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# MICROLOCAL ESTIMATES OF THE STATIONARY SCHRÖDINGER EQUATION IN SEMI-CLASSICAL LIMIT

*by*

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***Abstract.*** — We give a new proof for microlocal resolvent estimates for semi-classical Schrödinger operators, extending the known results to potentials with local singularity and to those depending on a parameter. These results are applied to the study of the stationary Schödinger equation with the approach of semi-classical measures. Under some weak regularity assumptions, we prove that the stationary Schrödinger equation tends to the Liouville equation in the semi-classical limit and that the associated semi-classical measure is unique with support contained in an outgoing region.

***Résumé* (Estimations microlocales de l'équation de Schrödinger stationnaire en limite semi-classique)**

Nous présentons une nouvelle démonstration pour les estimations microlocales de l'opérateur de Schrödinger semi-classique, qui permet de généraliser les résultats connus aux potentiels avec singularité locale et aux potentiels dépendant d'un paramètre. Nous appliquons ces résultats à l'étude de l'équation de Schödinger stationnaire par l'approche de mesure semi-classique. Sous des hypothèses faibles sur la régularité du potentiel, nous montrons que l'équation de Schrödinger stationnaire converge vers l'équation de Liouville en limite semi-classique et que la mesure semi-classique est unique et de support inclus dans une région sortante.

## 1. Introduction

Microlocal resolvent estimates for two-body Schrödinger operators were firstly studied by Isozaki and Kitada in [**19, 24**] for smooth potentials. These results are useful in the study of scattering problems. For semi-classical Schrödinger operators, under a non-trapping assumption on the classical Hamiltonian, microlocal resolvent estimates were obtained in [**36**]. The method of [**36**] consists in comparing the total resolvent with the free one, using the global parametrix in form of Fourier integral operators. Here we want to give a more elementary proof of such results which allows to treat

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potentials with local singularity or depending on a parameter. We will apply these estimates to study the semi-classical measure of stationary Schrödinger equation, which is motivated by the recent works on the high frequency Helmholtz equation with a source term having concentration or concentration-oscillation phenomena.

Let $P(h) = -h^2\Delta + V(x)$ with $V$ a smooth long-range potential verifying $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$ and for some $\rho > 0$

$$|\partial^\alpha V(x)| \le C_\alpha \langle x\rangle^{-\rho-|\alpha|}, \quad x \in \mathbb{R}^n, \tag{1.1}$$

for any $\alpha \in \mathbb{N}^n$. Here $h > 0$ is a small parameter and $\langle x\rangle = (1+|x|^2)^{1/2}$. $P(h)$ is self-adjoint in $L^2(\mathbb{R}^n)$. Let $R(z,h) = (P(h)-z)^{-1}$ for $z \notin \sigma(P(h))$. Let $b_\pm(.,.)$ be bounded smooth symbols with supp $b_\pm \subset \{(x,\xi) \in \mathbb{R}^{2n};\ \pm x\cdot\xi > -(1-\epsilon)|x||\xi|\}$ for some $\epsilon > 0$. Denote by $b_\pm(x,hD)$ the $h$-pseudo-differential operators with symbol $b_\pm$ defined by

$$(b_\pm(x,hD)u)(x) = \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^n} e^{ix\cdot\xi/h} b_\pm(x,\xi)\hat{u}(\xi)\ d\xi, \tag{1.2}$$

where $u \in \mathcal{S}(\mathbb{R}^d)$ and $\hat{u}$ is the Fourier transform of $u$. We denote by $b^w(x,hD)$ the Weyl quantization of $b$

$$(b^w_\pm(x,hD)u)(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi/h} b_\pm((x+y)/2,\xi)u(y)\ d\xi dy. \tag{1.3}$$

At the level of principal symbols in the semi-classical limit $h \to 0$, the two quantizations are equivalent.

Let $p(x,\xi)$ denote the classical Hamiltonian $p(x,\xi) = \xi^2 + V(x)$ and

$$t \to (x(t;y,\eta),\xi(t;y,\eta))$$

be solutions of the Hamiltonian system associated with $p(x,\xi)$:

$$\left\{\begin{array}{rcl} \frac{\partial x}{\partial t} &=& \partial_\xi p(x,\xi), \quad x(0;y,\eta) = y, \\ \frac{\partial \xi}{\partial t} &=& -\partial_x p(x,\xi), \quad \xi(0;y,\eta) = \eta. \end{array}\right. \tag{1.4}$$

$E > 0$ is called a non-trapping energy for the classical Hamiltonian $p(x,\xi) = |\xi|^2 + V(x)$ if

$$\lim_{|t|\to\infty} |x(t;y,\eta)| = \infty, \quad \forall\ (y,\eta) \in p^{-1}(E). \tag{1.5}$$

The one-sided microlocalized resolvent estimate says that if $E > 0$ is a non-trapping energy, then one has for any $s > 1/2$

$$\|\langle x\rangle^{s-1} b_\mp(x,hD)R(E\pm i0,h)\langle x\rangle^{-s}\| \le C_s h^{-1} \tag{1.6}$$

uniformly in $h > 0$ small enough. Here

$$R(E\pm i0,h) = \lim_{\epsilon\downarrow 0}(P(h)-E\mp i\epsilon)^{-1},$$

and $\|\cdot\|$ denotes the norm of bounded operators on $L^2(\mathbb{R}^n)$. Recall that without microlocalization, one can only have an estimate like

$$\|\langle x\rangle^{-s}R(E\pm i0,h)\langle x\rangle^{-s}\|\leq C_s h^{-1}. \tag{1.7}$$

See [**33**]. With microlocalization, one can overcome some difficulties related to the lack of decay. There are also two-sided microlocal resolvent estimates in semi-classical limit. See [**37**] for potentials satisfying (1.1).

The recent interest in uniform resolvent estimates arises from the study of propagation of semi-classical measure related to the high frequency Helmoltz equation. Recall that the Helmholtz equation describes the propagation of light wave in material medium. It appears in the design of very high power laser devices such as Laser Méga-Joule in France or the National Ignition Facility in the USA. The laser field, $A(x)$, can be very accurately modelled and computed by the solution of the Helmholtz equation

$$\Delta A(x)+k_0^2(1-N(x))A(x)+ik_0\nu(x)A(x)=0 \tag{1.8}$$

where $k_0$ is the wave number of laser in vacuum, $N(x)$ is a smooth positive function representing the electronic density of material medium and $\nu(x)$ is positive smooth function representing the absorption coefficient of the laser energy by material medium. Since laser can not propagate in the medium with the electronic density bigger than 1, it is assumed that $0\leq N(x)<1$. The equation (1.8) may be posed in an unbounded domain with a known incident excitation $A_0$. The equation is then complemented by a radiation condition. The highly oscillatory behavior of the solution to the Helmholtz equation makes the numerical resolution of (1.8) unstable and rather expensive. See [**3**]. Fortunately, the wave length of laser in vacuum, $\frac{2\pi}{k_0}$, is much smaller than the scale of $N$. It is therefore natural and important to study the Helmholtz equation in the high frequency limit $k_0\to\infty$. To be simple, instead of studying boundary value problem related to a non-self-adjoint operator, one studies the high frequency Helmholtz equation with a source term

$$(\Delta+\epsilon^{-2}n(x)^2+i\epsilon^{-1}\alpha_\epsilon)u_\epsilon(x)=-S_\epsilon(x) \tag{1.9}$$

in $\mathbb{R}^d$, $d\geq 1$. Here $n(x)$ is the refraction index, $\epsilon\sim\frac{1}{k_0}>0$ is regarded as a small parameter, $\alpha_\epsilon\geq 0$ and

$$\lim_{\epsilon\to 0}\alpha_\epsilon=\alpha\geq 0. \tag{1.10}$$

In [**4, 8, 40**], $\alpha_\epsilon$ is assumed to be a regularizing parameter :

$$\text{if } \alpha=0, \exists\gamma\in]0,1[ \text{ such that } \alpha_\epsilon\geq\epsilon^\gamma. \tag{1.11}$$

Motivated by this model, we study in this work the Schrödinger equation

$$(-h^2\Delta+V(x)-(E+i\kappa))u_h=S^h(x) \tag{1.12}$$

by the Wigner's approach or the approach of semi-classical measures. Here $E>0$, $\kappa=\kappa(h)\geq 0$ and $\kappa\to 0$ as $h\to 0$. To prove the existence of a limiting Liouville equation when $h \to 0$, we assume that $\alpha_h = \kappa h^{-1}$ satisfies (1.10) with $\epsilon = h \to 0$. The condition (1.11) is not needed in this work: when $\kappa = 0$, $u_h$ is defined as the unique outgoing (or incoming) solution of (1.12) for each $h \in ]0,1]$. Note that (1.12) is a scattering problem, since the behavior of $(-h^2\Delta + V - (E + i\kappa))^{-1}$ for $\kappa$ near 0 is closely related to the long-time behavior of the unitary group

$$U(t,h) = e^{-itP(h)/h}$$

as $t \to \infty$.

