1}{\tau}\right) = \frac{x(\tau) - 1}{x(\tau)}.$$

The example above is our toy model to the small quantum cohomology case. Indeed, we provide similar results by replacing the universal covering of $\mathbb{C} \setminus \{0,1\}$ to the universal covering of $\mathbb{C} \setminus \left\{\mathbb{Z} \oplus e^{\frac{\pi i}{3}} \mathbb{Z}\right\}$. Here, we will summarise some results of [5, Sections 1 and 2] which is also related to [10], and [24]. Consider the group of affine transformation on $\mathbb{C}$ which can be represented by

$$Aff(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}(2, \mathbb{C}),$$

where the action in $\mathbb{C}$ is given by

$$A(z) = \operatorname{proj}_1 \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = az + b, \qquad z \in \mathbb{C}, \qquad A \in \operatorname{Aff}(\mathbb{C}).$$

Define a group homomorphism $\psi\colon \mathrm{SL}_2(\mathbb{Z}) \mapsto \operatorname{Aff}(\mathbb{C})$ by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto \bar{S} = \begin{pmatrix} -1 & \zeta_6 + \zeta_6^2 \\ 0 & 1 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \bar{T} = \begin{pmatrix} \zeta_6 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \zeta_6 = \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}.$$

Then, due to the results of [5, Section 1 and 2], the following facts hold true:

(1) $\mathrm{SL}_2(\mathbb{Z}) / \operatorname{Ker} \psi \cong \operatorname{Im} \psi$.

(2) The $\operatorname{Im}\psi$ acts on $\mathbb{C}$ and do preserve the lattice $\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}$.

(3) The $\operatorname{Im}\psi$ acts properly discontinuously on $\mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}\}$ and have the interior of the triangle with corners 0, ζ_6, ζ_6^2 as fundamental chamber.

(4) $\mathbb{H} / \operatorname{Ker}\psi \cong \mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}\}$, hence $\pi_1(\mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}\}) = \operatorname{Ker}\psi$.

(5) The Weierstrass $\wp$ function with respect to the lattice $\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}$ has the following transformation laws due to (3.9):

$$\wp\big(v + m + n\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big) = \wp\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big), \qquad m, n \in \mathbb{Z},$$
$$\wp\left(\frac{v}{-\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}}, \frac{-1}{\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}}\right) = \big(\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)^2 \wp\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big).$$

Moreover, the $\operatorname{Im}\psi$ action induces the following:

$$\wp\big(-v + \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} + \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big) = \wp\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big), \qquad \wp\left(\frac{v}{-\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}}, \frac{-1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}}\right) = -\mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \wp\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big).$$

Then, the function $\wp'\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)^2$ is invariant with respect the $\operatorname{Im}\psi$ action.

(6) Let $F_{\operatorname{Im}\psi}$ and F_0 be the fundamental chamber of $\operatorname{Im}\psi$ and $\mathrm{SL}_2(\mathbb{Z})$, respectively. Then, the function $v\big|_{F_0}\colon F_0 \mapsto F_{\operatorname{Im}\psi}$ given by the composition of the j-function and the inverse of $\wp'\big(v, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)^2$, i.e., $v\big|_{F_0}(\tau) = \big(\big(\wp'_{\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}}\big)^2\big)^{-1}(J(\tau))$ is analytic and bijective.

(7) The function $v\big|_{F_0}$ extends to an analytic covering map

$$v(\tau)\colon\ \mathbb{H} \mapsto \mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}\}, \tag{3.64}$$

which is the universal covering map of $\mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}\}$.

As a consequence, we have the Cohn identities derived in [10, Section 9] (see also [5, 24])

$$1 - J(\tau) = 4\wp^3\big(v(\tau), \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big) + 1, \qquad (v'(\tau))^6 = \Delta(\tau). \tag{3.65}$$

Here, we state and prove the main result of this subsection.

Theorem 3.10. *The Landau–Ginzburg superpotential of small affine quantum cohomology of* $\mathbb{CP}^2$ *is a family of functions* $\lambda(\tilde{\tau}, v_0, \omega)\colon \mathbb{H} \mapsto \mathbb{C}$ *with holomorphic dependence in the parameter space* $(v_0, \omega) \in \big(\mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}\}\big) \times \mathbb{C}^*$ *and given by*

$$\lambda(\tilde{\tau}, v_0, \omega) = \frac{\wp\big(v(\tilde{\tau}), \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^2} - \frac{\wp\big(v_0, \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\big)}{(2\omega)^2},$$

where $v(\tilde{\tau})$ *is the universal covering of* $\mathbb{C} \setminus \{\mathbb{Z} \oplus \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \mathbb{Z}\}$. *In addition, the correspondent Abelian differential* ϕ *is given by* $\phi = 2\omega \mathrm{d}v(\tilde{\tau})$.

Proof. Substituting (3.59) and (3.65) in (3.60), we obtain the LG superpotential (3.51). Indeed,

$$\lambda(\tilde{\tau}, v_0, \omega) = \frac{\wp\big(v(\tilde{\tau}), e^{\frac{\pi i}{3}}\big)}{(2\omega)^2} - \frac{\wp\big(v_0, e^{\frac{\pi i}{3}}\big)}{(2\omega)^2} = -3Q^{\frac{1}{3}}4^{\frac{1}{3}}\wp\big(v(\tilde{\tau}), e^{\frac{\pi i}{3}}\big) + t^1 = 3Q^{\frac{1}{3}}J^{\frac{1}{3}}(\tilde{\tau}) + t^1.$$

Substituting the second equation of (3.65) and (3.59) in (3.52), the Abelian differential ϕ becomes

$$\phi = -\frac{2^{\frac{5}{2}}}{2\pi}\frac{\Delta^{\frac{1}{6}}(\tilde{\tau})}{Q^{\frac{1}{6}}}d\tilde{\tau} = -\frac{2^{\frac{17}{6}} \times 3^{\frac{1}{2}}}{2\pi}(2\omega)\Delta^{\frac{1}{6}}(\tilde{\tau})d\tilde{\tau} = -\frac{2^{\frac{17}{6}} \times 3^{\frac{1}{2}}}{2\pi}(2\omega)v'(\tilde{\tau})d\tilde{\tau}.$$

Summarising, up a constant, we have $\phi = 2\omega dv(\tilde{\tau}) = 2\omega v'(\tilde{\tau})d\tilde{\tau}$, as a consequence

$$\phi = d\tilde{w}_2 = 2\omega dv(\tilde{\tau}) \implies \tilde{w}_2 = 2\omega v(\tilde{\tau}).$$

In other words, the Abelian differential ϕ of affine small quantum cohomology of $\mathbb{CP}^2$ is the differential of the universal covering (3.64). ∎

Remark 3.11. The composition of universal covering (3.64) with the functions $\wp$, $\wp'$ with respect to the equianharmonic lattice can be expressed in terms of the Weber functions, i.e.,

$$\gamma_2(\tau) = 3 \times 4^{\frac{4}{3}}\wp\big(v(\tau), e^{\frac{\pi i}{3}}\big), \qquad \gamma_3(\tau) = 3^{\frac{3}{2}} \times 4^{\frac{3}{2}}\wp'\big(v(\tau), e^{\frac{\pi i}{3}}\big).$$

Remark 3.12. In the Givental setting [23], the elliptic curve (3.58), up affine shift, is precisely the 0-fiber of the Givental 2D superpotential of small quantum cohomology of $\mathbb{CP}^2$. Indeed, the 0 fiber of Givental superpotential $\lambda(x, y, Q) = x + y + \frac{Q}{xy}$, is the cubic

$$0 = x^2y + xy^2 + Q. \tag{3.66}$$

Substituting $\tilde{y} = \sqrt{x}y$ in (3.66), we have that $\tilde{y}^2 + x^{\frac{3}{2}}\tilde{y} + Q = 0$. Defining $\hat{y} = \tilde{y} + \frac{x^{\frac{3}{2}}}{2}$ and completing square, we obtain $\hat{y}^2 = \frac{x^3}{4} - Q$. After the rescaling $\hat{y} \mapsto \frac{\hat{y}}{2\sqrt{27}}$, $x \mapsto \frac{2^{\frac{2}{3}}}{3}x$, we obtain $\hat{y}^2 = 4x^3 - (4 \times 27)Q$.

4 Big quantum cohomology of $\mathbb{CP}^2$

4.1 Milanov deformation

The aim of this section is to derive a t^3-deformation of the superpotential (3.51).

In [29, Section 5], Milanov considered the t^3-deformation of abstract periods of small quantum cohomology in the following sense. Consider the power series

$$w_i\big(t^1, t^2, t^3\big) = w_i\big(t^1, Q, 0\big) + \sum_{n=1}^{\infty} w_{i,n}\big(t^1, Q\big)\big(t^3\big)^n, \qquad i = 1, 2, 3, \tag{4.1}$$

which solves the abstract Gauss–Manin connection flat coordinate system (1.8) of quantum cohomology of $\mathbb{CP}^2$. The solution (4.1) is a multivalued map that is locally analytic in the locus

$$D \setminus \Sigma = \left\{\big(t^1, Q, t^3\big) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C} \mid \big|Q\big(t^3\big)^3\big| < \frac{1}{a}, \det\begin{pmatrix} 3F_{33} & 2F_{23} & t^1 \\ 2F_{23} & F_{22} & 3 \\ t^1 & 3 & -t^3 \end{pmatrix} \neq 0\right\}, \tag{4.2}$$

because the coefficients of (1.8) with respect $QH^*\big(\mathbb{CP}^2\big)$ are analytic in the domain (1.25), excluding the discriminant locus (1.7).

In [29, Section 5.1], Milanov constructed an auxiliary period domain, which extends the map (3.49) for $t^3 \neq 0$ aiming to locally invert the map (4.1). More specifically, consider the map

$$\pi^{\text{aux}}\colon\ \mathscr{U} \mapsto D \setminus \Sigma \subset \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}, \tag{4.3}$$

given by

$$t^1 = -2\frac{(2\pi)^2}{r^2}E_4(\tau), \qquad Q = \frac{8}{27}\frac{(2\pi)^6}{r^6}\big(E_4^3(\tau) - E_6^2(\tau)\big), \qquad t^3 = s\frac{E_6^2(\tau)}{r^6}, \tag{4.4}$$

where $D \setminus \Sigma$ is defined in (4.2) and

$$\mathscr{U} = \{(\tau, r, s) \in \mathbb{H} \times \mathbb{C}^* \times \mathbb{C} \mid |s| < \delta(\tau, r)\},$$

the function $\delta(\tau, r)\colon \mathbb{H} \times \mathbb{C}^* \mapsto \mathbb{R}_{>0}$ is chosen in such way that the preimage of the discriminant by the map (4.3) is the zero locus of $E_6(\tau)$. More specifically, we express the discriminant as power series in t^3 as follows:

$$\det E_\bullet = \big(t^1\big)^3 + 27Q + t^3 f\big(t^1, Q, t^3\big) = 0 \tag{4.5}$$

for some $f\big(t^1, Q, t^3\big)$ holomorphic function in $D\setminus\Sigma$. Substituting the change of coordinates (4.4) in (4.5), we obtain

$$E_6^2(\tau)\big(8(2\pi)^2 + sf \bullet \pi^{\text{aux}}(\tau, r, s)\big) = 0.$$

For fixed $(\tau, r) \in \mathbb{H} \times \mathbb{C}^*$ the function $\delta(\tau, r)$ is chosen such that $\pi^{\text{aux}}(\tau, r, s) \in D$ and $|sf \bullet \pi^{\text{aux}}(\tau, r, s)| < 8(2\pi)^2$.

The domain of (4.3) is a tubular neighbourhood of the domain of (3.49), which keeps the group of Deck transformations of (3.49) constant. More precisely, the monodromy action (3.50) extends trivially on the direction s, i.e.,

$$A(\tau, r, s) = \left(\frac{a\tau + b}{c\tau + d}, r(c\tau + d)^2, s\right), \qquad B(\tau, r, s) = (\tau, -r, s).$$

Composing (4.4) with (4.1), we obtain a power series in t^3

$$w_i\big(\tau, r, t^3\big) = w_i(\tau, r, 0) + \sum_{n=1}^{\infty} w_{i,n}(\tau, r)\big(t^3\big)^n, \qquad i = 1, 2, 3, \tag{4.6}$$

where the coefficients are functions

$$w_{i,n}\colon\ \{(\tau, r) \in \mathbb{H} \times \mathbb{C}^* \mid E_6(\tau) \neq 0\} \mapsto \mathbb{C}.$$

Then, we can state in our setting the following [29, Proposition 5.8].

Lemma 4.1 ([29])**.** *Let the power series w_1, w_2, w_3 be defined in (4.6). Then the power series w_1, w_2, w_3 define holomorphic functions in the domain (4.3). Moreover, its Taylor coefficients $(w_{1,n}(\tau, r), w_{2,n}(\tau, r), w_{3,n}(\tau, r))$ have the following property:*

$$(w_{1,n}(\tau, r), w_{2,n}(\tau, r), w_{3,n}(\tau, r)) \in r^{1-2n}E_6^{-2n}\mathbb{C}[E_2, E_4, E_6], \qquad n > 0.$$

The sketch of the proof of Lemma 4.1 is given by the following the key points:

(1) The power series defined in (4.6) are holomorphic in the domain (4.3) since they are analytic continuation of the multivalued functions (4.1) due to [29, Proposition 5.3].

(2) Due to the quasi homogeneous condition (1.13) and the charge d of $QH^*\mathbb{CP}^2$ be equal 2, the coordinates w_1, w_2, w_3 have degree $-\frac{1}{2}$ and the coordinates r, τ, t^3 have degree $-\frac{1}{2}$, 0, -1, respectively. Then $w_{i,n}(\tau, r)$ is proportional to r^{1-2n}, i.e., $w_{i,n}(\tau, r)$ factorise as follows

$$w_{i,n}(\tau, r) = r^{1-2n} w_{i,n}(\tau).$$

(3) Under the change of coordinates (4.4) and due to the Ramanujan identities (3.56), the vector fields $t^1\partial_1$, $Q\partial_Q$ have the following form:

$$t^1\partial_1 = \frac{E_4}{E_6}\partial_\tau + \frac{1}{6E_6}(E_2E_4)r\partial_r, \qquad Q\partial_Q = \frac{E_4}{E_6}\partial_\tau + \frac{1}{6E_6}(E_2E_4 - E_6)r\partial_r. \tag{4.7}$$

The action of vector field ∂_3 can also be written in terms of the vector fields E_2, E_4, E_6 due to the vector field (4.7) and the quasi homogeneous condition (1.13). The coefficients of Gauss–Manin connection of $QH^*\mathbb{CP}^2$ are polynomial in t^1, t^3, $\frac{1}{t^3}$ and $\Phi(X)$, $\Phi'(X)$, $\Phi''(X)$, $\Phi'''(x)$ where $\Phi(X)$ is defined in (1.23). The function $\Phi(X)$ is a holomorphic power series around $X := \ln\big(Q\big(t^3\big)^3\big) \mapsto -\infty$. Hence, the Gauss–Manin connection of $QH^*\mathbb{CP}^2$ gives rise to a infinite list of differential equation for $w_{i,n}(\tau)$ with rational coefficients in E_2, E_4, E_6. Since, the ring of quasi-modular forms $\mathbb{C}[E_2, E_4, E_6]$ is closed under derivation, the functions $w_{i,n}(\tau)$ must be rational in E_2, E_4, E_6.

(4) The function $w_{i,n}(\tau) \in E_6^{-2n}\mathbb{C}[E_2, E_4, E_6]$, because of its holomorphic behaviour in (4.3), i.e., the series

$$w_i\big(\tau, r, t^3\big) = \sum_{n=0}^{\infty} w_{i,n}(\tau, r)\left(\frac{E_6^2(\tau)}{r^6}\right)^n s^n, \qquad i = 1, 2, 3,$$

is holomorphic in the domain (4.4). For more details, see [29, Proposition 5.8].

For a deeper understanding of the correspondence between (τ, r, s) and (w_1, w_2, w_3), it becomes essential to construct an auxiliary coordinate system (τ_1, τ_2, y). To achieve this, it is crucial to recall that the intersection form resulting from $S + S^T$ in specific flat coordinates is described by an indefinite bilinear form

$$\begin{pmatrix} \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \end{pmatrix} \begin{pmatrix} 2 & 3 & -3 \\ 3 & 2 & -3 \\ -3 & -3 & 2 \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}.$$

Then, through a linear change of coordinates

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 2 \\ 2 & 1 & -2 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \\ \mathrm{d}w_3 \end{pmatrix},$$

and after rescaling, the intersection form becomes

$$g^* = \mathrm{d}w_2^2 - 4\mathrm{d}w_1\mathrm{d}w_3. \tag{4.8}$$

Due to change of endomorphism (3.27), the monodromy action on the coordinates $(w_1, w_2, w_3) \in (\mathbb{C}^*)^3$ is given by the generators (3.28).