In the study of semi-classical measures associated to $u_h$, the uniform resolvent estimate plays an important role. See [**4, 9, 8, 10, 40**]. Under some technical conditions, the microlocal estimates are used in [**40**] to overcome the difficulty due to the lack of decay for the source term with concentration-oscillation over a hyperplane.

In these notes, we recall in Section 2 some abstract results on the uniform limiting absorption principle. In Section 3, we give a new proof of microlocal resolvent estimates in the semi-classical limit, using the Mourre's method and symbolic calculus of $h$-pseudo-differential operators. For fixed $h$, related ideas have appeared in [**12, 21, 34, 38**]. Our approach combines these ideas and the method used in the semi-classical resolvent estimates [**11, 13, 33, 38**]. The same ideas can be applied to potentials with singularities and potentials depending on a parameter. In Subsection 4.3, we apply the results on uniform resolvent estimates to the study of the equation (1.12) with the second hand side concentrated near one point. We prove that the outgoing solution of (1.12), when microlocalized in an incoming region, is uniformly bounded in $L^2$. The convergence of (1.12) to the limiting Liouville equation is proved under the assumption on the uniform continuity of $V$, $\nabla V$ and $x \cdot \nabla V$. The microlocal estimates for (1.12) give rise to some strong radiation property of the semi-classical measure associated with $u_h$, from which we derive the uniqueness of the semi-classical measure. The decay of the potential $V$ is not needed. The results of Subsection 4.3 hold for a large class of $N$-body potentials of the form

$$V(x) = \sum_a V_a(x^a),$$

where $x^a$ is part of the variables $x \in \mathbb{R}^d$.

The pre-requests of these lecture notes are contained in the books [**18, 31**] and [**32**]. The symbolic and functional calculi for $h$-pseudo-differential operators will be frequently used and can be found in [**31**]. To be self-contained, some known results are recalled here. In particular, the results of Section 2 are contained in a joint work with P. Zhang [**40**] and those of Subsections 4.1 and 4.2 are based on [**14, 16, 26**].

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## 2. Some abstract results

**2.1. Mourre's method depending on a parameter.** — We first state a parameter dependent version of Mourre's method which is an important tool in quantum scattering theory. Given two families $\{P_\epsilon\}$, $\{A_\epsilon\}$, $\epsilon \in ]0,1]$, in some Hilbert space $H$, we shall say $A_\epsilon$ is uniformly conjugate operator of $P_\epsilon$ on an interval $I \subset \mathbb{R}$ if the following properties are satisfied:

1. Domains of $P_\epsilon$ and $A_\epsilon$ are independent of $\epsilon$: $D(P_\epsilon) = D_1, D(A_\epsilon) = D_2$. For each $\epsilon$, $D = D_1 \cap D_2$ is dense in $D_1$ in the graph norm
$$\|u\|_{\Gamma_\epsilon} = \|P_\epsilon u\| + \|u\|.$$
2. The unitary group $e^{i\theta A_\epsilon}$, $\theta \in \mathbb{R}$ is bounded from $D_1$ into itself and
$$\sup_{\epsilon\in]0,1],|\theta|\leq 1} \|e^{i\theta A_\epsilon}u\|_{\Gamma_\epsilon} < \infty, \quad \forall u \in D_1.$$
3. The quadratic form $i[P_\epsilon, A_\epsilon]$ defined on $D$ is bounded from below and extends to a self-adjoint operator $B_\epsilon$ with $D(B_\epsilon) \supset D_1$ and $B_\epsilon$ is uniformly bounded from $D_1$ to $H$, *i.e.* $\exists C > 0$ such that
$$\|B_\epsilon u\| \leq C\|u\|_{\Gamma_\epsilon}, \quad u \in D_1$$
uniformly in $\epsilon$.
4. The quadratic form defined by $i[B_\epsilon, A_\epsilon]$ on $D$ extends to a uniformly bounded operator from $D_1$ to $H$.
5. (Uniform Mourre's estimate) There is $m_\epsilon > 0$ such that

$$E_I(P_\epsilon)i[P_\epsilon, A_\epsilon]E_I(P_\epsilon) \geq m_\epsilon E_I(P_\epsilon) \tag{2.13}$$

Remark that the usual Mourre's estimate is of the form

$$E_I(P)i[P, A]E_I(P) \geq E_I(P)(c_0 + K)E_I(P), \tag{2.14}$$

for some $c_0 > 0$ and $K$ a compact operator. If $E \notin \sigma_p(P)$, $E_I(P)$ tends to 0 strongly, as the length of $I$ tends to 0. So, one can take $\delta > 0$ small enough so that $E_I(P)i[P, A]E_I(P) \geq c_1 E_I(P)$ for $I = [E - \delta, E + \delta]$ with $\delta > 0$ sufficiently small and for some $c_1 > 0$. For Mourre's method independent of parameter, see [**21, 22, 27, 28**] and also [**2**] for more information.

In some estimates, we need the following condition on multiple commutators:

(2.15) $\quad (P_\epsilon + i)^{-1}B_j(\epsilon)(P_\epsilon + i)^{-1}$ extends to uniformly bounded operators on $H$

for $1 \leq j \leq n$, $n \in \mathbb{N}^*$. Here $B_0(\epsilon) = B_\epsilon$ and $B_j(\epsilon) = [B_{j-1}(\epsilon), A_\epsilon]$ for $j \geq 1$. The following parameter-dependent estimates are useful in many situations.

***Theorem 2.1***. — (The uniform limiting absorption principle) *Assume that $A_\epsilon$ is a uniform conjugate operator of $P_\epsilon$ on $I = ]a, b[$. Let $R_\epsilon(z) = (P_\epsilon - z)^{-1}$ and $E \in I$.*

(i). *For any $s > 1/2$, and $\delta > 0$, there exists $C > 0$ such that*

$$\|\langle A_\epsilon\rangle^{-s} R_\epsilon(E \pm i\kappa)\langle A_\epsilon\rangle^{-s}\| \le C m_\epsilon^{-1} \tag{2.16}$$

*Assume in addition that the condition (2.15) is satisfied for some $n \ge 2$. One has the following estimates*

(ii). *Let $c_\pm \in \mathbb{R}$ and let $\chi_\pm$ denote the characteristic functions of $]-\infty, c_-[$ and $]c_+, +\infty[$, respectively. For any $1/2 < s < n$, there exists $C > 0$ such that*

$$\|\langle A_\epsilon\rangle^{s-1} \chi_\mp(A_\epsilon) R_\epsilon(E \pm i\kappa)\langle A_\epsilon\rangle^{-s}\| \le C m_\epsilon^{-1}. \tag{2.17}$$

(iii). *For any $r, s \in \mathbb{R}$, with $(r)_+ + (s)_+ < n - 1$ and $(s)_+ = \max\{s, 0\}$, there is $C > 0$*

$$\|\langle A_\epsilon\rangle^{r} \chi_\mp(A_\epsilon) R_\epsilon(E \pm i\kappa)\chi_\pm(A_\epsilon)\langle A_\epsilon\rangle^{s}\| \le C m_\epsilon^{-1}. \tag{2.18}$$