Consider the map, which works as a local change of coordinates

$$\phi\colon\ \mathbb{H}^2 \times \mathbb{C}^* \mapsto (\mathbb{C}^*)^3, \qquad (\tau_1, \tau_2, y) \mapsto (w_1, w_2, w_3) = (\tau_1\tau_2 y, (\tau_1 + \tau_2)y, y). \tag{4.9}$$

Due to symmetric square representation (3.32), the monodromy action in $\mathbb{H}^2 \times \mathbb{C}^*$ is given by

$$A(\tau_1, \tau_2, y) = \left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a\tau_2 + b}{c\tau_2 + d}, y(c\tau_1 + d)(c\tau_1 + d)\right),$$
$$B(\tau_1, \tau_2, y) = (\tau_1, \tau_2, -y). \tag{4.10}$$

The change of coordinates of the intersection form (4.8) from its flat coordinates (w_1, w_2, w_3) to Saito flat coordinates (t^1, t^2, t^3) give rise to a relationship between both coordinates. In particular, we have [29, Lemma 5.9], which we state here for the convenience of the reader.

Lemma 4.2 ([29])**.** *Let (t^1, t^2, t^3) and (w_1, w_2, w_3) be the Saito flat coordinates and intersection form flat coordinates of $QH^*(\mathbb{CP}^2)$, respectively. Then $t^3 = w_2^2 - 4w_1w_3$. Moreover, due to the map (4.9), the Saito flat coordinate t^3 is described as*

$$t^3 = y^2(\tau_1 - \tau_2)^2 \tag{4.11}$$

in the coordinates (τ_1, τ_2, y).

Remark 4.3. The Lemma 4.2 can be understood as consequence of (1.14).

Corollary 4.4 ([29])**.** *Let the power series w_1, w_2, w_3 be defined in (4.6). Then the map*

$$F\colon\ \mathscr{U} \mapsto \mathbb{H} \times \mathbb{C}^* \times \mathbb{C}, \qquad (\tau, r, s) \mapsto (\tau_1 + \tau_2, y, t^3) \tag{4.12}$$

given by the power series

$$\frac{1}{2}(\tau_1 + \tau_2)(\tau, r, s) = \frac{-w_2}{2w_3}(\tau, r, t^3) = \tau + \sum_{n=1}^{\infty} \tau_{12,n}(\tau)\left(t^3 r^{-2}\right)^n,$$
$$y(\tau, r, s) = w_3(\tau, r, t^3) = r\left(1 + \sum_{n=1}^{\infty} r_n(\tau)\left(t^3 r^{-2}\right)^n\right), \qquad t^3(\tau, r, s) = s\frac{E_6^2(\tau)}{r^6} \tag{4.13}$$

is holomorphic in the domain (4.3). Moreover,

$$\tau_{12,n}(\tau), r_n(\tau) \in E_6^{-2n}\mathbb{C}[E_2, E_4, E_6]. \tag{4.14}$$

At this state, we can sketch the derivation of the inverse period map of quantum cohomology of $\mathbb{CP}^2$ done in [29, Section 5.4]. Indeed, inverting the second power series (4.13) in r, we obtain

$$r = y\left(1 + \sum_{n=1}^{\infty} \hat{r}_n(\tau)\left(t^3 y^{-2}\right)^n\right). \tag{4.15}$$

Substituting (4.15) in the first equation of (4.13),

$$\tau_{12} := \frac{\tau_1 + \tau_2}{2} = \tau + \sum_{n=1}^{\infty} \hat{\tau}_{12,n}(\tau)\left(t^3 y^{-2}\right)^n. \tag{4.16}$$

Inverting (4.15), (4.16) in τ and using equation (4.11), we obtain the following:

$$\tau(\tau_1, \tau_2, y) = \tau_{12} + \sum_{n=1}^{\infty} \tau_n(\tau_{12})(\tau_1 - \tau_2)^{2n},$$
$$r(\tau_1, \tau_2, y) = y\left(1 + \sum_{n=1}^{\infty} y_n(\tau_{12})(\tau_1 - \tau_2)^{2n}\right), \qquad t^3(\tau_1, \tau_2, y) = y^2(\tau_1 - \tau_2)^2. \tag{4.17}$$

Composing (4.17) with (4.4),

$$t^1 = -2(2\pi)^2 \frac{E_4\big(\tau_{12} + \sum_{n=1}^{\infty} \tau_n(\tau_{12})(\tau_1-\tau_2)^{2n}\big)}{y^2\big(1+\sum_{n=1}^{\infty} y_n(\tau_{12})(\tau_1-\tau_2)^{2n}\big)^2},$$
$$Q = \frac{8}{27}(2\pi)^6 \frac{\big(E_4^3 - E_6^2\big)\big(\tau_{12} + \sum_{n=1}^{\infty} \tau_n(\tau_{12})(\tau_1-\tau_2)^{2n}\big)}{y^6\big(1+\sum_{n=1}^{\infty} y_n(\tau_{12})(\tau_1-\tau_2)^{2n}\big)^6}, \qquad t^3 = y^2(\tau_1-\tau_2)^2.$$

Milanov describes the inverse period map in [29, Theorem 2.4], which we state here as follows.

Theorem 4.5 ([29]). *Let $\mathscr{D}$ be the image of the following map:*

$$t \bullet F\colon\ \mathscr{U} \mapsto F(\mathscr{U}) \mapsto \mathscr{D} \subset \mathbb{H}\times\mathbb{C}^*\times\mathbb{C}$$

given by

$$(\tau, r, s) \mapsto \big(\tau_1+\tau_2, y, t^3\big) \mapsto \left(\tau_1+\tau_2, y, \frac{t^3}{y^2} = (\tau_1-\tau_2)^2\right).$$

Then,

(1) *The inverse period map of big quantum cohomology of $\mathbb{CP}^2$ is a holomorphic map $t\colon \mathscr{D} \subset \mathbb{H}^2\times\mathbb{C}^* \mapsto D$ given by*

$$t^1(\tau_1,\tau_2,y) = -2\frac{(2\pi)^2}{y^2}\sum_{n=0}^{\infty} t_n^1(\tau_{12})(\tau_1-\tau_2)^{2n},$$
$$Q(\tau_1,\tau_2,y) = \frac{8}{27}\frac{(2\pi)^6}{y^6}\sum_{n=0}^{\infty} Q_n(\tau_{12})(\tau_1-\tau_2)^{2n}, \qquad t^3 = y^2(\tau_1-\tau_2), \tag{4.18}$$

where $\tau_{12} = \frac{\tau_1+\tau_2}{2}$.

(2) *The coefficients $t_n^1(\tau_{12}), Q_n(\tau_{12})$ are quasi-modular forms, i.e., $t_n^1(\tau_{12}), Q_n(\tau_{12}) \in \mathbb{C}[E_2, E_4, E_6]$.*

(3) *The inverse period map (4.18) is $A_1\times \mathrm{PSL}_2(\mathbb{Z})$-invariant with respect to the action (4.10).*

The first coefficients $t_n^1(\tau_{12})$, $Q_n(\tau_{12})$ are given by

$$t_0^1(\tau_{12}) = E_4(\tau_{12}), \qquad t_1^1(\tau_{12}) = \frac{1}{40}\partial_{\tau_{12}}^2 E_4(\tau_{12}),$$
$$t_2^1(\tau_{12}) = \frac{1}{4480}\partial_{\tau_{12}}^4 E_4(\tau_{12}) - \frac{\pi^4}{2016}Q_0(\tau_{12}),$$

and

$$Q_0(\tau_{12}) = E_4^3(\tau_{12}) - E_6^2(\tau_{12}), \qquad Q_1(\tau_{12}) = \frac{1}{140}\partial_{\tau_{12}}^2 Q_0(\tau_{12}) + \frac{\pi^2}{26}E_4(\tau_{12})Q_0(\tau_{12}),$$
$$Q_2(\tau_{12}) = \frac{1}{24960}\partial_{\tau_{12}}^4 Q_0(\tau_{12}) + \frac{\pi^2}{2704}E_4(\tau_{12})\partial_{\tau_{12}}^2 Q_0(\tau_{12})$$
$$+ \frac{\pi^2}{1040}Q_0(\tau_{12})\partial_{\tau_{12}}^2 E_4(\tau_{12}) + \frac{17\pi^4}{20280}E_4^2(\tau_{12})Q_0(\tau_{12}).$$

Remark 4.6. The multivalued map $w_i\big(t^1, Q, t^3\big)$ (4.1) can be locally invertible away from the discriminant locus. Then, the inverse period map is given by

$$t\colon\ \hat{\mathscr{D}} \mapsto \left\{\big(t^1, Q, t^3\big) \in \mathbb{C}\times\mathbb{C}^*\times\mathbb{C} \mid \big|Q\big(t^3\big)^3\big| < \frac{1}{a},\ \det g \neq 0\right\}.$$

Furthermore, using proof of [29, Proposition 5.3], i.e., using the fact that the inverse period map is bounded at the discriminant and Riemann extension theorem, we can extends analytically the inverse period map over the discriminant. Therefore, the largest image of the inverse period map is the set

$$\left\{\left(t^1, Q, t^3\right) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C} \mid \left|Q\left(t^3\right)^3\right| < \frac{1}{a}\right\}.$$

Remark 4.7. Even though, the flat coordinates with respect to the intersection form of $QH^* \times \left(\mathbb{CP}^2\right)$ is given by (w_1, w_2, w_3), it is convenient to write the inverse period map in terms of (τ_1, τ_2, y), which is related to (w_1, w_2, w_3) via the map (4.9).

Our next goal is to derive a t^3 deformation of the Landau–Ginzburg superpotential for affine small quantum cohomology, which is represented by the Weber function $\gamma_2(\tau)$ defined in (3.23). The t^3 deformation involves composing the Weber function $\gamma_2(\tau)$ with the local inverses of t^3-deformed periods (4.13).

Lemma 4.8. *Let the power series w_1, w_2, w_3 be defined in (4.6). Then, the composition of the functions*

$$\tilde{\tau}_{12} := -\frac{w_2}{2w_3}\left(t^1 - \lambda, Q, t^3\right), \qquad \tilde{y} := w_3\left(t^1 - \lambda, Q, t^3\right), \tag{4.19}$$

with the change of coordinates

$$\begin{aligned} t^1 &= -2\frac{(2\pi)^2}{r^2}E_4(\tau), & Q &= \frac{8}{27}\frac{(2\pi)^6}{r^6}\left(E_4^3(\tau) - E_6^2(\tau)\right), & t^3 &= s\frac{E_6^2(\tau)}{r^6}, \\ \lambda &= t^1 + 2\frac{(2\pi)^2}{\tilde{r}^2}E_4(\tilde{\tau}), & \tilde{r} &= r\frac{\left(E_4^3(\tilde{\tau}) - E_6^2(\tilde{\tau})\right)^{\frac{1}{6}}}{\left(E_4^3(\tau) - E_6^2(\tau)\right)^{\frac{1}{6}}}, & \tilde{s} &= \frac{\tilde{r}^6 t^3}{E_6^2(\tilde{\tau})}, \end{aligned} \tag{4.20}$$

are holomorphic functions

$$\begin{aligned} \tilde{\tau}_{12}&\colon\ \tilde{\mathscr{U}} = \left\{\left(\tilde{\tau}, Q^{\frac{1}{3}}t^3\right) \in \mathbb{H} \times \mathbb{C} \mid \left|Q^{\frac{1}{3}}t^3\right| < \epsilon,\ E_6(\tilde{\tau}) \neq 0\right\} \mapsto \mathbb{H}, \\ \tilde{r}&\colon\ \tilde{\mathscr{U}} = \left\{\left(\tilde{\tau}, Q^{\frac{1}{3}}t^3\right) \in \mathbb{H} \times \mathbb{C} \mid \left|Q^{\frac{1}{3}}t^3\right| < \epsilon,\ E_6(\tilde{\tau}) \neq 0\right\} \mapsto \mathbb{C}, \end{aligned}$$

given by the power series

$$\tilde{\tau}_{12} = \tilde{\tau} + \sum_{n=1}^{\infty} \frac{\tau_{12,n}(\tilde{\tau})}{\Delta^{\frac{n}{3}}(\tilde{\tau})}\left(Q^{\frac{1}{3}}t^3\right)^n, \qquad \tilde{y} = \tilde{r}\left(1 + \sum_{n=1}^{\infty} \frac{r_n(\tilde{\tau})}{\Delta^{\frac{n}{3}}(\tilde{\tau})}\left(Q^{\frac{1}{3}}t^3\right)^n\right), \tag{4.21}$$

where ϵ has the following properties:

$$\epsilon \leq \left(\frac{1}{a}\right)^{\frac{1}{3}}, \qquad \left|Q^{\frac{1}{3}}t^3\right| < \epsilon \implies \det g = 0 \iff E_6(\tilde{\tau}) = 0,$$

and $\tau_{12,n}(\tilde{\tau})$, $r_n(\tilde{\tau})$ are given in (4.12).

Moreover, for fixed $Q^{\frac{1}{3}}t^3$, the function $\tilde{\tau}\colon \mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z})(i) \mapsto \mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z})(i)$ given by the inverse function of (4.21),

$$\tilde{\tau} = \tilde{\tau}_{12} + \sum_{n=1}^{\infty} \frac{\tilde{\tau}_{12,n}(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\left(Q^{\frac{1}{3}}t^3\right)^n \tag{4.22}$$

is a holomorphic function.

Proof. Consider the t^3-deformed affine extended variables

$$\tilde{\tau}_{12} := \frac{w_2}{2w_3}\big(t^1-\lambda, Q, t^3\big), \qquad \tilde{y} := w_3\big(t^1-\lambda, Q, t^3\big). \tag{4.23}$$

Then, the composition of (4.23) with (4.20) is equivalent to replace τ to $\tilde{\tau}$ in the first power series of (4.13). More precisely,

$$\tilde{\tau}_{12} = \tilde{\tau} + \sum_{n=1}^{\infty} \tau_{12,n}(\tilde{\tau}) \left(\frac{t^3}{\tilde{r}^2}\right)^n, \qquad \tilde{y} = \tilde{r}\left(1 + \sum_{n=1}^{\infty} r_n(\tilde{\tau}) \left(\frac{t^3}{\tilde{r}^2}\right)^n\right). \tag{4.24}$$

In particular, the functions (4.24) are holomorphic in the domain

$$\hat{\mathscr{U}} = \{(\tilde{\tau}, \tilde{r}, \tilde{s}) \in \mathbb{H} \times \mathbb{C}^* \times \mathbb{C} \mid |\tilde{s}| < \delta(\tilde{\tau}, \tilde{r})\}.$$

Using the last equation of (4.20) in (4.24), we obtain

$$\tilde{\tau}_{12} = \tilde{\tau} + \sum_{n=1}^{\infty} \frac{\tau_{12,n}(\tilde{\tau})}{\Delta^{\frac{n}{3}}(\tilde{\tau})}\big(Q^{\frac{1}{3}}t^3\big)^n, \qquad \tilde{y} = \tilde{r}\left(1 + \sum_{n=1}^{\infty} \frac{r_n(\tilde{\tau})}{\Delta^{\frac{n}{3}}(\tilde{\tau})}\big(Q^{\frac{1}{3}}t^3\big)^n\right), \tag{4.25}$$

which are holomorphic function in the domains

$$\tilde{\mathscr{U}} = \left\{\big(\tilde{\tau}, Q^{\frac{1}{3}}t^3\big) \in \mathbb{H} \times \mathbb{C} \mid \big|Q^{\frac{1}{3}}t^3\big| < \epsilon,\ E_6(\tilde{\tau}) \neq 0\right\},$$

because the domain $\tilde{\mathscr{U}}$ corresponds to the domain $\hat{\mathscr{U}}$ when written in the coordinates $\big(\tilde{\tau}, Q^{\frac{1}{3}}t^3\big)$. Here ϵ is such that

$$\det\big(g^{\alpha\beta} - \lambda\eta^{\alpha\beta}\big) = \Big(\big(t^1-\lambda\big)^3 + 27Q\Big)\big(1 + O\big(t^3\big)\big) = 0 \iff \big(t^1-\lambda\big)^3 + 27Q = 0,$$

because ϵ is proportional to $\delta(\tilde{\tau}, \tilde{r})$, which have the property above. Furthermore, using (4.20), we have

$$\big(t^1-\lambda\big)^3 + 27Q = -8(2\pi)^6 \frac{E_6^2(\tilde{\tau})}{\tilde{r}^6}. \tag{4.26}$$

In addition, for fixed $Q^{\frac{1}{3}}t^3$, consider the function

$$\begin{aligned} \tilde{\tau}_{12}\colon\ \mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z})(i) &\mapsto \tilde{\tau}_{12}(\mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z})(i)) \subset \mathbb{H}, \\ \tilde{\tau} &\mapsto \tilde{\tau}_{12} = \tilde{\tau} + \sum_{n=1}^{\infty} \frac{\tau_{12,n}(\tilde{\tau})}{\Delta^{\frac{n}{3}}(\tilde{\tau})}\big(Q^{\frac{1}{3}}t^3\big)^n, \end{aligned} \tag{4.27}$$

and the inverse function of (4.27) $\tilde{\tau}\colon \tilde{\tau}_{12}(\mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z})(i)) \mapsto \mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z})(i)$ given by

$$\tilde{\tau} = \tilde{\tau}_{12} + \sum_{n=1}^{\infty} \frac{\tilde{\tau}_{12,n}(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\big(Q^{\frac{1}{3}}t^3\big)^n, \tag{4.28}$$

where the coefficients $\tilde{\tau}_{12,n}(\tilde{\tau}_{12})$ are obtained by the relation

$$\begin{aligned} \tilde{\tau}_{12}(\tilde{\tau}(\tilde{\tau}_{12})) = \tilde{\tau}_{12} &= \sum_{n=0}^{\infty} \frac{\tau_{12,n}(\tilde{\tau}(\tilde{\tau}_{12}))}{\Delta^{\frac{n}{3}}(\tilde{\tau}(\tilde{\tau}_{12}))}\big(Q^{\frac{1}{3}}t^3\big)^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{\partial^k}{\partial\tilde{\tau}_{12}^k}\left(\frac{\tau_{12,n}(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\right) B_{n,k}\left(\frac{\tilde{\tau}_{12,1}}{\Delta^{\frac{1}{3}}}, \frac{\tilde{\tau}_{12,2}}{\Delta^{\frac{2}{3}}}, \ldots, \frac{\tilde{\tau}_{12,n}}{\Delta^{\frac{n}{3}}}\right)\right)\frac{\big(Q^{\frac{1}{3}}t^3\big)^n}{n!}, \end{aligned} \tag{4.29}$$

where $B_{n,k}$ are the Bell polynomials defined in (A.1) and we have used the Faa di Bruno formula (A.2). Since $\tilde{\tau}_{12,n}$ is algebraically dependent with the coefficients $\tau_{12,n}$ due to equation (4.29), we have that $\tilde{\tau}_{12,n} \in \mathbb{C}\big[E_2, E_4, E_6, E_6^{-1}\big]$, because of condition (4.14) in the Corollary 4.4.

Composing (4.28) with the second equation of (4.25), we obtain a power series of the following form:

$$\tilde{y} = \tilde{r}\left(1 + \sum_{n=1}^{\infty} \frac{\tilde{r}_n(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\big(Q^{\frac{1}{3}}t^3\big)^n\right). \tag{4.30}$$

The image of the discriminant $E_6(\tilde{\tau}) = 0$ under the change of coordinates (4.22) is obtained by substituting (4.28) and (4.30) in (4.26),

$$\begin{aligned}\frac{E_6\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big)}{\tilde{y}^3} &:= \frac{E_6\big(\tilde{\tau}_{12} + \sum_{n=1}^{\infty} \frac{\tilde{\tau}_{12,n}(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\big(Q^{\frac{1}{3}}t^3\big)^n\big)}{\tilde{y}^3\big(1 + \sum_{n=1}^{\infty} \frac{\tilde{r}_n(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\big(Q^{\frac{1}{3}}t^3\big)^n\big)^{-3}}, \\ &= \frac{1}{\tilde{y}^3}\left[E_6(\tilde{\tau}_{12}) + \sum_{n=1}^{\infty} E_{6,n}(\tilde{\tau}_{12})\big(Q^{\frac{1}{3}}t^3\big)^n\right].\end{aligned} \tag{4.31}$$

Then, $\tilde{\tau}_{12}(\mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z})(i)) = \mathbb{H} \setminus \big\{E_6\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big) = 0\big\}$. Moreover, $\tilde{\tau}_{12}^0$ is zero of (4.31) if the function

$$\frac{1}{\tilde{y}^3}\left[E_6\big(\tilde{\tau}_{12}^0\big) + \sum_{n=1}^{\infty} E_{6,n}\big(\tilde{\tau}_{12}^0\big)\big(Q^{\frac{1}{3}}t^3\big)^n\right]$$

is the zero function in the variable $Q^{\frac{1}{3}}t^3$, i.e., $E_{6,n}(\tilde{\tau}_{12}) = 0$, $\forall n \geq 0$. In particular, the zero order coefficient of the Taylor expansion (4.31) is zero.

Therefore, $\tilde{\tau}_{12}\big(\tilde{\mathscr{U}}\big) = \mathbb{H} \setminus \{\mathrm{SL}_2(\mathbb{Z})(i)\}$. Lemma proved. ■

Here, we make the following deformation of the modular discriminant. Composing the t^3 deformations (4.22) and (4.30) with the last equation of (4.20), we obtain

$$Q = \frac{\Delta\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big)}{\tilde{y}^6},$$

where

$$\begin{aligned}\Delta\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big) &= \frac{\Delta\big(\tilde{\tau} = \tilde{\tau}_{12} + \sum_{n=1}^{\infty} \frac{\tilde{\tau}_{12,n}(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\big(Q^{\frac{1}{3}}t^3\big)^n\big)}{\big(1 + \sum_{n=1}^{\infty} \frac{\tilde{r}_n(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\big(Q^{\frac{1}{3}}t^3\big)^n\big)^6}, \\ &= \Delta(\tilde{\tau}_{12}) + \sum_{n=1}^{\infty} \Delta_n(\tilde{\tau}_{12})\big(Q^{\frac{1}{3}}t^3\big)^n.\end{aligned} \tag{4.32}$$

Lemma 4.9. *Let the power series w_1, w_2, w_3 be defined in (4.6). Then, the function*

$$\tilde{\tau}_1\tilde{\tau}_2 := \frac{\tilde{w}_1\big(t^1 - \lambda, Q, t^3\big)}{\tilde{w}_3\big(t^1 - \lambda, Q, t^3\big)}$$

satisfies the following relation:

$$t^3 = \tilde{w}_2^2 - 4\tilde{w}_1\tilde{w}_3. \tag{4.33}$$

Moreover, consider $(\tilde{\tau}_1 - \tilde{\tau}_2)^2 := \frac{t^3}{\tilde{w}_3^2}$. Hence, the following relation holds true:

$$\tilde{\tau}_1\tilde{\tau}_2 = \tilde{\tau}_{12}^2 - \frac{1}{4}\frac{Q^{\frac{1}{3}}t^3}{\Delta^{\frac{1}{3}}\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big)}.$$

Proof. The relation (4.33) holds true due to Lemma 4.2. Furthermore,

$$\tilde{\tau}_1\tilde{\tau}_2=\tilde{\tau}_{12}^2-\frac{(\tilde{\tau}_1-\tilde{\tau}_2)^2}{4}=\tilde{\tau}_{12}^2-\frac{\left(t^3\right)^2}{4\tilde{w}_3^2}=\tilde{\tau}_{12}^2-\frac{1}{4}\frac{Q^{\frac{1}{3}}t^3}{\Delta^{\frac{1}{3}}\left(\tilde{\tau}_{12},Q^{\frac{1}{3}}t^3\right)}. \qquad \blacksquare$$

Since, the extended Gauss–Manin connection (1.10) has the same monodromy group of the Gauss–Manin connection (1.8), the periods $\tilde{\tau}_1$, $\tilde{\tau}_2$, $\tilde{y}$ have the following transformation law due to (4.10):

$$\begin{aligned}&A(\tilde{\tau}_1,\tilde{\tau}_2,\tilde{y})=\left(\frac{a\tilde{\tau}_1+b}{c\tilde{\tau}_1+d},\frac{a\tilde{\tau}_2+b}{c\tilde{\tau}_2+d},\tilde{y}(c\tilde{\tau}_1+d)(c\tilde{\tau}_1+d)\right),\\&B(\tilde{\tau}_1,\tilde{\tau}_2,\tilde{y})=(\tilde{\tau}_1,\tilde{\tau}_2,-\tilde{y}),\end{aligned}\tag{4.34}$$

where $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$. Using the change of coordinates (4.9), we have that $\tilde{\tau}_{12}$ has the following transformation law under (4.34):

$$\begin{aligned}&\text{If}\quad(\tilde{\tau}_1,\tilde{\tau}_2)\mapsto(\tilde{\tau}_1+1,\tilde{\tau}_2+1),\implies\tilde{\tau}_{12}\mapsto\tilde{\tau}_{12}+1,\\&\text{If}\quad(\tilde{\tau}_1,\tilde{\tau}_2)\mapsto\left(\frac{-1}{\tilde{\tau}_1},\frac{-1}{\tilde{\tau}_2}\right),\implies\tilde{\tau}_{12}\mapsto\frac{-\tilde{\tau}_{12}}{\tilde{\tau}_1\tilde{\tau}_2}.\end{aligned}\tag{4.35}$$

Alternatively, due to Lemma 4.9 the group action (4.35) can be written as

$$\begin{aligned}&\text{If}\quad(\tilde{\tau}_1,\tilde{\tau}_2)\mapsto(\tilde{\tau}_1+1,\tilde{\tau}_2+1),\implies\tilde{\tau}_{12}\mapsto\tilde{\tau}_{12}+1,\\&\text{If}\quad(\tilde{\tau}_1,\tilde{\tau}_2)\mapsto\left(\frac{-1}{\tilde{\tau}_1},\frac{-1}{\tilde{\tau}_2}\right),\implies\tilde{\tau}_{12}\mapsto\frac{-\tilde{\tau}_{12}}{\left(\tilde{\tau}_{12}^2-\frac{1}{4}\frac{Q^{\frac{1}{3}}t^3}{\Delta^{\frac{1}{3}}(\tilde{\tau}_{12},Q^{\frac{1}{3}}t^3)}\right)}.\end{aligned}\tag{4.36}$$

Note that for t^3 small enough the group action (4.36) gives rise to an automorphism of $\mathbb{H}$. On another hand, the $A_1\times\mathrm{PSL}_2(\mathbb{Z})$ action in the periods $\tilde{\tau}$, $\tilde{r}$, t^3 is the following:

$$A\left(\tilde{\tau},\tilde{r},t^3\right)=\left(\frac{a\tilde{\tau}+b}{c\tilde{\tau}+d},\tilde{r}(c\tilde{\tau}+d)^2,t^3\right),\qquad B\left(\tilde{\tau},\tilde{r},t^3\right)=\left(\tilde{\tau},-\tilde{r},t^3\right),$$

where $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$. Therefore, the map (4.22) has the following transformation law:

$$\begin{aligned}&\tilde{\tau}\left(\tilde{\tau}_{12}+1,Q^{\frac{1}{3}}t^3\right)=\tilde{\tau}\left(\tilde{\tau}_{12},Q^{\frac{1}{3}}t^3\right)+1,\\&\tilde{\tau}\left(\frac{-\tilde{\tau}_{12}}{\left(\tilde{\tau}_{12}^2-\frac{1}{4}\frac{Q^{\frac{1}{3}}t^3}{\Delta^{\frac{1}{3}}(\tilde{\tau}_{12},Q^{\frac{1}{3}}t^3)}\right)},Q^{\frac{1}{3}}t^3\right)=\frac{-1}{\tilde{\tau}\left(\tilde{\tau}_{12},Q^{\frac{1}{3}}t^3\right)}.\end{aligned}$$

At this stage, we can prove the main theorem of this section.

Theorem 4.10.

(1) *Let the function* $\tilde{\tau}_{12}\colon\mathbb{H}\setminus\mathrm{SL}_2(\mathbb{Z})(\mathrm{i})\mapsto\mathbb{H}\setminus\mathrm{SL}_2(\mathbb{Z})(\mathrm{i})$ *given by*

$$\tilde{\tau}\left(\tilde{\tau}_{12},Q^{\frac{1}{3}}t^3\right)=\tilde{\tau}_{12}+\sum_{n=1}^{\infty}\frac{\tilde{\tau}_n(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\left(Q^{\frac{1}{3}}t^3\right)^n$$

be defined in (4.22)*. Then, the Landau–Ginzburg superpotential of big quantum cohomology of* $\mathbb{CP}^2$ *is a family of functions* $\lambda\left(\tilde{\tau}_{12},t^1,Q,t^3\right)\colon\mathbb{H}\mapsto\mathbb{C}$ *with holomorphic dependence in the parameter space*

$$\left\{\left(t^1,Q^{\frac{1}{3}},Q^{\frac{1}{3}}t^3\right)\in\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}\mid\left|Q^{\frac{1}{3}}t^3\right|<\left(\frac{1}{a}\right)^{\frac{1}{3}}\right\}$$

and given by

$$\lambda\big(\tilde\tau_{12}, t^1, Q, t^3\big) = t^1 + 3Q^{\frac13} J^{\frac13}\big(\tilde\tau_{12}, Q^{\frac13} t^3\big), \tag{4.37}$$

where

$$\begin{aligned} J^{\frac13}\big(\tilde\tau_{12}, Q^{\frac13} t^3\big) &:= J^{\frac13}\left(\tilde\tau_{12} + \sum_{n=1}^{\infty} \frac{\tilde\tau_n(\tilde\tau_{12})}{\Delta^{\frac{n}{3}}(\tilde\tau_{12})}\big(Q^{\frac13}t^3\big)^n\right), \\ &= J^{\frac13}(\tilde\tau_{12}) + \sum_{n=1}^{\infty} J_n^{\frac13}(\tilde\tau_{12})\big(Q^{\frac13}t^3\big)^n, \end{aligned} \tag{4.38}$$

and

$$J_n^{\frac13}(\tilde\tau_{12}) := \frac{1}{n!}\sum_{k=0}^{n} k!\frac{\partial^k J^{\frac13}}{\partial\tilde\tau_{12}}(\tilde\tau_{12}) B_{n,k}\left(1!\frac{\tilde\tau_1(\tilde\tau_{12})}{\Delta^{\frac13}(\tilde\tau_{12})}, 2!\frac{\tilde\tau_2(\tilde\tau_{12})}{\Delta^{\frac23}(\tilde\tau_{12})}, \dots, n!\frac{\tilde\tau_n(\tilde\tau_{12})}{\Delta^{\frac{n}{3}}(\tilde\tau_{12})}\right).$$

The functions $B_{n,k}(x_1,\dots,x_n)$ are the Partial Bell polynomials defined in (A.1). The coefficients $J_n^{\frac13}(\tilde\tau_{12})$ belong to the following ring $J_n^{\frac13}(\tilde\tau_{12}) \in \Delta^{\frac{-n}{3}}\mathbb{C}[E_2, E_4, E_6]$. In addition, the correspondent Abelian differential ϕ is given by

$$\phi = -\frac{2^{\frac52}}{2\pi}\frac{\Delta^{\frac16}\big(\tilde\tau_{12}, Q^{\frac13}t^3\big)}{Q^{\frac16}}\mathrm{d}\tilde\tau_{12}, \tag{4.39}$$

where $\Delta\big(\tilde\tau_{12}, Q^{\frac13}t^3\big)$ is defined in (4.32).

(2) *Denote by $\Gamma^{(3)}_{Q^{1/3}t^3}$ the image of the group $\Gamma^{(3)}$ under the group homomorphism (4.36). Then the Landau–Ginzburg superpotential of big quantum cohomology of $\mathbb{CP}^2$ (4.37) is $\Gamma^{(3)}_{Q^{1/3}t^3}$-invariant.*

Proof. We obtain the Landau–Ginzburg superpotential of $QH^*\big(\mathbb{CP}^2\big)$ for $t^3 \neq 0$ by applying the Dubrovin construction of Landau–Ginzburg superpotential, see Theorem 2.5. More specifically, in order to obtain the LG superpotential, we invert (4.19) in λ, which is done in practice by composing (4.22) with (3.51).

Using Faa di Bruno formula (A.2)

$$\begin{aligned} \lambda\big(\tilde\tau_{12}, t^1, Q, t^3\big) &= t^1 + 3Q^{\frac13}J^{\frac13}\left(\tilde\tau_{12} + \sum_{n=1}^{\infty}\frac{\tilde\tau_n(\tilde\tau_{12})}{\Delta^{\frac{n}{3}}(\tilde\tau_{12})}\big(Q^{\frac13}t^3\big)^n\right) \\ &= t^1 + 3Q^{\frac13}J^{\frac13}\left(\tilde\tau_{12} + \sum_{n=1}^{\infty} n!\frac{\tilde\tau_n(\tilde\tau_{12})}{\Delta^{\frac{n}{3}}(\tilde\tau_{12})}\frac{\big(Q^{\frac13}t^3\big)^n}{n!}\right) \\ &= t^1 + 3Q^{\frac13}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k!\frac{\partial^k J^{\frac13}}{\partial\tilde\tau_{12}} B_{n,k}\left(1!\frac{\tilde\tau_1}{\Delta^{\frac13}}, 2!\frac{\tilde\tau_2}{\Delta^{\frac23}}, \dots, n!\frac{\tilde\tau_n}{\Delta^{\frac{n}{3}}}\right)\right)\frac{\big(Q^{\frac13}t^3\big)^n}{n!} \\ &= t^1 + 3Q^{\frac13}\sum_{n=0}^{\infty} J_n^{\frac13}(\tilde\tau_{12})\big(Q^{\frac13}t^3\big)^n \\ &= t^1 + 3Q^{\frac13}J^{\frac13}\big(\tilde\tau_{12}, Q^{\frac13}t^3\big). \end{aligned} \tag{4.40}$$

The coefficients $J_n^{\frac13}(\tilde\tau_{12})$ are polynomial in $\frac{\tau_n(\tilde\tau_{12})}{\Delta^{n/3}(\tilde\tau_{12})}$, which belongs to $\Delta^{\frac{-n}{3}}\mathbb{C}\big[E_2, E_4, E_6, E_6^{-1}\big]$ due to Corollary 4.4 and Lemma 4.8. But the Taylor expansion of any Landau–Ginzburg superpotential near the discriminant is of the form

$$\lambda = u_i + \lambda''(\tilde\tau_{12}^0)\frac{\big(\tilde\tau_{12} - \tilde\tau_{12}^0\big)^2}{2} + O\big(\tilde\tau_{12} - \tilde\tau_{12}^0\big)^3,$$

which is holomorphic. Hence, we can extend the domain of Landau–Ginzburg superpotential from $\mathbb{H} \setminus \{E_6((\tilde{\tau}_{12})=0\}$ to $\mathbb{H}$, which also implies that the coefficients $J_n^{\frac{1}{3}}(\tilde{\tau}_{12})$ are not rational in E_6. Moreover, the Landau–Ginzburg superpotential is holomorphic with respect its parameters $\big(t^1, Q^{\frac{1}{3}}, Q^{\frac{1}{3}}t^3\big)$ in the space

$$\left\{\big(t^1, Q^{\frac{1}{3}}, Q^{\frac{1}{3}}t^3\big) \in \mathbb{C}\times\mathbb{C}^*\times\mathbb{C} \mid \big|Q^{\frac{1}{3}}t^3\big| < \left(\frac{1}{a}\right)^{\frac{1}{3}}\right\}$$

because, we can extend the parameter space over the discriminant locus as it was discussed in Remark 4.6.