*The above estimates are all uniform in $\epsilon, \kappa \in ]0, 1]$ and locally uniform for $E \in I$.*

(i) of Theorem 2.1 implies the point spectrum of $P_\epsilon$ is absent in $I$ and the spectrum of $P_\epsilon$ is absolutely continuous. The proof of Theorem 2.1 as stated is not written explicitly in the literature, but it can be derived by following the Mourre's original functional differential inequality method [**27**] and its subsequent improvement [**2, 13, 21, 37, 38**]. The conditions in parts (ii) and (iii) imply that for each $\epsilon$, $P_\epsilon$ is 2-smooth with respect to $A_\epsilon$ in sense of [**21**]. By the arguments of the above works, one sees that the boundary values

$$R_\epsilon(E \pm i0) = \lim_{\kappa \to 0_+} R_\epsilon(E \pm i\kappa)$$

exist in suitable topology and satisfy the same uniform estimates. As in the case of fixed $\epsilon$, one can state a similar version of Theorem 2.1 in terms of quadratic forms which allows to include stronger local singularity of potential in Schrödinger operators. See [**2**].

**2.2. Uniform resolvent estimates in Besov spaces.** — The Mourre's method can be used to obtain uniform resolvent estimates in Besov spaces for operators depending on a small parameter. This idea goes back to Mourre [**27, 28**] and was used in [**23, 42**] for operators without small parameter. One can follow the same idea in taking care of the dependence on the small parameter. See [**40**].

Let $F$ be a self-adjoint operator in $H$. Let $F_j$, $j \in \mathbb{N}$, denote the spectral projector of $F$ onto the set $\Omega_j$, where $\Omega_j = \{\lambda \in \mathbb{R}; 2^{j-1} \le |\lambda| < 2^j\}$ for $j \ge 1$ and $\Omega_0 = \{\lambda \in \mathbb{R}; |\lambda| < 1\}$. Introduce the abstract Besov spaces, $B_s(F)$, defined in terms of the operator $F$:

$$B_s(F) = \Big\{u \in H; \sum_{k=0}^{\infty} 2^{ks}\|F_k u\| < \infty\Big\}, \quad s \ge 0.$$

Its dual space $(B_s^F)^*$ with respect to the scalar product on $H$ is a Banach space with the norm given by

$$\|u\|_{B_s(F)^*} = \sup_{j\in\mathbb{N}} 2^{-js}\|F_j u\|.$$

When $F$ is equal to the multiplication by $|x|$, one recovers the usual Besov spaces denoted by $B_s$ and $B_s^*$. Note that in this case, the $B_s^*$-norm is equivalent with the norm

$$|||u|||_{B_s^*} = \sup_{R>1} \frac{1}{R^s}\Big\{\int_{|x|<R} |f(x)|^2\, dx\Big\}^{1/2}.$$

**Theorem 2.2 ([40]).** — *Let $P_\epsilon$ and $A_\epsilon$ be two families of self-adjoint operators in $H$. Assume that $A_\epsilon$ is uniformly conjugate to $P_\epsilon$ on an interval $I = ]a,b[$ and that $(P_\epsilon + i)^{-1}[[B_\epsilon, A_\epsilon], A_\epsilon](P_\epsilon + i)^{-1}$ extends to uniformly bounded operator on $H$. Let $E \in I$ and $s \geq \frac{1}{2}$. One has:*

$$\|R_\epsilon(E \pm i\kappa)\|_{\mathcal{L}(B_s(A_\epsilon), B_s(A_\epsilon)^*)} \leq C m_\epsilon^{-1} \tag{2.19}$$

*uniformly in $0 < \epsilon, \kappa < 1$. Here $m_\epsilon$ is the constant in the uniform Mourre estimate (2.13) and $R_\epsilon(z) = (P_\epsilon - z)^{-1}$.*

Let $l^{2,\infty}$ denote the space of measurable functions $g(t)$ on $\mathbb{R}$ such that

$$\|g\|_{2,\infty} = \Big\{\sum_{k\in\mathbb{Z}} |g|_k^2\Big\}^{\frac{1}{2}}$$

where $|g|_k =$ ess sup $\{|g(t)|; k \leq t < k+1\}$, $k \in \mathbb{Z}$. The following result with $\epsilon = 1$ is due to [**23**].

**Proposition 2.3.** — *Let $f_1, f_2 \in l^{2,\infty}$.*

$$\|f_1(A_\epsilon)R_\epsilon(E \pm i\kappa)f_2(A_\epsilon)\| \leq C m_\epsilon^{-1}\|f_1\|_{2,\infty}\|f_2\|_{2,\infty}, \tag{2.20}$$

*uniformly in $0 < \kappa < 1$.*

*Proof.* — We follow the Mourre's argument used in the proof of (III) of Theorem 1.2 in [**28**] (see also [**23**]), checking the $\epsilon$-dependence. Let $\chi_n$ ($\chi_\pm$, resp.) denote the characteristic function of $[n, n+1[$, $n \in \mathbb{Z}$, ($[0, +\infty[$, $]-\infty, 0[$, resp.). Then for $u$, $v \in H$,

$$\begin{aligned}
&| (f_1(A_\epsilon)R_\epsilon(E \pm i\kappa)f_2(A_\epsilon)u,\ v) | \\
&\leq \sum_{n,m\in\mathbb{Z}} |f_1|_n |f_2|_m \|\chi_n(A_\epsilon)v\| \, \|\chi_m(A_\epsilon)u\| \, \|\chi_n(A_\epsilon)R_\epsilon(E \pm i\kappa)\chi_m(A_\epsilon)\| \\
&\leq \|u\| \, \|v\| \, \|f_1\|_{2,\infty}\|f_2\|_{2,\infty} \sup_{n,m\in\mathbb{Z}} \|\chi_n(A_\epsilon)R_\epsilon(E \pm i\kappa)\chi_m(A_\epsilon)\|.
\end{aligned}$$

It remains to prove

$$\sup_{n,m} \|\chi_n(A_\epsilon)R_\epsilon(E \pm i\kappa)\chi_m(A_\epsilon)\| \leq Cm_\epsilon^{-1} \tag{2.21}$$

uniformly in $\kappa \in ]0, 1]$. Note that $A_\epsilon - n$ is still a conjugate operator of $P_\epsilon$ satisfying the uniform Mourre's estimate with the same lower bound. Theorem 2.1 (i) with $A_\epsilon p$ replaced by $A_\epsilon - n$ gives that

$$\|\chi_n(A_\epsilon)R_\epsilon(E \pm i\kappa)\chi_n(A_\epsilon)\| \leq Cm_\epsilon^{-1}$$

uniformly in $n$ and $\kappa$. Decompose $\chi_n(A_\epsilon)R_\epsilon(E + i\kappa)\chi_m(A_\epsilon)$ as

$$\begin{aligned}
&\chi_n(A_\epsilon)R_\epsilon(E + i\kappa)\chi_m(A_\epsilon) \\
&= \chi_n(A_\epsilon)\{\chi_-(A_\epsilon - m)R_\epsilon(E + i\kappa) + \chi_+(A_\epsilon - m)R_\epsilon(E - i\kappa) \\
&\quad +2i\kappa\chi_+(A_\epsilon - m)R_\epsilon(E - i\kappa)R_\epsilon(E + i\kappa)\}\chi_m(A_\epsilon)
\end{aligned}$$