Here, we conclude that the pair

$$\big(\lambda = t^1 + 3Q^{\frac{1}{3}}J^{\frac{1}{3}}\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big), \phi = \mathrm{d}\tilde{w}_2\big)$$

serve as a Landau–Ginzburg superpotential for big quantum cohomology of $\mathbb{CP}^2$, where $\tilde{w}_2$ is the pullback of $\tilde{w}_2 = w_2\big(t^1-\lambda, Q, t^3\big)$, via the map (4.37).

The domain of (4.40) is a family of $\mathbb{H}$ parametrized by

$$\big(\tau_{12}, y, Q^{\frac{1}{3}}t^3\big) \in (\mathbb{H}\setminus\{E_6(\tau_{12})=0\})\times\mathbb{C}^*\times D\left(0, \left(\frac{1}{a}\right)^{\frac{1}{3}}\right)$$

because of the parametrized LG superpotential (4.40) and the change of coordinates (4.18). Furthermore, this family of $\mathbb{H}$ is biholomorphic to the following family of manifolds:

$$\begin{aligned}\mathbb{H}_{(\tau_{12},y,Q^{\frac{1}{3}}t^3)} = \Bigg\{&\big(\tilde{\tau}_{12}, \tilde{y}, \tau_{12}, y, Q^{\frac{1}{3}}t^3\big) \in (\mathbb{H}\times\mathbb{C}^*)^2\times D\left(0, \left(\frac{1}{a}\right)^{\frac{1}{3}}\right): \\ &\tilde{y} = \frac{2^{\frac{3}{2}}}{(2\pi)^2}\left(\frac{\Delta\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big)}{Q}\right)^{\frac{1}{6}}, E_6(\tau_{12}) \neq 0\Bigg\}.\end{aligned} \tag{4.41}$$

Let the space $\Omega_{w_2,y}$ be defined by

$$\Omega_{w_2,y} = \left\{(w_2, y)\in\mathbb{C}^*\times\mathbb{C}^* \mid \operatorname{Im}\left(\frac{-w_2}{2y}\right) > 0\right\}.$$

Consider the following isomorphism $S\colon \mathbb{H}\times\mathbb{C}^*\times\mathbb{H}\times\mathbb{C}^* \mapsto \Omega_{\tilde{w}_2,\tilde{y}}\times\Omega_{w_2,y}$

$$S(\tilde{\tau}_{12}, \tilde{y}, \tau_{12}, y) = (\tilde{w}_2, \tilde{y}, w_2, y), \qquad \text{where} \quad \tilde{w}_2 = -2\tilde{y}\tilde{\tau}_{12}, \qquad w_2 = -2y\tau_{12}.$$

Hence, the section $\mathrm{d}\tilde{w}_2 \in \Gamma\big(T^*(\Omega_{\tilde{w}_2,\tilde{y}}\times\Omega_{w_2,y})\times D\big(0, \big(\frac{1}{a}\big)^{\frac{1}{3}}\big)\big)$ projected to the submanifold $\mathbb{H}_{(\tau_{12},y,Q^{\frac{1}{3}}t^3)}$ defined in (4.41) is the section $\mathrm{d}\tilde{w}_2 \in \Gamma\big(T^*\big(\mathbb{H}_{(\tau_{12},y,Q^{\frac{1}{3}}t^3)}\big)\big)$ and is given by

$$\mathrm{d}\tilde{w}_2 = -2\tilde{y}\mathrm{d}\tilde{\tau}_{12} = -\frac{2^{\frac{5}{2}}}{2\pi}\frac{\Delta^{\frac{1}{6}}\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big)}{Q^{\frac{1}{6}}}\mathrm{d}\tilde{\tau}_{12}.$$

Furthermore, by definition the function $J^{\frac{1}{3}}\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big)$ has the following form:

$$J^{\frac{1}{3}}\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big) = J^{\frac{1}{3}}\left(\tilde{\tau} = \tilde{\tau}_{12} + \sum_{n=1}^{\infty}\frac{\tilde{\tau}_n(\tilde{\tau}_{12})}{\Delta^{\frac{n}{3}}(\tilde{\tau}_{12})}\big(Q^{\frac{1}{3}}t^3\big)^n\right) = J^{\frac{1}{3}}(\tilde{\tau}(\tilde{\tau}_{12})). \tag{4.42}$$

Here, it is convenient to suppress the dependence of $Q^{\frac{1}{3}}t^3$ in $\tilde{\tau}(\tilde{\tau}_{12})$ to avoid a heavy notation. Then, by the construction of the group homomorphism (4.36), the action of $\gamma\in\Gamma^{(3)}_{Q^{\frac{1}{3}}t^3}$ in the domain of (4.42) is given by

$$J^{\frac{1}{3}}\big(\gamma\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big) = J^{\frac{1}{3}}(\tilde{\tau}(\gamma\tilde{\tau}_{12})),$$

$$= J^{\frac{1}{3}}\left(\frac{a\tilde{\tau}(\tilde{\tau}_{12})+b}{c\tilde{\tau}(\tilde{\tau}_{12})+d}\right), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^{(3)},$$
$$= J^{\frac{1}{3}}(\tilde{\tau}(\tilde{\tau}_{12})). \tag{4.43}$$

In the last line of (4.43), we have used that the function $J^{\frac{1}{3}}(\tilde{\tau})$ is $\Gamma^{(3)}$-invariant and the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^{(3)}$ in the second line is the inverse of $\gamma \in \Gamma^{(3)}_{Q^{\frac{1}{3}}t^3}$ under the group homomorphism (4.36). Theorem proved. ■

Remark 4.11. Due to Remark 3.8, the zeros of the Landau–Ginzburg superpotential (4.37) is achieved when $\tilde{\tau}_{12} = \tau_{12}$. Therefore, the coefficients $J_n^{\frac{1}{3}}(\tilde{\tau}_{12})$ can be also obtained by the relation

$$t^1 = 3Q^{\frac{1}{3}}J^{\frac{1}{3}}\big(\tau_{12}, Q^{\frac{1}{3}}t^3\big). \tag{4.44}$$

More specifically, given the coefficients t_n^1 and Q_n in (4.18), we can derive $J_n^{\frac{1}{3}}(\tilde{\tau}_{12})$ by using the Taylor expansions of t^1 and Q in $(\tau_1 - \tau_2)^2$ and substituting in (4.44).

4.2 From canonical to flat coordinates

The objective of this subsection is to provide a geometric interpretation of the t^3 deformation by considering the analytic continuation of solutions from the flat coordinate system of the Dubrovin connection. To achieve this, we begin with a brief overview of the theory of n-dimensional semisimple Dubrovin–Frobenius manifolds, we refer to [18, 25] for this part.

Let us consider flat sections of the Dubrovin connection constrained by the first condition of (1.1). We can then select the holomorphic part of (2.1) as the fundamental matrix solution

$$H(z,t) = \big(\eta^{\beta\gamma}\partial_\gamma\tilde{t}_\alpha\big) = \sum_{p=0}^{\infty} H_p(t)z^p,$$

if we prefer to work in Saito flat coordinates $\big(t^1, t^2, \dots, t^n\big)$, or alternatively, we can choose the holomorphic part of (2.2)

$$\Psi(z,t) = (Y_{i\alpha}) = \sum_{p=0}^{\infty} \Psi_p(u)z^p,$$

if we prefer the canonical coordinates $(u_1, u_2, \dots, u_n)$. The coefficients of the matrices are related by the following:

$$\Psi_{i\alpha} = \psi_{i\beta}\eta^{\beta\gamma}\partial_\gamma\tilde{t}_\alpha. \tag{4.45}$$

The flat coordinate system of the first condition of (1.1) deforms the Saito flat coordinates $\big(t^1, t^2, \dots, t^n\big)$ as follows:

$$\tilde{t}_\alpha(z,t) = t_\alpha + \sum_{p=1}^{\infty} H_{\alpha,p}(t)z^p, \qquad t_\alpha = \eta_{\alpha\beta}t^\beta. \tag{4.46}$$

Here, $H_{\alpha,p}$ is determined by the following recursion:

$$H_{\alpha,0} = t_\alpha = \eta_{\alpha\beta}t^\beta, \qquad \partial_\gamma\partial_\beta H_{\alpha,p+1} = c_{\gamma\beta}^{\epsilon}\partial_\epsilon H_{\alpha,p}, \qquad p = 0, 1, 2, \dots.$$

In this setting, based on [18, Exercise 2.7], the following identity holds true:

$$t_\alpha = \langle\nabla H_{\alpha,0}, \nabla H_{1,1}\rangle = \eta^{\mu\lambda}\partial_\mu H_{\alpha,0}\partial_\lambda H_{1,1}. \tag{4.47}$$

Thus, substituting the equation (4.46) into (4.45) and the change of basis

$$\frac{\partial}{\partial t^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}\frac{\partial}{\partial u_i}, \tag{4.48}$$

we obtain

$$\Psi_{i\alpha,p} = \frac{\partial_i H_{\alpha,p}}{\psi_{i1}}, \qquad \text{where} \quad \Psi_p = (\Psi_{i\alpha,p})\,. \tag{4.49}$$

Therefore, by substituting (4.49) and (4.48) into (4.47), we get

$$t_\alpha(u) = \sum_{i=1}^{n} \Psi_{i\alpha,0}(u)\Psi_{i1,1}(u), \qquad \alpha = 1, 2, \ldots, n. \tag{4.50}$$

The functions (4.50) are multivalued functions within the domain

$$\{(u_1, u_2, \ldots, u_n) \in \mathbb{C}^n \mid u_i \neq u_j, \text{ if } i \neq j\} = \mathbb{C}^n \setminus \{u_i = u_j\},$$

since they are derived from solutions of the Dubrovin flat coordinate system within the semisimple locus. To express (4.50) more clearly, consider the compatibility condition of the Dubrovin connection flat coordinate system in canonical coordinates (1.18), (1.19), given by $\frac{\partial^2 Y}{\partial z \partial u_i} = \frac{\partial^2 Y}{\partial u_i \partial z}$, which specifically appears as

$$\frac{\partial V}{\partial u_i} = [V_i, V], \qquad [U, V_i] = [E_i, V]. \tag{4.51}$$

Furthermore, due to the following fact

$$[U, V_k] = [E_k, V] \implies (V_k)_{ij} = \frac{\delta_{ki} - \delta_{kj}}{u_i - u_j} V_{ij},$$

the system (4.51) can be expressed as

$$\frac{\partial V}{\partial u_i} = [V_i, V], \qquad \frac{\partial \Psi}{\partial u_i} = V_i \Psi. \tag{4.52}$$

Remark 4.12. The first equation (4.52) can be presented as a time-dependent Hamiltonian system $\frac{\partial V}{\partial u_i} = \{V, H_i\}$, $i = 1, 2, \ldots, n$, with a quadratic Hamiltonian

$$H_i(V; u) = \frac{1}{2} \sum_{i \neq j} \frac{V_{ij}^2}{u_i - u_j}, \qquad i = 1, 2, \ldots, n, \tag{4.53}$$

employing the Poisson bracket

$$\{V_{ij}, V_{kl}\} = V_{il}\delta_{jk} - V_{jl}\delta_{ik} + V_{jk}\delta_{il} - V_{ik}\delta_{jl}. \tag{4.54}$$

The Hamiltonians $H_i(V; u)$ (4.53) pairwise commute with respect to the Poisson bracket (4.54), i.e., $\{H_i, H_j\} = 0$ for any i, j. Hence, the one-form $\sum_{i=1}^{n} H_i(V; u)\mathrm{d}u_i$ forms a closed form for any solution of (4.52). This implies the local existence of a function $\tau_I(u)$ such that

$$\frac{\partial \ln \tau_I(u)}{\partial u_i} = H_i(V; u), \qquad i = 1, 2, \ldots, n. \tag{4.55}$$

The function τ_I defined in (4.55) is referred to as the isomonodromic τ function of the system (4.52). In the context of semisimple quantum cohomology, τ_I satisfies a significant identity. Specifically, if F^1 represents the generating function of genus 1 Gromov–Witten invariants, then $F^1 = \log \frac{\tau_I}{J^{\frac{1}{24}}}$, where $J = \det\left(\frac{\partial t^\alpha}{\partial u_i}\right) = \psi_{11} \cdots \psi_{n1}$.

Here, our focus centers on the particular case of $3D$ Dubrovin–Frobenius manifolds. Therefore, we will use the results obtained by Guzzetti in [25, Section 6.1]. Consider the metric η and matrix $V(u_1,u_2,u_3)$,

$$\eta=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\qquad V(u)=\begin{pmatrix}0&-\Omega_3&\Omega_2\\\Omega_3&0&-\Omega_1\\-\Omega_2&\Omega_1&0\end{pmatrix}.$$

Recall that the columns of matrix Ψ are the eigenvectors of matrix V. Moreover, it satisfies $\Psi^T\Psi=\eta$. Therefore, we can write Ψ in the following form:

$$\Psi(u)=\begin{pmatrix}\frac{E_{11}}{f}&E_{12}&E_{13}f\\\frac{E_{21}}{f}&E_{22}&E_{23}f\\\frac{E_{31}}{f}&E_{32}&E_{33}f\end{pmatrix},$$

where

$$\begin{aligned}&E_{11}=\frac{\Omega_1\Omega_2-\mu\Omega_3}{2\mu^2},\qquad E_{12}=\frac{\Omega_1}{i\mu},\qquad E_{13}=-\frac{\Omega_1\Omega_2+\mu\Omega_3}{\Omega_1^2+\Omega_3^2},\\&E_{21}=-\frac{\Omega_1^2+\Omega_3^2}{2\mu^2},\qquad E_{22}=\frac{\Omega_2}{i\mu},\qquad E_{23}=1,\\&E_{31}=\frac{\Omega_2\Omega_3+\mu\Omega_1}{2\mu^2},\qquad E_{32}=\frac{\Omega_3}{i\mu},\qquad E_{33}=-\frac{\Omega_2\Omega_3-\mu\Omega_3}{\Omega_1^2+\Omega_3^2},\end{aligned}\tag{4.56}$$

and $f(u)$ is determined by $\frac{\partial\Psi}{\partial u_i}=V_i\Psi$. Here, it is convenient to make the following global change of coordinates $F\colon\mathbb{C}^3\setminus\{u_i=u_j\}\mapsto\mathbb{C}\times\mathbb{C}^*\times(\mathbb{C}\setminus\{0,1\})$, $F(u_1,u_2,u_3)=(v,H,x)$, where

$$v=u_1+u_2+u_3,\qquad H=u_2-u_1,\qquad x=\frac{u_3-u_1}{u_2-u_1}.\tag{4.57}$$

From the standard theory of Dubrovin–Frobenius manifolds [16, 18, 25], the action of the unit vector field e and Euler vector field E in the matrix V are given by

$$\sum_{i=1}^{3}\partial_iV=0,\qquad\sum_{i=1}^{3}u_i\partial_iV=0.\tag{4.58}$$

Hence, considering the change of coordinates (4.57) and the symmetries of the matrix V in (4.58), we have the following $V(u_1,u_2,u_3)=V\big(\frac{u_3-u_1}{u_2-u_1}\big)=V(x)$. Consequently, the system (4.51) reduces to

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}x}=\frac{\Omega_2\Omega_3}{x},\qquad\frac{\mathrm{d}\Omega_2}{\mathrm{d}x}=\frac{\Omega_1\Omega_3}{1-x},\qquad\frac{\mathrm{d}\Omega_3}{\mathrm{d}x}=\frac{\Omega_1\Omega_2}{x(x-1)}.\tag{4.59}$$

In the specific case of $QH^*(\mathbb{CP}^2)$, Guzzetti derived the Saito flat coordinates $\big(t^1,t^2,t^3\big)$ as functions of (v,H,x) in [25, Section 6.2.2], specifically in his equations (6.7), (6.8), and (6.9)

$$\begin{aligned}&t^1(u)=u_1+a(x)H,\qquad t^2(u)=3\ln(H)+3\int^x\frac{\mathrm{d}\zeta}{\zeta+\frac{E_{21}E_{22}}{E_{31}E_{32}}},\\&t^3(u)=-9\frac{c(x)}{b(x)^2H},\end{aligned}\tag{4.60}$$

where

$$a(x)=E_{21}E_{23}+xE_{31}E_{33},\qquad b(x)=E_{22}E_{21}+xE_{32}E_{31},\qquad c(x)=E_{21}^2+xE_{31}^2.\tag{4.61}$$

In [16, Appendix E], Dubrovin proved that any semisimple 3D Dubrovin–Frobenius manifold is associated with a special solution of P_{VI}. More concretely, consider the twisted Gauss–Manin connection $(U-\lambda I)\partial_\beta\chi + C_\beta\mu\chi = 0$, $(U-\lambda I)\partial_\lambda\chi - \mu\chi = 0$. In canonical coordinates, the corresponding 3×3 system of differential equations is given by

$$\frac{\mathrm{d}X}{\mathrm{d}\lambda} = -\mu\left[\frac{A_1}{\lambda-u_1} + \frac{A_2}{\lambda-u_2} + \frac{A_3}{\lambda-u_3}\right]X, \qquad \frac{\partial X}{\partial u_i} = \mu\frac{A_i}{\lambda-u_i}X, \tag{4.62}$$

where

$$A_i = \begin{pmatrix} \psi_{i1}\psi_{i3} & 0 & -\psi_{i3}^2 \\ \psi_{i1}\psi_{i2} & 0 & -\psi_{i2}\psi_{i3} \\ \psi_{i1}^2 & 0 & -\psi_{i1}\psi_{i3} \end{pmatrix}, \qquad \mu = \mu\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$
$$\frac{\partial u_i}{\partial t^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}.$$

The twisted Gauss–Manin connection (4.62) can be split into a 2×2 system of differential equations due to the fact that the matrix $\mu = \mathrm{diag}(\mu, 0, -\mu)$ has a 0 eigenvalue. Indeed,

$$\frac{\mathrm{d}X}{\mathrm{d}\lambda} = -\mu\left[\frac{A_1}{\lambda-u_1} + \frac{A_2}{\lambda-u_2} + \frac{A_3}{\lambda-u_3}\right]X, \qquad \frac{\partial X}{\partial u_i} = \mu\frac{A_i}{\lambda-u_i}X, \tag{4.63}$$

where

$$A_i = \begin{pmatrix} \psi_{i1}\psi_{i3} & -\psi_{i3}^3 \\ \psi_{i1}^2 & -\psi_{i1}\psi_{i3} \end{pmatrix}, \qquad \mu = \mu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}, \qquad \frac{\partial u_i}{\partial t^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}.$$

Using the new coordinates q, p given by the relations

$$0 = \left[\sum_{i=1}^{3}\frac{A_i}{q-u_i}\right]_{12}, \qquad p = \left[\sum_{i=1}^{3}\frac{A_i}{q-u_i}\right]_{11},$$

the compatibility conditions of the twisted Gauss–Manin connection (4.63) are given by

$$\frac{\partial q}{\partial u_i} = \frac{P(q)}{P'(u_i)}\left[2p + \frac{1}{q-u_i}\right], \qquad \frac{\partial p}{\partial u_i} = \frac{P'(q)p^2 + \left(2q + u_i - \sum_{i=1}^{3}u_i\right)p + \mu(1-\mu)}{P'(u_i)},$$

where $P(\lambda) = (\lambda-u_1)(\lambda-u_2)(\lambda-u_3)$.

Eliminating p from the system, we obtain a second-order differential equation for the function $q = q(u_1,u_2,u_3)$. Using a change of coordinates $x = \frac{u_3-u_1}{u_2-u_1}$, $y = \frac{q-u_1}{u_2-u_1}$, we have that $y = y(x)$ solves a particular one-parameter family of Painlevé VI equation, denoted by P_{VI_μ},

$$y'' = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right](y')^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]y'$$
$$+ \frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2 + \frac{x(x-1)}{(y-x)^2}\right].$$

The parameter μ coincides with the μ of the monodromy data (μ, R, S, C) of some 3D Dubrovin–Frobenius manifold.

Remark 4.13. A differential equation is said to possess the *Painlevé property* if the problematic singularities of its corresponding solutions, such as essential singularities and branch points, are always at the coefficients of the differential equation. Additionally, the movable singularities—those dependent on the integration constants—are poles. In particular, the *Painlevé* VI

(P_{VI}) equation has the Painlevé property. Consequently, the potential locations of essential singularities, branch points, and so on, are confined to 0, 1, and ∞, while the positions of the poles depend on the integration constants. In this context, the locus of movable poles of P_{VI_μ}, known as the Malgrange divisor MD_μ, coincides with the zeros and poles divisor associated with the isomonodromic τ function defined in (4.55) projected onto $\mathbb{C}\setminus\{0,1\}$. The functions $a(x)$, $b(x)$, $c(x)$ in (4.61) are rational functions of x, $y(x)$, $y'(x)$, see [25, Section 6.1]. Due to the Painlevé property, the Saito flat coordinates t^1, t^2, t^3 (4.60) are meromorphic functions in $\mathbb{C}\times\mathbb{C}^*\times(\mathbb{C}\setminus\{0,1\})$.

From here, we return to the case of $QH^*\big(\mathbb{CP}^2\big)$. Note that the hypersurface $t^3=0$ in canonical coordinates corresponds to the hypersurface $\big\{\big(v,H,\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\big)\in\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}\setminus\{0,1\}\big\}$. Indeed, substituting the small canonical coordinates (3.57) in (4.57), we obtain the desired result.

Hence, the big quantum cohomology is an expansion around $x=\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}$. Following [25, Section 6.5], define $s:=x-\mathrm{e}^{\frac{-\pi\mathrm{i}}{3}}$ and expand Ω_1, Ω_2, Ω_3 in (4.59) around $\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}$

$$\Omega_1(x)=-\frac{\mathrm{i}\sqrt{3}}{2}-\left(\frac{1}{6}+\frac{\mathrm{i}\sqrt{3}}{6}\right)s+\frac{\mathrm{i}\sqrt{3}}{9}s^2+O\big(s^3\big),$$
$$\Omega_2(x)=\frac{\mathrm{i}\sqrt{3}}{2}+\left(\frac{1}{6}-\frac{\mathrm{i}\sqrt{3}}{6}\right)s-\frac{\mathrm{i}\sqrt{3}}{9}s^2+O\big(s^3\big),$$
$$\Omega_3(x)=\frac{\mathrm{i}\sqrt{3}}{2}-\frac{1}{3}s+\frac{2\mathrm{i}\sqrt{3}}{9}s^2+O\big(s^3\big). \tag{4.64}$$

Furthermore, expanding $y(x)$,

$$y(x)=\frac{1}{2}-\frac{\mathrm{i}\sqrt{3}}{6}+\frac{1}{3}s-\frac{\mathrm{i}\sqrt{3}}{3}s^2+O\big(s^3\big).$$

Substituting (4.64) in (4.56), (4.61) and (4.60), the Saito flat coordinates have the following expansion:

$$t^1(u)=u_1+\left(\frac{1}{2}-\frac{\mathrm{i}\sqrt{3}}{6}s+O\big(s^2\big)\right)H,\qquad Q(u)=\frac{\mathrm{i}\sqrt{3}}{143}\big(1+\mathrm{i}\sqrt{3}s+O\big(s^2\big)\big)H^3,$$
$$t^3(u)=\big(-9s+O\big(s^2\big)\big)H^{-1}.$$

Then, the expansion of coordinates $Q^{\frac{1}{3}}t^3$ is given by

$$\begin{aligned}Q^{\frac{1}{3}}t^3(x)&=\sum_{n=1}^{\infty}Q_n\big(x-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\big)^n\\&=\left(\frac{3}{2}-\frac{i\sqrt{3}}{2}\right)\big(x-\mathrm{e}^{\frac{-\pi\mathrm{i}}{3}}\big)-\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\big(x-\mathrm{e}^{\frac{-\pi\mathrm{i}}{3}}\big)^2+O\big(\big(x-\mathrm{e}^{\frac{-\pi\mathrm{i}}{3}}\big)^3\big)\end{aligned} \tag{4.65}$$

is a holomorphic function in some neighbourhood $U\big(\mathrm{e}^{\frac{-\pi\mathrm{i}}{3}}\big)$ of $\mathrm{e}^{\frac{-\pi\mathrm{i}}{3}}\in\mathbb{C}\setminus\{0,1\}$.

At this stage, we can state the main result of this subsection.

Lemma 4.14. *The composition of the modular lambda function* (3.63) *restricted to a suitable neighborhood of* $\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$ *with the map* (4.65) *is a local biholomorphism given by the following power series*

$$Q^{\frac{1}{3}}t^3(z)=\sum_{n=0}^{\infty}\tilde{Q}_n\big(z-\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big)^n, \tag{4.66}$$

where

$$\tilde{Q}_n = \frac{\sum_{k=0}^{n} k! Q_k B_{n,k}(x_1, x_2, \ldots, x_n)}{n!},$$

x_m *are the coefficients of the modular lambda function* (3.63) *around* $\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$*, i.e.,*

$$x(z) = \sum_{m=0}^{\infty} x_m \frac{\left(z - \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right)^m}{m!},$$

and $B_{n,k}$ *are the partial Bell polynomials given by the formula* (A.1)*.*

Proof. The map defined by (4.65) is a local biholomorphism

$$Q^{\frac{1}{3}} t^3(x)\colon\; U\left(\mathrm{e}^{\frac{-\pi \mathrm{i}}{3}}\right) \subset \mathbb{C} \setminus \{0,1\} \mapsto Q^{\frac{1}{3}} t^3\left(U\left(\mathrm{e}^{\frac{-\pi \mathrm{i}}{3}}\right)\right) \subset D\left(0, \left(\frac{1}{a}\right)^{\frac{1}{3}}\right).$$

Indeed, the point $\mathrm{e}^{\frac{-\pi \mathrm{i}}{3}} \in \mathbb{C} \setminus \{0,1\}$ lies within the semisimple locus and serves as a holomorphic point for the functions given by (4.64). Since the Jacobian of the transition function between canonical and flat Saito coordinates

$$J = \det\left(\frac{\partial t^\alpha}{\partial u_i}\right) = \prod_{i=1}^{3} \psi_{i1}$$

does not vanish in the semisimple locus, the map (4.65) is a local biholomorphism.

Recall that the preimage of $\mathrm{e}^{\frac{-\pi \mathrm{i}}{3}}$ under the modular lambda function (3.63) is $\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} \bmod \Gamma(2)$. Moreover, the modular lambda function (3.63) is a local biholomorphism. In particular, the modular lambda function (3.63) is a local biholomorphism around $\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$. Therefore, composing the modular lambda function (3.63) restricted to a suitable neighborhood $U\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right)$ of $\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} \in \mathbb{H}$ with (4.65), we obtain the desire local biholomorphism

$$\begin{aligned} Q^{\frac{1}{3}} t^3(x(z))\colon\; & U\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right) \subset \mathbb{H} \mapsto U\left(\mathrm{e}^{\frac{-\pi \mathrm{i}}{3}}\right) \subset \mathbb{C} \setminus \{0,1\} \\ & \mapsto Q^{\frac{1}{3}} t^3\left(x\left(U\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right)\right)\right) \subset D\left(0, \left(\frac{1}{a}\right)^{\frac{1}{3}}\right). \end{aligned} \tag{4.67}$$

Furthermore, we use (4.65), (3.63) and using the Faa di Bruno formula (A.2) to compute the explicit power series composition

$$\begin{aligned} Q^{\frac{1}{3}} t^3(x(z)) &= \sum_{n=0}^{\infty} Q_n \left(x(z) - \mathrm{e}^{\frac{-\pi \mathrm{i}}{3}}\right)^n = \sum_{n=0}^{\infty} n! Q_n \frac{\left(x(z) - \mathrm{e}^{\frac{-\pi \mathrm{i}}{3}}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} n! \frac{Q_n}{n!} \left(\sum_{m=0}^{\infty} x_m \frac{\left(z - \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right)^m}{m!}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k! Q_k B_{n,k}(x_1, x_2, \ldots, x_n)\right) \frac{\left(z - \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \tilde{Q}_n \left(z - \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right)^n. \end{aligned}$$

Lemma proved. ■

Remark 4.15. The coefficients of (4.66) can be explicitly computed to any desired order using the Guzzetti algorithm (4.65) or by employing the alternative algorithm derived in Appendix A.3.

4.3 Canonical coordinates for big quantum cohomology of $\mathbb{CP}^2$

In this subsection, the goal is to establish a formula expressing the canonical coordinates of quantum cohomology for $\mathbb{CP}^2$ in terms of the Saito flat coordinates.

The canonical coordinates are the roots (1.15). In particular, $g^{\alpha\beta}$ and $\eta^{\alpha\beta}$ for big quantum cohomology are given by

$$g^{\alpha\beta} = E^{\epsilon} c_{\epsilon}^{\alpha\beta} = \begin{pmatrix} \frac{3}{(t^3)^3}[9\Phi''(X) - 9\Phi'(X) + 2\Phi(X)] & \frac{2}{(t^3)^2}[3\Phi''(X) - \Phi'(X)] & t^1 \\ \frac{2}{(t^3)^2}[3\Phi''(X) - \Phi'(X)] & t^1 + \frac{\Phi''(X)}{t^3} & 3 \\ t^1 & 3 & -t^3 \end{pmatrix}, \quad (4.68)$$

$$\eta^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (4.69)$$

where $\Phi(X)$ is defined by (1.23).

In [20, Example 3.10.34], Dubrovin and Zhang made the following ansatz:

$$u_i = t^1 + \frac{9 + \Phi''(X) - z_i}{t^3}. \quad (4.70)$$

Substituting (4.68), (4.69) and (4.70) in (1.15), we obtain that z_1, z_2, z_3 are the roots of the following cubic equation $z^3 - s_1 z^2 + s_2 z - s_3 = 0$, where

$$s_1 = 27 + 2\Phi'', \qquad s_2 = 243 + 6\Phi - 15\Phi' + 27\Phi'' + (\Phi'')^2,$$
$$s_3 = (27 + 2\Phi' - 3\Phi'')^2. \quad (4.71)$$

Summarising, we have the following lemma.

Lemma 4.16. *The canonical coordinates of quantum cohomology u_1, u_2, u_3 as function of the Saito flat coordinates t^1, t^2, t^3 are given by*

$$u_k = t^1 + \frac{1}{t^3} \sum_{n=1}^{\infty} A_n^k \big(Q^{\frac{1}{3}} t^3\big)^n, \quad (4.72)$$

where the coefficients A_k are given explicitly by

$$\tilde{A}_{3k} = \frac{k^2 N_k}{(3k-1)!},$$
$$\sum_{n_2=2}^{3n} 3\cos\left(\frac{2\pi(n_2-1)}{3}\right) \tilde{A}_{3n-n_2+1}\tilde{A}_{n_2-1} = (6 - 15n - 9n^2)\frac{N_n}{(3n-1)!},$$
$$\sum_{n_2=1}^{3n-2} \sum_{n_3=1}^{3n-n_2-1} 3\cos\left(\frac{2\pi(n_2+2n_3)}{3}\right) \tilde{A}_{3n-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3} = (54 - 243n)\frac{N_n}{(3n-1)!} + \delta_n,$$

where

$$A_n^k = \tilde{A}_n \big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big)^{nk}, \qquad \delta_n = \begin{cases} 0 & \textit{if } n = 1, \\ \tilde{\delta}_n & \textit{otherwise,} \end{cases}$$
$$\tilde{\delta}_n = \sum_{n_2=2}^{n} \frac{(6(n_2-1) - 3(n-n_2+1)(n_2-1)^2)}{(3n-3n_2+2)!(3n_2-4)!} N_{n-n_2+1}N_{n_2-1}$$

$$+\sum_{n_2=2}^{n}\frac{(-4(n-n_2+1)(n_2-1))}{(3n-3n_2+2)!(3n_2-4)!}N_{n-n_2+1}N_{n_2-1}$$
$$+\sum_{n_2=2}^{n}\frac{(-9(n_1-n_2+1)^2(n_2-1)^2)}{(3n-3n_2+2)!(3n_2-4)!}N_{n-n_2+1}N_{n_2-1}.$$

Proof. See Appendix A.2. ■

Moreover, the transition function matrix $\psi_{i\alpha}$ can be obtained by finding the eigenvectors related to (1.15) as done in [20, Example 3.10.34]. More precisely, the transition function matrix $\psi_{i\alpha}$ is the matrix such that the columns are eigenvectors of $g^*\eta$ which is given by

$$\psi_{i1}=h_i,\qquad \psi_{i2}=\frac{h_i\big(-27-2\Phi'+3\Phi''+3z_i\big)}{z_i t^3},$$
$$\psi_{i3}=\frac{h_i\big(81+6\Phi'-9\Phi''-18z_i+z_i^2-z_i^2\Phi''\big)}{z_i\big(t^3\big)^2},$$

where h_i have the form

$$h_1=\frac{t^3\sqrt{z_1}}{\sqrt{(z_1-z_2)}\sqrt{(z_1-z_3)}},\qquad h_2=\frac{t^3\sqrt{z_2}}{\sqrt{(z_2-z_1)}\sqrt{(z_2-z_3)}},$$
$$h_3=\frac{t^3\sqrt{z_3}}{\sqrt{(z_3-z_1)}\sqrt{(z_3-z_2)}}.$$