The first two terms can be bounded by $Cm_\epsilon^{-1}$ according to (2.17). For the third term, note that the operator $R_\epsilon(E - i\kappa)R_\epsilon(E + i\kappa)$ is positive. the Cauchy's inequality applied to the positive quadratic form

$$\varphi \to < \varphi, R_\epsilon(E - i\kappa)R_\epsilon(E + i\kappa)\varphi >$$

implies that

$$\begin{aligned}
&| < \varphi, R_\epsilon(E - i\kappa)R_\epsilon(E + i\kappa)\psi > | \\
&\leq | < \varphi, R_\epsilon(E - i\kappa)R_\epsilon(E + i\kappa)\varphi > |^{1/2} | < \psi, R_\epsilon(E - i\kappa)R_\epsilon(E + i\kappa)\psi > |^{1/2}.
\end{aligned}$$

This shows

$$\begin{aligned}
&2\kappa\|\chi_n(A_\epsilon)R_\epsilon(E - i\kappa)R_\epsilon(E + i\kappa)\chi_m(A_\epsilon)\| \\
&\leq 2\|\chi_n(A_\epsilon)R_\epsilon(E + i\kappa)\chi_n(A_\epsilon)\|^{\frac{1}{2}} \|\chi_m(A_\epsilon)R_\epsilon(E + i\kappa)\chi_m(A_\epsilon)\|^{\frac{1}{2}} \\
&\leq Cm_\epsilon^{-1}
\end{aligned}$$

uniformly in $n, m$ and $\kappa$. (2.21) is proved. □

*Proof of Theorem 2.2.* — Let $f \in B_s(A_\epsilon)$. By Proposition 2.3, one has for $s \geq \frac{1}{2}$

$$\begin{aligned}
&2^{-js}\|F_j R(E \pm i\kappa) f\| \\
\leq\ & \sum_{k=0}^{\infty} 2^{-js}\|F_j R(E \pm i\kappa) F_k\| \|F_k f\| \\
\leq\ & C m_\epsilon^{-1} \sum_{k=0}^{\infty} 2^{-j(s-\frac{1}{2})} 2^{k/2} \|F_k f\| \leq C m_\epsilon^{-1} \|f\|_{B_s(A_\epsilon)},
\end{aligned}$$

uniformly in $\epsilon, \kappa$ and $j$. This proves Theorem 2.2. □

## 3. Uniform microlocal resolvent estimates

The purpose of this Section is to prove uniform microlocal resolvent estimates for a large class of Schrödinger operators depending on a parameter. In Subsection 3.1, we give a new proof of the result of [**36**]. The idea is to construct a uniform conjugate operator $F(h)$ in the form

$$F(h) = h(x \cdot D + D \cdot x)/2 + \mu s_{h,\tau}(x) + r^w(x, hD)$$

where $\mu$ and $\tau$ are to choose appropriately, and $r^w(x, hD)$ is an $h$-pseudo-differential operator with compactly supported symbol. It remains then to turn the spectral localizations of Theorem 2.1 into microlalizations. In Subsections 3.2 and 3.3, we show that the same ideas can be applied to potentials with repulsive Coulomb singularity and to potentials depending on a parameter.

**3.1. Microlocal estimates in semi-classical limit.** — An interesting application of the abstract results of Section 2 is the resolvent estimate of semi-classical Schrödinger operators $P(h) = -h^2\Delta + V(x)$ near a non-trapping energy. For two-body Schrödinger operators, under the non-trapping condition, the semi-classical resolvent estimate (1.7) was firstly proved in [**33**] by method of global parametrix. The necessity of non-trapping condition to obtain (1.7) was proved in [**35**]. Its proof based on Mourre's method was given in [**13**]. Since then, there are many extensions and new proofs, among which we mention an interesting proof using method of semi-classical measures (see [**6, 20**]). The same method is also used by Castella-Jecko in [**7**] to prove the semi-classical estimates in homogenous Besov (or Morrey-Campanato) spaces for $C^2$ potentials. This result is particularly useful in the study of concentration phenomenon. For N-body Schrödinger operators, the semi-classical resolvent estimate was proved in [**11**] for $N = 3$ and in [**37**] for general $N$. For microlocal resolvent estimates, see [**19, 21, 24, 12, 38**] for the case $h > 0$ is fixed and [**36**] in semi-classical limit.

Let $V \in C^\infty$ satisfying

$$|\partial_x^\alpha V(x)| \leq C_\alpha r(x) \langle x \rangle^{-|\alpha|}, \quad x \in \mathbb{R}^d, \quad \forall\ \alpha \in \mathbb{N}^{d_a}. \tag{3.22}$$

Here $r(x) \to 0$ as $x \to \infty$. Let $E \in \mathbb{R}_+$ such that

$$p \text{ is non-trapping at } E. \tag{3.23}$$

Under the assumptions (3.22) and (3.23), one can construct a uniform conjugate operator, $F(h)$, of $P(h)$ near $E$ in the form

$$F(h) = h(x \cdot D + D \cdot x)/2 + r^w(x, hD)$$

where $r^w(x, hD)$ is a self-adjoint bounded smoothing semi-classical pseudo-differential operator and one has

$$i\chi(P(h))[P(h), F(h)]\chi(P(h)) \geq c_0 h\chi(P(h))^2, \quad h \in ]0, 1], \tag{3.24}$$

where $c_0 > 0$ is independent of $h$ and $\chi$ is a smooth real function on $\mathbb{R}$ supported sufficiently near $E$. See [**13**]. From the abstract results of Section 2, one deduces easily the semi-classical resolvent estimates in Besov spaces.

***Theorem 3.1***. — *Let $s \geq \frac{1}{2}$. Under the assumptions (3.22) and (3.23), one has:*

$$\|R(E \pm i\kappa, h)\|_{\mathcal{L}(B_s, B_s^*)} \leq Ch^{-1} \tag{3.25}$$

*uniformly in* $0 < h, \kappa < 1$.

*Proof*. — Let $F(h)$ be fined above. Theorem 2.2 is true with $A_\epsilon$ replaced by $F(h)$ and $m_\epsilon$ by $h$. Let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(t) = 1$ for $t$ near $E$. $(1 - \chi(P(h))^2)R(E \pm i\kappa, h)$ is uniformly bounded in $\mathcal{L}(L^2, L^2)$, therefore also in $\mathcal{L}(B_s, B_s^*)$. Note that $F(h)$ is a semi-classical pseudo-differential operator with the Weyl symbol $x \cdot \xi + r(x, \xi)$ where $r$ is a bounded symbol. We can show that for $s \geq 0$,

$$\|\langle F(h)\rangle^s \chi(P(h))\langle x\rangle^{-s}\| \leq C \tag{3.26}$$

uniformly in $h$. An argument of interpolation ([**1, 18**]) gives then

$$\|\chi(P(h))\|_{\mathcal{L}(B_s, B_s(F(h))} \leq C$$

uniformly in $h$. By the duality, the same is true for $\chi(P(h))$ as operator from $(B_s^F)^*$ to $B_s^*$. It follows that

$$\|\chi(P(h))^2 R(E \pm i\kappa, h)\|_{\mathcal{L}(B_s, B_s^*)} \leq Ch^{-1}.$$

(3.25) follows from Theorem 2.2. ☐

Denote by $S_\pm$ the class of bounded symbols $a_\pm$ on $\mathbb{R}^{2d}$ satisfying, for some $\delta_\pm > 0$,

$$\text{supp} a_\pm \subset \{(x, \xi); \pm x \cdot \xi \geq -(1 - \delta_\pm)|x||\xi|\}, \tag{3.27}$$

and

$$a_\pm \in C^\infty(\mathbb{R}^{2d}), \ |\partial_x^\alpha \partial_\xi^\beta a_\pm(x, \xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|}.$$

For $\mu \in \mathbb{R}$, we denote by $S_\pm(\mu)$ the class of bounded symbols $a_\pm$ on $\mathbb{R}^{2d}$ satisfying

$$\text{supp} a_\pm \subset \{(x, \xi); \pm x \cdot \xi \geq \pm\mu\langle x\rangle\}, \tag{3.28}$$

and the same estimates on the derivatives. A family of symbols $a(h), h \in ]0, h_0]$, is said in the class $S_\pm$ or $S_\pm(\mu_\pm)$ if for any $N$, $a(h)$ admits an expansion of the form

$$a(h) = \sum_{j=0}^{N} h^j a_j + h^{N+1} r_{N+1}(h)$$

where each $a_j$ satisfies support properties required above and

$$|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)| \le C_{\alpha\beta} \langle x\rangle^{-j-|\alpha|} \langle \xi\rangle^{-j-|\beta|}, \quad \forall \alpha, \beta$$

and

$$|\partial_x^\alpha \partial_\xi^\beta r_N(x,\xi,h)| \le C_{\alpha\beta} \langle x\rangle^{-N-1-|\alpha|} \langle \xi\rangle^{-N-1-|\beta|}, \quad \forall \alpha, \beta$$

uniformly in $h$.

***Theorem 3.2***. — *Assume (3.22) and (3.23). Then one has the following estimates uniformly in $\kappa \in ]0,1]$ and $h > 0$ small enough.*

(i). *Let $b_\pm \in S_\pm$. For any $s > 1/2$, there exists $C > 0$ such that*

$$\| \langle x\rangle^{s-1} b_\mp(x, hD) R(E \pm i\kappa, h) \langle x\rangle^{-s} \| \le C h^{-1} \tag{3.29}$$

(ii). *Let $b_\pm \in S_\pm$ for some $\delta_\pm > 0$ such that $\delta_- + \delta_+ > 2$. Then for any $s, r \in \mathbb{R}$, there exists $C > 0$ such that*

$$\| \langle x\rangle^{s} b_\mp(x, hD) R(E \pm i\kappa, h) b_\pm(x, hD) \langle x\rangle^{r} \| \le C h^{-1} \tag{3.30}$$

The first step of the proof is to construct an appropriate uniform conjugate operator, combining ideas from [**13, 11, 37**] and [**12, 21, 34, 38**]. Let $\mu \in \mathbb{R}$, $\tau > 0$. Put $\tau' = \tau h$. Define the parameter-dependent function $s = s_{\tau,h}$ by

$$s(x) = \frac{x^2}{(x^2 + \tau'^2)^{1/2}}. \tag{3.31}$$

$\tau > 0$ is to be taken small enough. An additional parameter $\mu$ is used in order to obtain microlocal estimates with support as large as possible. See the proof of Corollary 3.6 for its choice.

***Lemma 3.3***. — *For any $\epsilon > 0$, there is $\tau_0$ such that*

$$i[-h^2\Delta, \mu s(x)] \ge -h(\mu^2(1+\epsilon) - h^2\Delta), \quad \forall\, h \in ]0,1] \tag{3.32}$$

*uniformly in $0 < \tau \le \tau_0$ and $\mu \in \mathbb{R}$.*

*Proof*. — We have:

$$\begin{aligned} i[-h^2\Delta, \mu s(x)] &= \mu h(\nabla s(x) \cdot hD + hD \cdot \nabla s(x)) \\ &\ge -h(-\mu^2(1+\sigma)|\nabla s(x)|^2 - \frac{1}{1+\sigma} h^2\Delta), \end{aligned}$$

where $\sigma$ is a positive number to be adjusted below. An easy calculation gives:

$$|\nabla s(x)|^2 = \frac{x^2(x^2 + 2\tau'^2)^2}{(x^2+\tau'^2)^3} \le 1 + \frac{x^2\tau'^2}{(x^2+\tau'^2)^2} \le 5/4. \tag{3.33}$$

For $|x| \geq R\tau'$, $|\nabla s(x)|^2 \leq 1 + R^{-2}$. Consequently,

$$\| |\nabla s|\ u\|^2_{L^2(|x|\geq R\tau')} \leq (1 + R^{-2})\|u\|^2.$$

Let $\rho \in C_0^\infty$ with $\rho(x) = 1$ for $|x| \leq 1$. Recall the following Hardy inequality

$$\| |x|^{-s} u\| \leq C_s \|u\|_{\dot{H}^s}, s \in ]0, d/2[, \tag{3.34}$$

where $\dot{H}^s$ is the homogeneous Sobolev space of the order $s$ equipped with the norm

$$\|v\|_{\dot{H}^s} = \{\int |\xi|^{2s} |\hat{v}(\xi)|^2 d\xi\}^{1/2},$$

and $\hat{v}$ is the Fourier transform of $v$. One can derive from (3.34) that for some $s' \in ]0, 1/2[$

$$\|(-\Delta + 1)^{-1/2} \rho(x/\eta)(-\Delta + 1)^{-1/2}\|_{\mathcal{L}(L^2)} \leq C\eta^{s'}.$$

By a dilation, we obtain

$$\|(-h^2\Delta + 1)^{-1/2} \rho(x/(\eta h))(-h^2\Delta + 1)^{-1/2}\|_{\mathcal{L}(L^2)} \leq C\eta^{s'},$$

uniformly in $h > 0$. For $|x| < R\tau'$ and $u \in \mathcal{D}(-\Delta)$, we can apply the above estimate to obtain that

$$\|(\nabla s)u\|^2_{L^2(|x|<R\tau')} \leq 5/4 \|\rho(x/(R\tau h)u\|^2 \leq C(R\tau)^{s'} < (-h^2\Delta + 1)u, u >$$

for some $s' > 0$. Therefore

$$< |\nabla s(x)|^2 u, u > \leq (1 + R^{-2} + C(R\tau)^{s'})\|u\|^2 + C(R\tau)^{s'} < -h^2\Delta u, u > .$$

This proves that

$$i[-h^2\Delta, \mu s(x)] \geq -h\Big( - \mu^2(1+\sigma)(1 + R^{-2} + C(R\tau)^{s'}) - \frac{1 + C(R\tau)^{s'}}{1+\sigma} h^2\Delta\Big).$$

Now taking $\sigma = C(R\tau)^{s'}$, (3.32) follows by choosing $R = R(\epsilon)$ large enough and $\tau_0 = \tau_0(R, \epsilon)$ small enough. ☐

Set

$$A_\mu(h) = A(h) + \mu s(x), \quad A(h) = h(x \cdot D + D \cdot x)/2. \tag{3.35}$$

A nice property of $A_\mu(h)$ is that for any $\mu \in \mathbb{R}$, $A(\mu)$ is unitarily equivalent with $A(h)$:

$$A_\mu(h) = e^{-i\frac{\mu}{h}(x^2+\tau'^2)^{1/2}} A(h) e^{i\frac{\mu}{h}(x^2+\tau'^4 2)^{1/2}}. \tag{3.36}$$

***Proposition 3.4***. — *Under the assumptions (3.22) and (3.23), for any $\mu \in \mathbb{R}$ with $|\mu| < \sqrt{E}$, there exists $r \in C_0^\infty(\mathbb{R}^{2d})$ and $\tau > 0$ small enough such that*

$$F(h) = A_\mu(h) + r^w(x, hD) \tag{3.37}$$

*is a uniform conjugate operator of $P(h)$ at the energy $E$ (with $P_\epsilon = P(h)$ and $A_\epsilon = F(h)$ in notation of Section 2) with the estimate*

$$iE_I(P(h))[P(h), F(h)]E_I(P) \geq chE_I(P(h)), \tag{3.38}$$

*and (2.15) is satisfied for any $n$. Here $c > 0$, $I = ]E - \delta_0, E + \delta_0[$ for some $\delta_0 > 0$ and $E_I(P(h))$ denotes the spectral projection of $P(h)$ onto the interval $I$.*