4.4 Geometric isomonodromic deformation

The aim of this subsection is to give a geometric interpretation for the isomonodromic deformation of the Hauptmodul $\gamma_2(\tilde\tau)$ (3.23). For this purpose, we derive a t^3 deformation of the equianharmonic elliptic curve

$$y^2=4(x-u_1)(x-u_2)(x-u_3)=4\big(x-t^1\big)^3-27Q.$$

Lemma 4.17. *Consider the canonical coordinates (u_1,u_2,u_3) of the quantum cohomology of $\mathbb{CP}^2$, obtained from (4.70) and (4.71), along with the cubic equation*

$$y^2=4(\lambda-u_1)(\lambda-u_2)(\lambda-u_3). \tag{4.73}$$

Then, the pair (λ,y) satisfying (4.73) can be expressed as follows:

$$\lambda=t^1+\frac{\Phi''\big(Q\big(t^3\big)^3\big)}{3t^3}-\frac{\wp(v,z)}{(2\tilde\omega)^2},\qquad y=i\frac{\wp'(v,z)}{(2\tilde\omega)^3}. \tag{4.74}$$

Proof. Substituting (4.70) in (4.73), we obtain

$$\tilde y^2=4\big(\tilde\lambda-z_1\big)\big(\tilde\lambda-z_2\big)\big(\tilde\lambda-z_3\big)=4\big(\tilde\lambda^3-s_1\tilde\lambda^2+s_2\tilde\lambda-s_3\big), \tag{4.75}$$

where s_1, s_2, s_3 are given by (4.71) and

$$y=\frac{i\tilde y}{\big(t^3\big)^{\frac{3}{2}}},\qquad \lambda=t^1+\frac{9+\Phi''-\tilde\lambda}{t^3}. \tag{4.76}$$

To express (4.75) in the depressed form of the cubic equation, we make in (4.75) the following substitution:

$$\tilde\lambda=\hat\lambda+\frac{s_1}{3},\qquad \hat y=\tilde y,\qquad z_i=\hat e_i+\frac{s_1}{3}, \tag{4.77}$$

then we obtain

$$\hat{y}^2 = 4\big(\hat{\lambda} - \hat{e}_1\big)\big(\hat{\lambda} - \hat{e}_2\big)\big(\hat{\lambda} - \hat{e}_3\big) = 4\hat{\lambda}^3 - g_2\hat{\lambda} - g_3, \tag{4.78}$$

where $\hat{g}_2 = -4(\hat{e}_1\hat{e}_2 + \hat{e}_1\hat{e}_3 + \hat{e}_2\hat{e}_3)$, $\hat{g}_3 = 4\hat{e}_1\hat{e}_2\hat{e}_3$. Moreover, consider the following rescaling:

$$\hat{\lambda} = \frac{\lambda^r}{(2\tilde{\omega})^2}, \qquad \hat{e}_i = \frac{e_i}{(2\tilde{\omega})^2}, \qquad \hat{y} = \frac{y^r}{(2\tilde{\omega})^3}. \tag{4.79}$$

Hence, substituting (4.79) in (4.78),

$$(y^r)^2 = 4(\lambda^r - e_1)(\lambda^r - e_2)(\lambda^r - e_3) = 4(\lambda^r)^3 - g_2\lambda^r - g_3, \tag{4.80}$$

where $g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3)$, $g_3 = 4e_1e_2e_3$.

Due to the uniformization of elliptic curves, we can parametrize the cubic equation (4.80) using the Weierstrass $\wp$ function and its derivatives. In other words, we express (4.80) in the form

$$(\wp'(v,z))^2 = 4\wp(v,z)^3 - g_2(z)\wp(v,z) - g_3(z). \tag{4.81}$$

Substituting (4.81), (4.79), (4.77) in (4.76), we obtain

$$\lambda = t^1 + \frac{\Phi''\big(Q\big(t^3\big)^3\big)}{3t^3} - \frac{\wp(v,z)}{(2\tilde{\omega})^2}, \qquad y = \mathrm{i}\frac{\wp'(v,z)}{(2\tilde{\omega})^3}.$$

Lemma proved. ■

Our objective is to extend the family of elliptic curves (4.74) to a LG superpotential for $QH^*\big(\mathbb{CP}^2\big)$. To begin, we examine the following auxiliary lemma.

Lemma 4.18. *The Landau–Ginzburg superpotential of quantum cohomology of $\mathbb{CP}^2$ is a family of functions $\lambda\big(\tilde{\tau}_{12}, t^1, Q^{\frac{1}{3}}, z\big)\colon \mathbb{H} \mapsto \mathbb{C}$ parametrized by*

$$\big(t^1, Q^{\frac{1}{3}}, z\big) \in \big\{\big(t^1, Q^{\frac{1}{3}}, z\big) \in \mathbb{C} \times \mathbb{C}^* \times U\big(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big)\big\} \subset \mathbb{C} \times \mathbb{C}^* \times \mathbb{H}$$

and given by $\lambda\big(\tilde{\tau}_{12}, t^1, Q, z\big) = t^1 + 3Q^{\frac{1}{3}}J^{\frac{1}{3}}(\tilde{\tau}_{12}, z)$, where $J^{\frac{1}{3}}(\tilde{\tau}_{12}, z)$ is the composition of the function $J^{\frac{1}{3}}\big(\tilde{\tau}_{12}, Q^{\frac{1}{3}}t^3\big)$ defined in (4.38) and $Q^{\frac{1}{3}}t^3(z)$ defined in (4.67), i.e.,

$$\begin{aligned} J^{\frac{1}{3}}(\tilde{\tau}_{12}, z) &:= \sum_{n=0}^{\infty} J_n^{\frac{1}{3}}(\tilde{\tau}_{12})\big(Q^{\frac{1}{3}}t^3(z)\big)^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k!J_k^{\frac{1}{3}}(\tilde{\tau}_{12})B_{n,k}\big(1!\tilde{Q}_1, \dots, m!\tilde{Q}_m\big)\right)\frac{\big(z - \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big)^n}{n!}, \end{aligned}$$

where $B_{n,k}$ are the Bell polynomials defined in (A.1). In addition, the correspondent Abelian differential ϕ is given by

$$\phi = \frac{\Delta^{\frac{1}{6}}(\tilde{\tau}_{12}, z)}{Q^{\frac{1}{6}}}\mathrm{d}\tilde{\tau}_{12},$$

where $\Delta(\tilde{\tau}_{12}, z)$ is the composition of (4.67) with (4.32).

Proof. Doing the change of coordinates (4.67) in the Landau–Ginzburg superpotential (4.37) and in Abelian differential (4.39) we get the desired result. More explicitly, substituting (4.67) in (4.38), we obtain

$$\begin{aligned}\lambda\big(\tilde{\tau}_{12},t^1,Q,t^3\big) &= t^1+3Q^{\frac{1}{3}}\sum_{n=0}^{\infty}J_n^{\frac{1}{3}}(\tilde{\tau}_{12})\big(Q^{\frac{1}{3}}t^3\big)^n \\ &= t^1+3Q^{\frac{1}{3}}\sum_{n=0}^{\infty}n!J_n^{\frac{1}{3}}(\tilde{\tau}_{12})\frac{\big(\sum_{k=0}^{\infty}\tilde{Q}_k\big(z-\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big)^k\big)^n}{n!} \\ &= t^1+3Q^{\frac{1}{3}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}k!J_k^{\frac{1}{3}}(\tilde{\tau}_{12})B_{n,k}\big(1!\tilde{Q}_1,\dots,m!\tilde{Q}_m\big)\right)\frac{\big(z-\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big)^n}{n!} \\ &= t^1+3Q^{\frac{1}{3}}J^{\frac{1}{3}}(\tilde{\tau}_{12},z).\end{aligned} \tag{4.82}$$

That concludes the proof. ∎

Our next goal is to generalise the Cohn identity (3.65) for $z\in\mathbb{H}$ near $\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$.

Theorem 4.19. *The Landau–Ginzburg superpotential of big quantum cohomology of* $\mathbb{CP}^2$ *is a family of functions* $\lambda(\tilde{\tau}_{12},\tau_{12},\tilde{\omega},z)\colon\mathbb{H}\mapsto\mathbb{C}$ *parametrized by*

$$\big\{(\tau_{12},\tilde{\omega},z)\in\mathbb{H}\times\mathbb{C}^*\times\mathbb{H}\mid z\in U\big(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big)\big\}$$

and given by

$$\lambda(v(\tilde{\tau}_{12},z),\tau_{12},\tilde{\omega},z)=\frac{\wp(v(\tau_{12},z),z)}{(2\tilde{\omega})^2}-\frac{\wp(v(\tilde{\tau}_{12},z),z)}{(2\tilde{\omega})^2},$$

where for $z\in\mathbb{H}$ *close enough to* $\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$, $v(\tilde{\tau}_{12},z)$ *is the universal covering of* $\mathbb{C}\setminus\{\mathbb{Z}\oplus z\mathbb{Z}\}$. *In addition, the correspondent Abelian differential* ϕ *is given by* $\phi=2\tilde{\omega}\mathrm{d}v(\tilde{\tau}_{12},z)$, *where* $\Delta(\tilde{\tau}_{12},z)$ *is the composition of* (4.67) *with* (4.32).

Proof. Recall that the critical values of the uniformization (4.74) are the canonical coordinates (4.72) by construction. The Abelian differential which gives the correct Landau–Ginzburg superpotential is $\phi=\mathrm{d}\tilde{w}_2$. But, in order to implement the Abelian differential in the spectral curve (4.73), we need to construct, for a fixed $z\in\mathbb{H}$, a change of coordinates

$$v_z\colon\ \mathbb{H}\mapsto\mathbb{C}\setminus\mathbb{Z}\oplus z\mathbb{Z},\qquad \tilde{\tau}_{12}\mapsto v_z(\tilde{\tau}_{12}):=v(\tilde{\tau}_{12},z) \tag{4.83}$$

such that

$$\lambda\big(\tilde{\tau}_{12},t^1,Q,z\big)=t^1+Q^{\frac{1}{3}}\frac{\Phi''\big(Q\big(t^3\big)^3(z)\big)}{3Q^{\frac{1}{3}}t^3(z)}-\frac{\wp(v(\tilde{\tau}_{12},z),z)}{(2\tilde{\omega})^2}. \tag{4.84}$$

Then, the change of coordinates (4.83) is concretely constructed by comparing (4.84) with (4.82). More explicitly, by the relation below

$$\begin{aligned}&3Q^{\frac{1}{3}}J^{\frac{1}{3}}(\tilde{\tau}_{12},z)=Q^{\frac{1}{3}}\frac{\Phi''\big(Q\big(t^3\big)^3(z)\big)}{3Q^{\frac{1}{3}}t^3(z)}-\frac{\wp(v(\tilde{\tau}_{12},z),z)}{(2\tilde{\omega})^2},\\ &\frac{\Delta^{\frac{1}{6}}(\tilde{\tau}_{12},z)}{Q^{\frac{1}{6}}}=2\tilde{\omega}v'(\tilde{\tau}_{12},z).\end{aligned} \tag{4.85}$$

The identity (4.85) can be understood as generalised Cohn identity. Indeed, the map (4.83) is defined by a composition of the deformed $J^{\frac{1}{3}}$ function (4.82) restricted to $\mathbb{H}\setminus \mathrm{SL}_2(\mathbb{Z})(\mathrm{i})$ and the inverse of the Weierstrass $\wp$ function restricted to $\mathbb{C}\setminus\{e_1(z),e_2(z),e_3(z)\}$, for fixed z close enough to $\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$. More precisely, consider the functions

$$J_z^{\frac{1}{3}}\colon\ \mathbb{H}\setminus \mathrm{SL}_2(\mathbb{Z})(i)\mapsto \mathbb{C}\setminus\{e_1^*(z),\mathrm{e}_2^*(z),\mathrm{e}_3^*(z)\},\qquad \tilde{\tau}_{12}\mapsto J^{\frac{1}{3}}(\tilde{\tau}_{12},z),$$
$$\wp\colon\ (\mathbb{C}\setminus\mathbb{Z}\oplus z\mathbb{Z})\setminus\left\{\frac{\mathbb{Z}}{2}\oplus z\frac{\mathbb{Z}}{2}\right\}\mapsto\mathbb{C}\setminus\{e_1(z),e_2(z),e_3(z)\},\qquad v\mapsto\wp(v,z),$$

where

$$e_i^*(z)=\frac{\Phi''\big(Q\big(t^3\big)^3(z)\big)}{9Q^{\frac{1}{3}}t^3(z)}-4^{\frac{1}{3}}e_i(z).$$

Then, for small enough z, the composition

$$v(\tilde{\tau}_{12},z)\colon\ \mathbb{H}\setminus\{E_6=0\}\mapsto(\mathbb{C}\setminus\mathbb{Z}\oplus z\mathbb{Z})\setminus\left\{\frac{\mathbb{Z}}{2}\oplus z\frac{\mathbb{Z}}{2}\right\}\tag{4.86}$$

is a holomorphic surjective function, because it is a composition of holomorphic surjective functions. Moreover, for z close enough to $\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$, the derivative of the function (4.86)

$$v'(\tilde{\tau}_{12},z)=v'(\tilde{\tau}_{12})+\sum_{n=1}^{\infty}v_n(\tau_{12})\big(z-\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\big)^n$$

is close to its leading term $v'(\tilde{\tau}_{12})=\Delta^{\frac{1}{6}}(\tilde{\tau}_{12})$, which is a non-vanishing function on $\mathbb{H}$. Therefore, we can extend (4.86) to a local biholomorphism on $\mathbb{H}$

$$v(\tilde{\tau}_{12},z)\colon\ \mathbb{H}\mapsto\ \mathbb{C}\setminus(\mathbb{Z}\oplus z\mathbb{Z}).\tag{4.87}$$

The function (4.87) is a $\pi_1(\mathbb{C}\setminus(\mathbb{Z}\oplus z\mathbb{Z}))$-invariant function. Indeed, let $\gamma\in\Gamma^{(3)}$ and $\tilde{\tau}_{12}(\tilde{\tau})$ the function defined in (4.21), then consider the group homomorphism defined by

$$\gamma_z\tilde{\tau}_{12}(\tilde{\tau}):=\tilde{\tau}_{12}(\gamma\tilde{\tau}).\tag{4.88}$$

Since (4.21) is an isomonodromic deformation equation, the following property holds:

$$J^{\frac{1}{3}}(\tilde{\tau}_{12}(\tilde{\tau}),z)=J^{\frac{1}{3}}(\tilde{\tau}).$$

In addition,

$$J^{\frac{1}{3}}(\gamma_z\tilde{\tau}_{12}(\tilde{\tau}),z)=J^{\frac{1}{3}}(\tilde{\tau}_{12}(\gamma\tilde{\tau}),z)=J^{\frac{1}{3}}(\gamma\tilde{\tau})=J^{\frac{1}{3}}(\tilde{\tau}).$$

The group homomorphism (4.88) induces another group homomorphism

$$\bar{\gamma}_zv(\tilde{\tau}_{12}(\tilde{\tau}),z):=v(\tilde{\tau}_{12}(\gamma\tilde{\tau}),z).\tag{4.89}$$

Denoting the image of (4.88) by $\Gamma_z^{(3)}$ and $\operatorname{Im}\Psi_z$, $\operatorname{Ker}\Psi_z$ the image and kernel of (4.89), respectively, we have that

$$\mathbb{H}/\Gamma_z^{(3)}=(\mathbb{H}/\operatorname{Ker}\Psi_z)/\operatorname{Im}\Psi_z=\mathbb{C}\setminus(\mathbb{Z}\oplus z\mathbb{Z})/\operatorname{Im}\Psi_z,$$
$$\mathbb{H}/\operatorname{Ker}\Psi_z=\mathbb{C}\setminus(\mathbb{Z}\oplus z\mathbb{Z})=\mathbb{H}/\pi_1(\mathbb{C}\setminus(\mathbb{Z}\oplus z\mathbb{Z})).$$

Hence, the map (4.87) is a local biholomorphism, which is $\pi_1(\mathbb{C}\setminus(\mathbb{Z}\oplus z\mathbb{Z}))$-invariant. Then, the map (4.87) can be identified with the quotient map $\pi\colon \mathbb{H}\mapsto \mathbb{H}/\pi_1(\mathbb{C}\setminus\mathbb{Z}\oplus z\mathbb{Z})$, which is a covering map, since $\pi_1(\mathbb{C}\setminus\mathbb{Z}\oplus z\mathbb{Z})$ acts properly discontinuously on $\mathbb{H}$.

Therefore, (4.83) is a family of universal covering of $\mathbb{C}\setminus\mathbb{Z}\oplus z\mathbb{Z}$, which is an isomonodromic deformation of the universal covering of the equianharmonic lattice.