*Proof.* — One has the formula

$$i[P(h), A_\mu(h)] = h(2P(h) - 2V - x\cdot\nabla V) + i[-h^2\Delta, \mu s].$$

By (3.22) and Lemma 3.3, for $P(h)$ localized near $E$, $\mu^2 < E$ and $|x| > R_0$ with $R_0 = R_0(\mu)$ large enough, we can take $\tau > 0$ small enough such that

$$i[P(h), A_\mu(h)] \geq ch > 0.$$

Making use of the non-trapping condition, we can construct as in [**13**] a smooth function, $r$, with compact support such that $F(h) = A_\mu(h) + r^w(x, hD)$ is a uniform conjugate operator of $P(h)$ near $E$. More explicitly, let $\delta > 0$ be small enough such that the condition (3.23) remains true for any energy in $]E-2\delta, E+2\delta[$. Let $g \in C_0^\infty$ with $0 \leq g \leq 1$ and $g(x) = 1$ for $|x| \leq 1$ , 0 for $|x| > 2$. Set

$$r(y,\eta) = \chi_1(p(y,\eta))R_2 g(\frac{x}{R_2^2}) \int_0^\infty g\Big(\frac{x(t;y,\eta)}{R_1}\Big)\,dt.$$

Here $\chi_1 \in C_0^\infty(]E-2\delta, E+2\delta[)$ and is equal to 1 on $[E-\delta, E+\delta]$. For $R_1, R_2$ large enough, one can estimate the Poisson bracket

$$\{p(x,\xi), x\cdot\xi + \mu s(x) + r(x,\xi)\} \geq c > 0$$

for all $(x,\xi) \in p^{-1}([E-\delta, E+\delta])$. Let $\chi \in C_0^\infty(]E-\delta, E+\delta[)$ , equal to 1 near $E$. By the result on functional calculus of $h$-pseudo-differential operators, $\chi(P(h))$ is an $h$-pseudo-differential operator with the principal symbol $\chi(p(x,\xi))$. See [**31**]. One can estimate that

$$i\chi(P(h)[P(h), F(h)]\chi(P(h)) \geq \frac{c}{2}h\chi(P(h))^2$$

for $h > 0$ small enough. The lower bound in (3.38) follows. Since $r$ is of compact support and $A_\mu(h)$ is unitarily equivalent with $A(h)$, the other conditions for uniform conjugate operator can easily verified. In particular, remark that $s$ is $h$-dependent. One has the control $\partial^\alpha\nabla s(x) = O(h^{-|\alpha|})$, or equivalently, $(h\partial)^\alpha\nabla s(x) = O(1)$ uniformly in $x$ and $h$. We can check that (2.15) is verified for any $n$ uniformly in $h$. □

Theorem 2.1 shows that for any $s > 1/2$

$$\|\langle F(h)\rangle^{s-1}\chi_\mp(F(h))R(E\pm i\kappa, h)\langle F(h)\rangle^{-s}\| \leq Ch^{-1} \tag{3.39}$$

and for any $r, s \in \mathbb{R}$,

$$\|\langle F(h)\rangle^{r}\chi_\mp(F(h))R(E\pm i\kappa, h)\chi_\pm(F(h))\langle F(h)\rangle^{s}\| \leq Ch^{-1}, \tag{3.40}$$

uniformly in $\kappa \in ]0,1]$ and $h > 0$ small enough. It remains to convert spectral localizations into microlocalizations. The following Proposition is the main technical issue in this step. See also [**21**] for the special case $f = 0$ and $h = 1$.

***Proposition 3.5***. — *Let $\mu \in \mathbb{R}$ be the parameter used in the definition of $F(h)$ and let $b_\pm \in S_\pm(\mu_\pm)$ with supp $b_\pm \subset \{|x| \geq 1\}$. Then one has*

(i) *For any $\pm\mu_\pm > \mp\mu$, one has for any $s \geq 0$*

$$\|\langle x\rangle^s b_\pm(x, hD)\langle F(h)\rangle^{-s}\| \leq C \tag{3.41}$$

*uniformly in $h$.*

(ii) *Let $\chi_\pm \in C^\infty(\mathbb{R})$ with $\chi_+(r) = 0$ if $r < c_1$; $\chi_+(r) = 1$ if $r > c_2$ (resp., $\chi_-(r) = 0$ if $r > c_2$; $\chi_-(r) = 0$ if $r < c_1$ ) for some $c_1 < c_2$. For any $s_1, s_2 \in \mathbb{R}$, one has:*

$$\|\langle x\rangle^{s_1} b_\pm(x, hD)\chi_\mp(F(h))\langle F(h)\rangle^{s_2}\| \leq C \tag{3.42}$$

*uniformly in $h$.*

*Proof*

(i) Since $r$ is of compact support, $\langle F(h)\rangle^{-s}\langle A_\mu(h)\rangle^s$ is uniformly bounded. It suffices to prove (3.41) with $F(h)$ replaced by $A_\mu(h)$. Note that

$$A_\mu(h) = e^{-i\mu f(x)/h} A(h) e^{i\mu f(x)/h},$$

where $f(x) = (x^2 + \tau'^2)^{1/2}$.

Let $\chi(\cdot)$ be a cut-off function on $\mathbb{R}$ such that $\chi(t) = 1$, if $t \leq 4$; 0 if $t > 5$. Put:

$$b_{\pm,1}(x, \xi) = b_\pm(x, \xi)(1 - \chi(|\xi|/\langle\mu\rangle)), \ b_{\pm,2}(x, \xi) = b_\pm(x, \xi)\chi(|\xi|/\langle\mu\rangle).$$

Let us first consider $b_{\pm,1}$. Noticing that $A_\mu(h)$ is unitarily equivalent with $A(h)$, we obtain

$$\|\langle x\rangle^s b_{\pm,1}(x, hD)\langle A_\mu(h)\rangle^{-s}\| = \|\langle x\rangle^s b^\mu_\pm(x, hD; h)\langle A(h)\rangle^{-s}\|, \tag{3.43}$$

where

$$b^\mu_\pm(x, hD; h) = e^{i\mu f(x)/h} b_{\pm,1}(x, hD) e^{-i\mu f(x)/h}.$$

Writing $f(x) - f(y) = (x - y) \cdot \nabla f(x, y)$, we have:

$$\begin{aligned} b^\mu_\pm(x, hD; h)u(x) &= \frac{1}{(2\pi h)^d} \iint e^{\frac{i}{h}[(x-y)\cdot\xi + \mu(f(x)-f(y))]} b_{\pm,1}(x, \xi)u(y)\ d\xi dy \\ &= \frac{1}{(2\pi h)^d} \iint e^{\frac{i}{h}(x-y)\cdot\xi} b_{\pm,1}(x, \xi - \mu\nabla f(x, y))u(y)\ d\xi dy. \end{aligned}$$

Using the Taylor expansion of $b_{\pm,1}(x, \xi - \mu\nabla f(x, y))$ around $y = 0$, we obtain for any $M \in \mathbb{N}$:

$$b^\mu_\pm(x, hD; h) = \sum_{j=0}^{M} h^j c_{\pm,j}(x, hD) + h^{M+1} r_{\pm,M}(x, hD; h),$$

where

$$c_{\pm,j}(x, \xi) = \sum_{|\alpha|=j} C_\alpha \partial_y^\alpha D_\xi^\alpha b_{\pm,1}(x, \xi - \mu\nabla f(x, y))\ |_{y=0},\ j = 0, 1, \cdots, M.$$

Let us look at $c_{+,0} = b_{+,1}(x, \xi - \mu\nabla f(x))$ carefully. Assume without loss that $\mu_+ < 0$ and $\mu > 0$. By the choice of $b_{+,1}$,

$$\text{supp } b_{+,1} \subset \{x \cdot \xi \geq \mu_+|x|, |x| > 1 \text{ and } |\xi| \geq 4\langle\mu\rangle\}.$$

Consequently, the support of $c_{+,0}$ is contained in

$$\{x \cdot (\xi - \mu\nabla f(x)) \geq \mu_+|x|, |x| > 1 \text{ and } |\xi - \mu\nabla f(x)| \geq 4\langle\mu\rangle\}.$$