Moreover, the correspondent Abelian differential $\phi = \mathrm{d}\tilde{w}_2$ in these coordinates is given by

$$\phi = \mathrm{d}\tilde{w}_2 = \tilde{y}\mathrm{d}\tilde{\tau}_{12} = \frac{\Delta^{\frac{1}{6}}(\tilde{\tau}_{12},z)}{Q^{\frac{1}{6}}}\mathrm{d}\tilde{\tau}_{12} = 2\tilde{\omega}\frac{\partial v(\tilde{\tau}_{12},z)}{\partial\tilde{\tau}_{12}}\mathrm{d}\tilde{\tau}_{12} = 2\tilde{\omega}\mathrm{d}v(\tilde{\tau}_{12},z).$$

Theorem proved. ■

Remark 4.20. The title of this subsection was inspired by Doran's paper [13], which utilizes the moduli space of elliptic curves over $\mathbb{CP}^1$ to construct algebraic-geometric solutions of Painlevé VI. In the context of this manuscript, we also have a Painlevé VI associated with isomonodromic deformation induced by the Gauss–Manin connection of $QH^*(\mathbb{CP}^2)$ as mentioned in Section 4.2. It would be interesting to compare both constructions in a more general setting, since geometric isomonodromic deformation coming from Gromov–Witten potential could give a good source of interesting moduli space of elliptic curves over $\mathbb{CP}^1$.

A Appendix

A.1 Composition of power series and Faa di Bruno formula

In this appendix, we state the necessary definitions and results on composition of power series, referencing [9, Chapter 11].

Definition A.1 ([9])**.** The polynomials $B_n(x_1,x_2,\dots,x_n)$, $B_{n,k}(x_1,x_2,\dots,x_n)$ in the variables $x_1,x_2,\dots,x_n$, defined by the sum

$$\begin{aligned} B_n &= \sum_{k_1+2k_2+3k_3+\dots+nk_n=n}\frac{n!}{k_1!(1!)k_2!(2!)^{k_2}\cdots k_n!(n!)^{k_n}}x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n},\\ B_{n,k} &= \sum_{\substack{k_1+\dots+k_n=k\\ k_1+2k_2+3k_3+\dots+nk_n=n}}\frac{n!}{k_1!(1!)k_2!(2!)^{k_2}\cdots k_n!(n!)^{k_n}}x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} \end{aligned} \tag{A.1}$$

are called exponential Bell partition polynomial and partial Bell partition polynomial respectively.

Theorem A.2 (Faa di Bruno formula, [9])**.** *Let $f(u)$ and $g(t)$ be two functions of real variables for which all the derivatives*

$$g_r = \left[\frac{\mathrm{d}^r g(t)}{\mathrm{d}t^r}\right]_{t=a},\qquad r=0,1,\dots,\qquad f_k = \left[\frac{\mathrm{d}^k f(u)}{\mathrm{d}u^k}\right]_{u=(g(a)},\qquad k=0,1,\dots,$$

exist. Then the derivatives of the composite function $h(t)=f(g(t))$,

$$h_n = \left[\frac{\mathrm{d}^n h(u)}{\mathrm{d}t^n}\right]_{t=a},\qquad n=0,1,\dots,$$

are given by

$$h_n = \sum_{k=0}^{n} f_k B_{n,k}(g_1,g_2,\dots,g_n) = B_n(fg_1,fg_2,\dots,fg_n). \tag{A.2}$$

Definition A.3 ([9])**.** The polynomial $C_{n,s} = C_{n,s}(x_1, x_2, \dots, x_n)$ in the variables $x_1, x_2, \dots, x_n$, defined for a real (or complex) number s by the sum

$$C_{n,s} = \sum_{k_1+2k_2+3k_3+\cdots+nk_n=n} \frac{n!}{k_1!(1!)k_2!(2!)^{k_2}\cdots k_n!(n!)^{k_n}} \binom{s}{k} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

is called potential partition polynomial, where

$$\binom{s}{k} = \frac{s(s-1)(s-2)\cdots(s-k+1))}{k!}.$$

Theorem A.4 ([9])**.** *The generating function of the potential partition polynomials $C_{n,s}(x_1, \dots, x_n)$, $n = 0, 1, \dots$, for fixed k, is given by*

$$C_s(t) = \sum_{n=0}^{\infty} C_{n,s}(x_1, \dots, x_n) \frac{t^n}{n!} = [1 + (g(t) - x_0)]^s,$$

where $g(t) = \sum_{r=0}^{\infty} x_r \frac{t^r}{r!}$.

A.2 Canonical coordinates as functions of Gromov–Witten invariants

Lemma A.5. *Let be $\zeta = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$ and*

$$b_{k_1,k_2} := \zeta^{k_1} + \zeta^{k_1+2k_2} + \zeta^{2k_1}, \qquad c_{n_1,n_2,n_3} = \zeta^{n_1+2n_2},$$
$$\tilde{c}_{n_1,n_2,n_3} := c_{n_1,n_2,n_3} + c_{n_1,n_3,n_2} + c_{n_3,n_2,n_1} + c_{n_3,n_1,n_2} + c_{n_2,n_1,n_3} + c_{n_2,n_3,n_1}$$

for $k, k_1, k_2 k_3 \in \mathbb{Z}$. Then the following identities holds:

$$\zeta^k + \zeta^{2k} + \zeta^{3k} = 1 + 2\cos\left(\frac{2\pi k}{3}\right), \qquad b_{k,k} = 1 + 2\cos\left(\frac{2\pi k}{3}\right), \qquad c_{k,k,k} = 1,$$
$$b_{k_1-k_2,k_2} + b_{k_2,k_1-k_2} = \begin{cases} 6\cos\left(\frac{2\pi k_2}{3}\right) & \text{if } k_1 \in 3\mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases}$$
$$\tilde{c}_{k_1-k_2-k_3,k_2,k_3} = \begin{cases} 6\cos\left(\frac{2\pi(k_2+2k_3)}{3}\right) & \text{if } k_1 \in 3\mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases} \tag{A.3}$$

Proof.

$$\zeta^k + \zeta^{2k} + \zeta^{3k} = \zeta^k + \zeta^{2k} + 1 = 2\left(\frac{\mathrm{e}^{\frac{2\pi \mathrm{i}k}{3}} + \mathrm{e}^{\frac{-2\pi \mathrm{i}k}{3}}}{2}\right) + 1 = 1 + 2\cos\left(\frac{2\pi k}{3}\right).$$

Define b_{k_1,k_2}, $b_{k_1,k_2} := \zeta^{k_1} + \zeta^{k_1+2k_2} + \zeta^{2k_1}$. Then,

$$b_{k,k} = \zeta^k + \zeta^{3k} + \zeta^{2k} = 1 + 2\cos\left(\frac{2\pi k}{3}\right).$$

Moreover,

$$b_{k_1-k_2,k_2} = \zeta^{k_1-k_2} + \zeta^{k_1+k_2} + \zeta^{2(k_1-k_2)}, \qquad b_{k_2,k_1-k_2} = \zeta^{k_2} + \zeta^{2k_1-k_2} + \zeta^{2k_2}.$$

If $k_1 \in 3\mathbb{Z}$,

$$b_{k_1-k_2,k_2} + b_{k_2,k_1-k_2} = \zeta^{-k_2} + \zeta^{k_2} + \zeta^{-2k_2} + \zeta^{k_2} + \zeta^{-k_2} + \zeta^{2k_2}$$

$$= \zeta^{2k_2} + \zeta^{k_2} + \zeta^{k_2} + \zeta^{k_2} + \zeta^{2k_2} + \zeta^{2k_2}$$
$$= 3\big(\zeta^{2k_2} + \zeta^{k_2}\big) = 6\cos\left(\frac{2\pi k_2}{3}\right).$$

If $k_1 \in 1 + 3\mathbb{Z}$,

$$\begin{aligned} b_{k_1-k_2,k_2} + b_{k_2,k_1-k_2} &= \zeta^{1-k_2} + \zeta^{k_2+1} + \zeta^{2-2k_2} + \zeta^{k_2} + \zeta^{2-k_2} + \zeta^{2k_2} \\ &= \zeta^{1+2k_2} + \zeta^{k_2+1} + \zeta^{k_2+2} + \zeta^{k_2} + \zeta^{2(k_2+1)} + \zeta^{2k_2} \\ &= 2\cos\left(\frac{2\pi(1-k_2)}{3}\right) + 2\cos\left(\frac{2\pi k_2}{3}\right) + 2\cos\left(\frac{2\pi(1+k_2)}{3}\right) \\ &= 0. \end{aligned}$$

If $k_1 \in 2 + 3\mathbb{Z}$,

$$\begin{aligned} b_{k_1-k_2,k_2} + b_{k_2,k_1-k_2} &= \zeta^{2-k_2} + \zeta^{k_2+2} + \zeta^{1-2k_2} + \zeta^{k_2} + \zeta^{1-k_2} + \zeta^{2k_2} \\ &= \zeta^{2-2k_2} + \zeta^{k_2+2} + \zeta^{2(2-k_2)} + \zeta^{k_2} + \zeta^{2(k_2+2)} + \zeta^{2k_2} \\ &= 2\cos\left(\frac{2\pi(2-k_2)}{3}\right) + 2\cos\left(\frac{2\pi k_2}{3}\right) + 2\cos\left(\frac{2\pi(2+k_2)}{3}\right) \\ &= 0. \end{aligned}$$

Let be the tensor defined by $c_{n_1,n_2,n_3} = \zeta^{n_1+2n_2}$. It is straightforward the following $c_{n,n,n} = \zeta^{3n} = 1$.

Moreover, setting

$$\tilde{c}_{n_1,n_2,n_3} := c_{n_1,n_2,n_3} + c_{n_1,n_3,n_2} + c_{n_3,n_2,n_1} + c_{n_3,n_1,n_2} + c_{n_2,n_1,n_3} + c_{n_2,n_3,n_1},$$

we have

$$\tilde{c}_{n_1,n_2,n_3} := \zeta^{n_1+2n_2} + \zeta^{n_1+2n_3} + \zeta^{n_3+2n_2} + \zeta^{n_3+2n_1} + \zeta^{n_2+2n_1} + \zeta^{n_2+2n_3}.$$

In particular,

$$\begin{aligned} \tilde{c}_{n_1-n_2-n_3,n_2,n_3} = &\ \zeta^{n_1+n_2-n_3} + \zeta^{n_1-n_2+n_3} + \zeta^{n_3+2n_2} \\ &+ \zeta^{-n_3+2n_1-2n_2} + \zeta^{-n_2+2n_1-2n_3} + \zeta^{n_2+2n_3}. \end{aligned}$$

If $n_1 \in 3\mathbb{Z}$,

$$\begin{aligned} \tilde{c}_{n_1-n_2-n_3,n_2,n_3} &= \zeta^{n_2-n_3} + \zeta^{-n_2+n_3} + \zeta^{n_3+2n_2} + \zeta^{-n_3-2n_2} + \zeta^{-n_2-2n_3} + \zeta^{n_2+2n_3} \\ &= \zeta^{n_2+2n_3} + \zeta^{2n_2+n_3} + \zeta^{n_3+2n_2} + \zeta^{2n_3+n_2} + \zeta^{2n_2+n_3} + \zeta^{n_2+2n_3} \\ &= 3\big(\zeta^{n_2+2n_3} + \zeta^{2n_2+n_3}\big) = 3\big(\zeta^{n_2+2n_3} + \zeta^{-n_2-2n_3}\big) \\ &= 6\cos\left(\frac{2\pi\,(n_2+2n_3)}{3}\right). \end{aligned}$$

If $n_1 \in 1 + 3\mathbb{Z}$,

$$\begin{aligned} \tilde{c}_{n_1-n_2-n_3,n_2,n_3} = &\ \zeta^{1+n_2-n_3} + \zeta^{1-n_2+n_3} + \zeta^{n_3+2n_2} \\ &+ \zeta^{-n_3-2n_2+2} + \zeta^{-n_2-2n_3+2} + \zeta^{n_2+2n_3} \\ = &\ \zeta^{1+n_2-n_3} + \zeta^{2+2n_2-2n_3} + \zeta^{n_3+2n_2} + \zeta^{-n_3-2n_2+2} + \zeta^{-n_2-2n_3+1} + \zeta^{n_2+2n_3} \\ = &\ 2\cos\left(\frac{2\pi(1+n_2-n_3)}{3}\right) + 2\cos\left(\frac{2\pi(2n_2+n_3)}{3}\right) \end{aligned}$$

$$
\begin{aligned}
&\quad + 2\cos\left(\frac{2\pi(1-n_2+n_3)}{3}\right) \\
&= 0.
\end{aligned}
$$

If $n_1 \in 2 + 3\mathbb{Z}$.

$$
\begin{aligned}
\tilde{c}_{n_1-n_2-n_3,n_2,n_3} &= \zeta^{2+n_2-n_3} + \zeta^{2-n_2+n_3} + \zeta^{n_3+2n_2} + \zeta^{-n_3-2n_2+1} + \zeta^{-n_2-2n_3+1} + \zeta^{n_2+2n_3} \\
&= \zeta^{2+n_2-n_3} + \zeta^{1+2n_2-2n_3} + \zeta^{n_3+2n_2} + \zeta^{-n_3-2n_2+1} + \zeta^{-n_2-2n_3+2} + \zeta^{n_2+2n_3} \\
&= 2\cos\left(\frac{2\pi(2+n_2-n_3)}{3}\right) + 2\cos\left(\frac{2\pi(2n_2+n_3)}{3}\right) \\
&\quad + 2\cos\left(\frac{2\pi(2-n_2+n_3)}{3}\right) \\
&= 0.
\end{aligned}
$$

■

Lemma A.6. *Let be u_1, u_2, u_3 the canonical coordinates of quantum cohomology as function of the Saito flat coordinates t^1, t^2, t^3, i.e.,*

$$
u_k = t^1 + \frac{1}{t^3}\sum_{n=1}^{\infty} A_n^k \big(Q^{\frac{1}{3}} t^3\big)^n. \tag{A.4}
$$

Then, the coefficients A_k are given explicitly by

$$
\begin{aligned}
&\tilde{A}_{3n} = \frac{n^2 N_n}{(3n-1)!}, \\
&\sum_{n_2=2}^{3n} 3\cos\left(\frac{2\pi(n_2-1)}{3}\right)\tilde{A}_{3n-n_2+1}\tilde{A}_{n_2-1} = \big(6-15n-9n^2\big)\frac{N_n}{(3n-1)!}, \\
&\sum_{n_2=1}^{3n-2}\sum_{n_3=1}^{3n-n_2-1} 3\cos\left(\frac{2\pi(n_2+2n_3)}{3}\right)\tilde{A}_{3n-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3} \\
&\qquad = \big(54-243n+243n^2\big)\frac{N_n}{(3n-1)!} + \delta_n,
\end{aligned}
$$

where

$$
\begin{aligned}
&A_n^k = \tilde{A}_n\big(e^{\frac{2\pi i}{3}}\big)^{nk}, \qquad \delta_n = \begin{cases} 0 & \text{if } n=1, \\ \tilde{\delta}_n & \text{otherwise,}\end{cases} \\
&\tilde{\delta}_n = \sum_{n_2=2}^{n} \frac{\big(6(n_2-1)-3(n-n_2+1)(n_2-1)^2\big)}{(3n-3n_2+2)!(3n_2-4)!} N_{n-n_2+1}N_{n_2-1} \\
&\qquad + \sum_{n_2=2}^{n}\frac{(-4(n-n_2+1)(n_2-1))}{(3n-3n_2+2)!(3n_2-4)!}N_{n-n_2+1}N_{n_2-1} \\
&\qquad + \sum_{n_2=2}^{n}\frac{\big(-9(n_1-n_2+1)^2(n_2-1)^2\big)}{(3n-3n_2+2)!(3n_2-4)!}N_{n-n_2+1}N_{n_2-1}.
\end{aligned}
$$

Proof. The canonical coordinates u_1, u_2, u_3 can be written as

$$
u_i = t^1 + \frac{9+\Phi''-z_i}{t^3}, \tag{A.5}
$$

where z_i is are the roots of $(z-z_1)(z-z_2)(z-z_3)=z^3-s_1z^2+s_2z-s_3=0$, where

$$\begin{aligned} s_1 &= z_1+z_2+z_3 = 27+2\Phi'', \\ s_2 &= z_1z_2+z_2z_3+z_1z_3 = 243+6\Phi-15\Phi'+27\Phi''+\big(\Phi''\big)^2, \\ s_3 &= z_1z_2z_3 = \big(27+2\Phi'-3\Phi''\big)^2. \end{aligned} \tag{A.6}$$

Define

$$f_k := \frac{1}{t^3}\sum_{n=1}^{\infty} A_n^k \big(Q^{\frac{1}{3}}t^3\big)^n.$$

Substituting (A.5) in (A.4),

$$z_i = 9+\Phi''-f_i. \tag{A.7}$$

Substituting (A.7) in (A.6), we obtain

$$\begin{aligned} &f_1+f_2+f_3 = \Phi'', \qquad f_1f_2+f_2f_3+f_1f_3 = 6\Phi-15\Phi'-9\Phi'', \\ &f_1f_2f_3 = 54\Phi-243\Phi'+243\Phi''+6\Phi\Phi''-4\Phi'^2-3\Phi'\Phi''-9\Phi''^2. \end{aligned} \tag{A.8}$$

Using (1.21) in the first two equation of the right-hand side of (A.8), we have

$$\begin{aligned} \Phi'' &= \sum_{n=1}^{\infty} n^2 \frac{N_n}{(3n-1)!}\big(Q\big(t^3\big)^3\big)^n, \\ 6\Phi-15\Phi'-9\Phi'' &= \sum_{n=1}^{\infty}\big(6-15n-9n^2\big)\frac{N_n}{(3n-1)!}\big(Q\big(t^3\big)^3\big)^n. \end{aligned} \tag{A.9}$$