Recall that

$$x \cdot \nabla f(x) = s(x) \quad \text{ and } (1 - \tau'^2)^{1/2}|x| \leq s(x) \leq |x|$$

for $|x| > 1$ and $\tau' = \tau h$. On the support of $c_{+,0}$, one has for $\tau > 0$ small enough,

$$x \cdot \xi \geq (\mu_+ + (1 - \tau'^2)^{1/2}\mu)|x| \geq \delta|x|/2, \ |\xi| \geq 3\langle\mu\rangle$$

for some $\delta > 0$. This implies that on the support of $c_{+,0}$,

$$|\xi - \mu\nabla f(x)| \geq C(|\xi| + \langle\mu\rangle),$$

for some $C > 0$. Since $b_+ \in S_+(\mu_+)$, we can check that:

$$|\partial_x^\alpha \partial_\xi^\beta c_{+,0}(x, \xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}.$$

Similarly, we can verify that

$$|\partial_x^\alpha \partial_\xi^\beta c_{+,j}(x, \xi)| \leq C_{\alpha\beta}\langle x\rangle^{-j-|\alpha|}\langle\xi\rangle^{-j-|\beta|}, \text{ for } j = 1, \cdots, M. \tag{3.44}$$

To prove that $\|\langle x\rangle^s b_\pm^\mu(x, hD; h)\langle A(h)\rangle^{-s}\|$ is uniformly bounded, consider first the case $s = 1$. Setting $\langle x\rangle c_{+,j}(x, \xi) = c'_j(x, \xi)(x \cdot \xi + i)$ with

$$c'_j(x, \xi) = \frac{\langle x\rangle c_{+,j}(x, \xi)}{(x \cdot \xi + i)},$$

we have:

$$\langle x\rangle c_{+,j}(x, hD) = c'_j(x, hD)(A(h) + i) + hr_j(x, hD; h). \tag{3.45}$$

On the support of $c_{+,j}$, one has $x \cdot \xi \geq c|x|$. Consequently, the symbols $c'_j$ and $r_j(h)$ and their derivatives are all bounded. This proves:

$$\|c'_j(x, hD)\| \leq C, \ \|r_j(x, hD; h)\| \leq C, \ j = 0, \cdots, M,$$

uniformly in $h$. It follows that $\|\langle x\rangle c_{+,j}(x, hD)\langle A(h)\rangle^{-1}\| \leq C$. The case $s \in \mathbb{N}$, $s \geq 1$ can be proved in the same way. The result for any $s \geq 0$ follows from a complex interpolation. By the method of symbolic calculus of pseudo-differential operators, we can prove that the remainder term $r_{+,M}(h)$ satisfies estimates (3.44) with $j$ replaced by $M$ uniformly in $h$. Taking $M > s$, we derive that $\|\langle x\rangle^s r_+(x, hD; h)\langle A(h)\rangle^{-s}\|$ is also uniformly bounded. Consequently, one obtains

$$\|\langle x\rangle^s b_{+,1}(x, hD)\langle A_\mu(h)\rangle^{-s}\| \leq C. \tag{3.46}$$

To prove the similar estimates for $b_{+,2}$, we introduce

$$b_2 = \rho(x)\theta(x \cdot \xi/\mu|x|)\chi(|\xi|/\langle\mu\rangle) \in S_+(-\mu),$$

where supp $\rho \subset \{x; |x| > 1\}$ with $\rho(x) = 1$ for $|x| > 2$ and $\theta(t) = 0$ if $t \leq -1 + \epsilon/2$; 1 if $t > -1 + \epsilon$ for some $\epsilon > 0$ small enough. Since $b_{+,2}$ and $1 - b_2$ are of disjoint support, it suffices to prove the estimate with $b_{+,2}$ replaced by $b_2$. Let

$$b_2^\mu(x, hD; h) = e^{i\mu f(x)/h} b_2(x, hD) e^{-i\mu f(x)/h}.$$

We can expand the symbol $b_2^\mu$ by the method used before:

$$b_2^\mu(h) = \sum_{j=0}^{M} h^j d_j + h^{M+1} r_{2,M}(h),$$

where $d_j$ has a similar expression as $c_j$. Due to the choice of $b_2$, the support of $\partial_\xi d_0 = \partial_\xi b_2(x, \xi - \mu \nabla f(x))$ is contained in

$$\{-(1-\epsilon/2)\mu|x| \leq x \cdot (\xi - \mu\nabla f(x)) \leq -(1-\epsilon)\mu|x|\} \cup \{4\langle\mu\rangle \leq |\xi - \mu\nabla f(x)| \leq 5\langle\mu\rangle\}.$$

By an elementary analysis, one sees that on the both parts of the support of $\partial b_2(x, \xi - \mu\nabla f(x))$, $|\xi - \mu\nabla f(x)| \geq C\langle\xi\rangle$. This allows us to check that (3.44) holds for $d_j$ with $j = 0, \cdots, M$. The estimate (3.46) for $b_2$ follows from the arguments already used above. This finishes the proof of (i) for $b_+$. The proof for $b_-$ is the same.

(ii) Let $g(r) = \chi_+(r)\langle r\rangle^s$ $s < -1$. By the formula on functional calculus of Helffer-Sjöstrand (Proposition 7.2 of [**17**]), one has

$$g(P) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{g}}{\partial \bar{z}}(z)(P - z)^{-1} \; L(dz). \tag{3.47}$$

Here $P$ is a self-adjoint operator, $L(dz)$ denotes the Lebesgue's measure over $\mathbb{C}$ and $\tilde{g} \in C^\infty(\mathbb{C})$ satisfies $\tilde{g}(r) = g(r)$ for $r \in \mathbb{R}$ and $\partial_{\bar{z}}\tilde{g}(z) = O(|\Im z|^\infty)$ for $z$ near $\mathbb{R}$ (*i.e.*, an almost holomorphic extension of $g$). Since $F(h)$ and $A_\mu(h)$ differ only by an $h$-pseudo-differential operators with compactly supported symbol, for any $k \geq 1$, there exists $N_0 > 0$ such that

$$(\langle x\rangle^2 - h^2\Delta)^k((F(h) - z)^{-1} - (A_\mu(h) - z)^{-1}) = O\Big(\frac{1}{|\Im z|^{N_0}}\Big)$$

in $\mathcal{L}(L^2)$ norm. Applying (3.47) to $F(h)$ and $A_\mu(h)$, one sees that

$$\|(\langle x\rangle^2 - h^2\Delta)^k(g(F(h)) - g(A_\mu(h)))\| \leq C \tag{3.48}$$

uniformly in $h$. When $s \geq -1$, using the identity

$$\begin{aligned} &\chi_+(F(h))\langle F(h)\rangle^s - \chi_+(A_\mu(h))\langle A_\mu(h)\rangle^s \\ = \;& (\chi_+(F(h))\langle F(h)\rangle^{s-N} - \chi_+(A_\mu(h))\langle A_\mu(h)\rangle^{s-N})\langle F(h)\rangle^N \\ & + \chi_+(A_\mu(h))\langle A_\mu(h)\rangle^{s-N})(\langle F(h)\rangle^N - \langle A_\mu(h)\rangle^N) \end{aligned}$$

for some integer $N > s + 1$, one can apply (3.48) to show that

$$\|\langle x\rangle^{s'}\chi_+(F(h))\langle F(h)\rangle^s - \chi_+(A_\mu(h))\langle A_\mu(h)\rangle^s\| \leq C.$$

This estimate allows us to replace $F(h)$ by $A_\mu(h)$ in (3.42). To prove (3.42) for $A_\mu(h)$, we introduce the same decompositions for the symbols and make the same unitary transformation as in (i). We are reduced to prove that

$$\langle x\rangle^{s'} c(x,hD)\chi_-(A(h))\langle A(h)\rangle^s$$

is uniformly bounded in $\mathcal{L}(L^2)$, where $c$ is a bounded symbol with the same support properties as $c_{+,0}$. On the support of $c(x,\xi)$, one has $x\cdot\xi > \sigma|x|$ and $|\xi|\geq\sigma$ for some $\sigma>0$. Using (i), we may suppose that $\chi_-(r)=0$ for $r>-R$, $R>0$.