The third equation of the right-hand side of (A.8) is bit more involved

$$\begin{aligned} &54\Phi-243\Phi'+243\Phi''+6\Phi\Phi''-3\Phi'\Phi''-4\big(\Phi'\big)^2-9\big(\Phi''\big)^2 \\ &\quad = \sum_{n=1}^{\infty}\big(54-243n+243n^2\big)\frac{N_n}{(3n-1)!}\big(Q\big(t^3\big)^3\big)^n \\ &\qquad + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\big(6n_2-3n_1n_2^2-4n_1n_2-9n_1^2n_2^2\big)\frac{N_{n_1}}{(3n_1-1)!}\frac{N_{n_2}}{(3n_2-1)!}\big(Q\big(t^3\big)^3\big)^{n_1+n_2}. \end{aligned}$$

Using the following double infinite sum identity

$$\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} C_{k_1,k_2} = \sum_{k_1=2}^{\infty}\left(\sum_{k_2=2}^{k_1} C_{k_1-k_2+1,k_2-1}\right)$$

in the equation (A.8),

$$\begin{aligned} &554\Phi-243\Phi'+243\Phi''+6\Phi\Phi''-4\Phi'^2-3\Phi'\Phi''-9\Phi''^2 \\ &\quad = \sum_{n=1}^{\infty}\big(54-243n+243n^2\big)\frac{N_n}{(3n-1)!}\big(Q\big(t^3\big)^3\big)^n \\ &\qquad + \sum_{n_1=2}^{\infty}\left(\sum_{n_2=2}^{n_1}\frac{(6(n_2-1)-3(n_1-n_2+1)(n_2-1)^2)}{(3n_1-3n_2+2)!(3n_2-4)!}N_{n_1-n_2+1}N_{n_2-1}\right)\big(Q\big(t^3\big)^3\big)^{n_1} \end{aligned}$$

$$+\sum_{n_1=2}^{\infty}\left(\sum_{n_2=2}^{n_1}\frac{(-4(n_1-n_2+1)(n_2-1))}{(3n_1-3n_2+2)!(3n_2-4)!}N_{n_1-n_2+1}N_{n_2-1}\right)\left(Q\left(t^3\right)^3\right)^{n_1}$$
$$+\sum_{n_1=2}^{\infty}\left(\sum_{n_2=2}^{n_1}\frac{(-9(n_1-n_2+1)^2(n_2-1)^2)}{(3n_1-3n_2+2)!(3n_2-4)!}N_{n_1-n_2+1}N_{n_2-1}\right)\left(Q\left(t^3\right)^3\right)^{n_1}. \tag{A.10}$$

On another hand, setting $A_n^k=\tilde{A}_n\bigl(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\bigr)^{nk}$ and using the first equation of (A.3), we have the first equation of the left-hand side of (A.8) can be written as

$$f_1+f_2+f_3=\sum_{n=1}^{\infty}\tilde{A}_n\Bigl(\bigl(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\bigr)^n+\bigl(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\bigr)^{2n}+\bigl(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\bigr)^{3n}\Bigr)\bigl(Q^{\frac{1}{3}}t^3\bigr)^n$$
$$=\sum_{n=1}^{\infty}\tilde{A}_n\left(1+\cos\left(\frac{2\pi n}{3}\right)\right)\bigl(Q^{\frac{1}{3}}t^3\bigr)^n=\sum_{n=1}^{\infty}3\tilde{A}_{3n}\bigl(Q\bigl(t^3\bigr)^3\bigr)^n. \tag{A.11}$$

Comparing equation (A.11) with (A.9) and (A.8), we have $\tilde{A}_{3n}=\frac{n^2N_n}{3(3n-1)!}$.

The second equation of the left-hand side of (A.8) can be written as

$$f_1f_2+f_2f_3+f_1f_3=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\tilde{A}_{n_1}\tilde{A}_{n_2}b_{n_1,n_2}\bigl(Q^{\frac{1}{3}}t^3\bigr)^{n_1+n_2}$$
$$=\sum_{n_1=2}^{\infty}\left(\sum_{n_2=2}^{n_1}\tilde{A}_{n_1-n_2+1}\tilde{A}_{n_2-1}b_{n_1-n_2+1,n_2-1}\right)\bigl(Q^{\frac{1}{3}}t^3\bigr)^{n_1}, \tag{A.12}$$

where $b_{n_1,n_2}=\bigl(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\bigr)^{n_1+2n_2}+\bigl(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\bigr)^{n_1}+\bigl(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\bigr)^{2n_1}$ using Lemma A.5 in the equation (A.12)

$$\sum_{n_1=2}^{\infty}\left(\sum_{n_2=2}^{n_1}\tilde{A}_{n_1-n_2+1}\tilde{A}_{n_2-1}b_{n_1-n_2+1,n_2-1}\right)\bigl(Q^{\frac{1}{3}}t^3\bigr)^{n_1}$$
$$=\sum_{n_1=1}^{\infty}\left(\sum_{n_2=2}^{3n_1}3\cos\left(\frac{2\pi(n_2-1)}{3}\right)\tilde{A}_{3n_1-n_2+1}\tilde{A}_{n_2-1}\right)\bigl(Q^{\frac{1}{3}}t^3\bigr)^{3n_1}$$
$$=\sum_{n_1=1}^{\infty}\left(\sum_{n_2=2}^{3n_1}3\cos\left(\frac{2\pi(n_2-1)}{3}\right)\tilde{A}_{3n_1-n_2+1}\tilde{A}_{n_2-1}\right)\bigl(Q\bigl(t^3\bigr)^3\bigr)^{n_1}. \tag{A.13}$$

Comparing (A.13) with (A.9),

$$\sum_{n_2=2}^{3n}3\cos\left(\frac{2\pi(n_2-1)}{3}\right)\tilde{A}_{3n-n_2+1}\tilde{A}_{n_2-1}=\bigl(6-15n-9n^2\bigr)\frac{N_n}{(3n-1)!}.$$

Using the triple infinite sum identity was used in

$$\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}C_{k_1,k_2,k_3}=\sum_{k_1=3}^{\infty}\left(\sum_{k_2=1}^{k_1-2}\sum_{k_3=1}^{k_1-k_2-1}C_{k_1-k_2-k_3,k_2,k_3}\right).$$

The third equation of the left-hand side of (A.8) can be written as

$$f_1f_2f_3=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\tilde{A}_{n_1}\tilde{A}_{n_2}\tilde{A}_{n_3}c_{n_1,n_2,n_3}\bigl(Q^{\frac{1}{3}}t^3\bigr)^{n_1+n_2+n_3}$$
$$=\sum_{n_1=3}^{\infty}\left(\sum_{n_2=1}^{n_1-2}\sum_{n_3=1}^{n_1-n_2-1}\tilde{A}_{n_1-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3}c_{n_1-n_2-n_3,n_2,n_3}\right)\bigl(Q^{\frac{1}{3}}t^3\bigr)^{n_1},$$

where $c_{n_1,n_2,n_3} = \big(e^{\frac{2\pi i}{3}}\big)^{n_1+2n_2+3n_3}$. Then

$$\begin{aligned}
&\sum_{n_1=3}^{\infty}\left(\sum_{n_2=1}^{n_1-2}\sum_{n_3=1}^{n_1-n_2-1}\tilde{A}_{n_1-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3}c_{n_1-n_2-n_3,n_2,n_3}\right)\big(Q^{\frac{1}{3}}t^3\big)^{n_1}\\
&\quad=\sum_{n_1=3}^{\infty}\left(\sum_{n_2=1}^{n_1-2}\sum_{n_3=1}^{n_1-n_2-1}\tilde{A}_{n_1-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3}\big(e^{\frac{2\pi i}{3}}\big)^{n_1+n_2+2n_3}\right)\big(Q^{\frac{1}{3}}t^3\big)^{n_1}\\
&\quad=\sum_{n_1=1}^{\infty}\left(\sum_{n_2=1}^{3n_1-2}\sum_{n_3=1}^{3n_1-n_2-1}3\cos\left(\frac{2\pi(n_2+2n_3)}{3}\right)\tilde{A}_{3n_1-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3}\right)\big(Q^{\frac{1}{3}}t^3\big)^{3n_1}\\
&\quad=\sum_{n_1=1}^{\infty}\left(\sum_{n_2=1}^{3n_1-2}\sum_{n_3=1}^{3n_1-n_2-1}3\cos\left(\frac{2\pi(n_2+2n_3)}{3}\right)\tilde{A}_{3n_1-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3}\right)\big(Q\big(t^3\big)^3\big)^{n_1}.
\end{aligned}\tag{A.14}$$

Comparing (A.14) with (A.10),

$$\begin{aligned}
&\sum_{n_2=1}^{3n-2}\sum_{n_3=1}^{3n-n_2-1}3\cos\left(\frac{2\pi(n_2+2n_3)}{3}\right)\tilde{A}_{3n-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3}\\
&\quad=\big(54-243n+243n^2\big)\frac{N_n}{(3n-1)!}+\delta_n,
\end{aligned}$$

where

$$\begin{aligned}
\delta_n &= \begin{cases}0 & \text{if } n=1,\\ \tilde{\delta}_n & \text{otherwise},\end{cases}\\
\tilde{\delta}_n &= \sum_{n_2=2}^{n}\frac{\big(6(n_2-1)-3(n-n_2+1)(n_2-1)^2\big)}{(3n-3n_2+2)!(3n_2-4)!}N_{n-n_2+1}N_{n_2-1}\\
&\quad+\sum_{n_2=2}^{n}\frac{(-4(n-n_2+1)(n_2-1))}{(3n-3n_2+2)!(3n_2-4)!}N_{n-n_2+1}N_{n_2-1}\\
&\quad+\sum_{n_2=2}^{n}\frac{\big(-9(n_1-n_2+1)^2(n_2-1)^2\big)}{(3n-3n_2+2)!(3n_2-4)!}N_{n-n_2+1}N_{n_2-1}.
\end{aligned}$$

■

Corollary A.7. *Let be u_1, u_2, u_3 the canonical coordinates of quantum cohomology as function of the Saito flat coordinates t^1, t^2, t^3, i.e.,*

$$u_k = t^1 + \frac{1}{t^3}\sum_{n=1}^{\infty}A_n^k\big(Q^{\frac{1}{3}}t^3\big)^n.$$

Then, the coefficients A_k are given recursively by

$$\begin{aligned}
\tilde{A}_{3n} &= \frac{n^2N_n}{(3n-1)!},\\
\tilde{A}_{3n-1} &= \frac{1}{3\tilde{A}_1}\left[\big(6-15n-9n^2\big)\frac{N_n}{(3n-1)!}-\sum_{n_2=3}^{3n-1}3\cos\left(\frac{2\pi(n_2-1)}{3}\right)\tilde{A}_{3n-n_2+1}\tilde{A}_{n_2-1}\right],\\
\tilde{A}_{3n-2} &= \frac{1}{9\tilde{A}_1^2}\left[\big(54-243n+243n^2\big)\frac{N_n}{(3n-1)!}+\delta_n\right.
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{n_3=2}^{3n-2} 3\cos\left(\frac{2\pi(1+2n_3)}{3}\right)\tilde{A}_{3n-1-n_3}\tilde{A}_1\tilde{A}_{n_3}\Bigg] \\
&- \frac{1}{9\tilde{A}_1^2}\left[\sum_{n_2=2}^{3n-3}\sum_{n_3=1}^{3n-n_2-1} 3\cos\left(\frac{2\pi(n_2+2n_3)}{3}\right)\tilde{A}_{3n-n_2-n_3}\tilde{A}_{n_2}\tilde{A}_{n_3}\right],
\end{aligned}
$$

where

$$
A_n^k = \tilde{A}_n\big(e^{\frac{2\pi i}{3}}\big)^{nk}, \qquad \delta_n = \begin{cases} 0 & \textit{if } n=1, \\ \tilde{\delta}_n & \textit{otherwise}, \end{cases}
$$

$$
\begin{aligned}
\tilde{\delta}_n = &\sum_{n_2=2}^{n} \frac{\big(6(n_2-1) - 3(n-n_2+1)(n_2-1)^2\big)}{(3n-3n_2+2)!(3n_2-4)!} N_{n-n_2+1}N_{n_2-1} \\
&+ \sum_{n_2=2}^{n} \frac{\big(-4(n-n_2+1)(n_2-1)\big)}{(3n-3n_2+2)!(3n_2-4)!} N_{n-n_2+1}N_{n_2-1} \\
&+ \sum_{n_2=2}^{n} \frac{\big(-9(n_1-n_2+1)^2(n_2-1)^2\big)}{(3n-3n_2+2)!(3n_2-4)!} N_{n-n_2+1}N_{n_2-1}.
\end{aligned}
$$

A.3 Coefficients of the cross ratio function

Lemma A.8. *Let the cross ratio function be defined by the formula*

$$
f\big(Q^{\frac{1}{3}}t^3\big) = \frac{u_3-u_1}{u_2-u_1} = \sum_{n=0}^{\infty} f_n\big(Q^{\frac{1}{3}}t^3\big)^n. \tag{A.15}
$$

Then, the Taylor series of (A.15) *is given by*

$$
f\big(Q^{\frac{1}{3}}t^3\big) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{A_{n+1-m}^3 - A_{n+1-m}^1}{A_1^2 - A_1^1}\right) C_{m,-1}(y_1,y_2,\dots,y_m)\big(Q^{\frac{1}{3}}t^3\big)^n,
$$

where $y_n = \frac{A_{n+1}^2 - A_{n+1}^1}{A_1^2 - A_1^1}$*, and* A_n *is defined in* (4.72).

Proof. The coefficients f_n (A.15) is obtained by substituting (4.72) in (A.15) and using Theorem A.4. Indeed,

$$
\begin{aligned}
\frac{u_3-u_1}{u_2-u_1} &= \frac{\sum_{n=1}^{\infty}\big(A_n^3 - A_n^1\big)\big(Q^{\frac{1}{3}}t^3\big)^n}{\sum_{n=1}^{\infty}\big(A_n^2 - A_n^1\big)\big(Q^{\frac{1}{3}}t^3\big)^n} \\
&= \left(\sum_{n=0}^{\infty}\left(\frac{A_{n+1}^3 - A_{n+1}^1}{A_1^2 - A_1^1}\right)\big(Q^{\frac{1}{3}}t^3\big)^n\right)\left(1 + \sum_{n=1}^{\infty}\left(\frac{A_{n+1}^2 - A_{n+1}^1}{A_1^2 - A_1^1}\right)\big(Q^{\frac{1}{3}}t^3\big)^n\right)^{-1} \\
&= \left(\sum_{n=0}^{\infty}\left(\frac{A_{n+1}^3 - A_{n+1}^1}{A_1^2 - A_1^1}\right)\big(Q^{\frac{1}{3}}t^3\big)^n\right)\left(\sum_{n=0}^{\infty} C_{n,-1}(y_1,y_2,\dots,y_n)\big(Q^{\frac{1}{3}}t^3\big)^n\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{A_{n+1}^3 - A_{n+1}^1}{A_1^2 - A_1^1}\right) C_{m,-1}(y_1,y_2,\dots,y_m)\big(Q^{\frac{1}{3}}t^3\big)^{n+m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{A_{n+1-m}^3 - A_{n+1-m}^1}{A_1^2 - A_1^1}\right) C_{m,-1}(y_1,y_2,\dots,y_m)\big(Q^{\frac{1}{3}}t^3\big)^n,
\end{aligned}
$$

where $y_n = \frac{A^2_{n+1} - A^1_{n+1}}{A^2_1 - A^1_1}$, and the following Double sum identity was used:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} C_{n-m,m}.$$

Lemma proved. ■

A.4 Coefficients of modular lambda function

Recall the Eisenstein series $E_2(\tau)$, $E_4(\tau)$, $E_6(\tau)$ at the point $\tau = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$ have the following values:

$$E_2\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big) = \frac{2\sqrt{3}}{\pi}, \qquad E_4\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big) = 0, \qquad E_6\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big) = \frac{\big(\Gamma\big(\frac{1}{3}\big)\big)^{18}}{4^6}. \tag{A.16}$$

Moreover, there exist the following relation between x, x' and Δ:

$$\frac{2^8}{3^3}(2\pi)^{-12}\Delta(\tau) = \frac{(x')^6}{x^4(x-1)^4}. \tag{A.17}$$

The special values of x at $\tau = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$ is

$$x\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big) = \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}. \tag{A.18}$$

Substituting (A.18), (A.16) in (A.17), we compute $x'\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\big)$. Computing

$$\frac{2^8}{3^3}(2\pi)^{-12}\frac{\mathrm{d}^n \log\Delta(\tau)}{\mathrm{d}\tau^n} = \frac{\mathrm{d}^n}{\mathrm{d}\tau^n}\left(\frac{(x')^6}{x^4(x-1)^4}\right)n, \tag{A.19}$$

we obtain in the left-hand side (A.19) a polynomial expression in terms of E_2, E_4, E_6 and in the right-hand side a rational expression of $\frac{\mathrm{d}^n x(\tau)}{\mathrm{d}\tau^n}$. Then, we can derive $\frac{\mathrm{d}^n x(\tau)}{\mathrm{d}\tau^n}$ for any order n by using the recursive relation (A.19).

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