Let $\mathcal{M}$ be the Mellin transform defined by

$$\mathcal{M}(f)(\lambda,\omega) = \frac{1}{\sqrt{2\pi h}}\int_0^\infty f(r\omega)r^{d/2-1-i\lambda/h}dr, f\in C_0^\infty(\mathbb{R}^d). \tag{3.49}$$

Then $\mathcal{M}$ extends to a unitary map from $L^2(\mathbb{R}^d;dx)$ onto $L^2(\mathbb{R},L^2(\mathbf{S}^{d-1});d\lambda d\omega)$ and is a spectral representation of $A(h)$

$$(\mathcal{M}A(h)f)(\lambda,\omega) = \lambda\mathcal{M}(f)(\lambda,\omega)$$

for $f\in D(A(h))$. See [**29**]. One has

$$\mathcal{F}^*A(h)\mathcal{F} = -A(h),$$

where $\mathcal{F}$ is the $h$-dependent Fourier transform. For $u\in C_0^\infty(\mathbb{R}^d)$, we can write

$$\begin{aligned}&\mathcal{M}(\mathcal{F}^*(\langle x\rangle^{s'}c(x,hD)\chi_-(A(h))\langle A(h)\rangle^s)^*u)(\lambda,\omega)\\ &= \frac{1}{(2\pi h)^{(d+1)/2}}\langle\lambda\rangle^s\chi_-(-\lambda)\int_0^\infty\int e^{(d/2-1+i\lambda/h)\log r+irx\cdot\omega/h}\overline{c(x,r\omega)}\langle x\rangle^{s'}u(x)dxdr\end{aligned}$$

The phase function $r\to\Phi(r)=\lambda\log r+rx\cdot\omega$ has no critical point in $]0,+\infty[$ when $\lambda>R>0$ and $x\cdot\omega\geq\sigma|x|$ for $\sigma>0$. The method of non-stationary phase shows that

$$\|\langle x\rangle^{s'}c(x,hD)\chi_-(A(h))\langle A(h)\rangle^s\| \leq C_N h^N \tag{3.50}$$

for any $N\in\mathbb{N}$ and $s,s'>0$. This estimate, together with the reduction used before, finishes the proof of (3.42). ☐

***Corollary 3.6***. — *Assume the conditions (3.22) and (3.23). Let $b_\pm\in S_\pm(\mu_\pm)$ with $\pm\mu_\pm > -\sqrt{E}$. Then one has for any $s>1/2$*

$$\|\langle x\rangle^{s-1}b_\mp(x,hD)R(E\pm i\kappa,h)\langle x\rangle^{-s}\| \leq Ch^{-1} \tag{3.51}$$

*If $\mu_-<\mu_+$, then one has for $r,s\in\mathbb{R}$,*

$$\|\langle x\rangle^r b_\mp(x,hD)R(E\pm i\kappa,h)b_\pm(x,hD)\langle x\rangle^s\| \leq Ch^{-1}, \tag{3.52}$$

*uniformly in $\kappa\in]0,1]$ and $h>0$ small enough.*

*Proof*. — Let $b_-\in S(\mu_-)$ with $\mu_-<\sqrt{E}$. Take $\mu_-<\mu<\sqrt{E}$ so that Propositions 3.4 and 3.5 can be applied. Let $\chi\in C_0^\infty(]E-\delta,E+\delta[)$ with $0\leq\chi\leq 1$ and $\chi=1$ on $[E-\delta/2,E+\delta/2]$. $\delta=\delta(\epsilon_0)$ is small enough. $\chi(P(h))$ is an $h$-pseudo-differential operators with bounded symbols whose support is contained in $p^{-1}(]E-\delta,E+\delta[)$.

$\langle x\rangle^{-s}\chi(P(h))\langle F(h)\rangle^s$ is uniformly bounded for any $s \geq 0$. Let $\chi_+ + \chi_- = 1$ with $\chi_\pm$ having the similar properties as in (ii) of Proposition 3.5. One can then estimate for any $s > 1/2$

$$\begin{aligned}
&\|\langle x\rangle^{s-1}b_-(x,hD)R(E+i\kappa,h)\langle x\rangle^{-s}\| \\
&\leq \|\langle x\rangle^{s-1}b_-(x,hD)R(E+i\kappa,h)\langle F(h)\rangle^{-s}\|\|\langle F(h)\rangle^s\chi(P(h)\langle x\rangle^{-s}\| \\
&\quad +\|\langle x\rangle^{s-1}b_-(x,hD)R(E+i\kappa,h)(1-\chi(P(h)))\langle x\rangle^{-s}\| \\
&\leq C\|\langle F(h)\rangle^{-s}R(E+i\kappa,h)\langle F(h)\rangle^{-s}\| \\
&\quad +C\|\langle F(h)\rangle^{s-1}\chi_-(F(h))R(E+i\kappa,h)\langle F(h)\rangle^{-s}\| + C \\
&\leq C'h^{-1}.
\end{aligned}$$

This proves (3.51) for $b_-$. The other cases in Corollary 3.6 can be proved similarly. Note that under the conditions of (3.52), we can construct a uniform conjugate operator $F(h)$ for some $\mu$ satisfying $\mu_- < \mu < \mu_+$ and $|\mu| < \sqrt{E}$. ☐

Note that the classes of symbols used in Corollary 3.6 are sufficient for the construction of the partition of unity in the phase space. But their supports are not as large as those in $S_\pm$. Using the decay assumption (3.22), we can derive Theorem 3.2 from Corollary 3.6 by a localization in energy.

*Proof of Theorem 3.2.* — Let us first prove (3.29) for $b_-$. Let $\epsilon_0 > 0$ be such that supp $b_- \subset \{x\cdot\xi < (1-\epsilon_0)|x||\xi|\}$. Let $\chi$ be a cut-off around $E$ as above with $\delta = \delta(\epsilon_0)$ small enough. On $supp b_- \cap p^{-1}(]E-\delta, E+\delta[)$,

$$x\cdot\xi \leq (1-\epsilon_0)|x||\xi|,\ E-2\delta < |\xi|^2 < E+2\delta$$

for $|x|$ large enough. This shows that $b_-(x,hD)\chi(P(h))$ is of symbol supported in

$$\{x\cdot\xi \leq (1-\epsilon_0)(E+2\delta)^{1/2}|x|\} \cup \{|x| > R\}$$

for some $R$ large enough. Taking $\delta > 0$ so small that $\mu = (1-\epsilon_0)(E+2\delta)^{1/2} < E^{1/2}$, one can then apply Theorem 3.1 and Corollary 3.6 to obtain for any $s > 1/2$

$$\|\langle x\rangle^{s-1}b_-(x,hD)\chi(P(h))R(E+i\kappa,h)\langle x\rangle^{-s}\| \leq Ch^{-1}.$$

Clearly, one has

$$\|\langle x\rangle^{s-1}b_-(x,hD)(1-\chi(P(h)))R(E+i\kappa,h)\langle x\rangle^{-s}\| \leq C.$$

This proves (3.29) for $b_-$. (3.29) for $b_+$ can be derived in the same way.

To prove (3.30), let $b_\pm \in S_\pm$ be a pair of symbols with the property of disjoint support. Then, there exists $\delta_\pm > 0$ with $\delta_+ + \delta_- > 2$ such that

$$\text{supp } b_\pm(.,.) \subset \{\pm x\cdot\xi > -(1-\delta_\pm)|x||\xi|\}.$$

For $(x,\xi) \in$ supp $b_- \cap p^{-1}(]E-\delta, E+\delta[$ and $|x|$ large enough, one has

$$x\cdot\xi \leq (1-\delta_-)(E+2\delta)^{1/2}|x|,$$

while for $(x,\xi) \in$ supp $b_+ \cap p^{-1}(]E-\delta, E+\delta[$ and $|x|$ large enough one has

$$x\cdot\xi \geq -(1-\delta_+)(E-2\delta)^{1/2}|x|.$$

Since $\delta_- + \delta_+ > 2$, we can take $\delta > 0$ small enough such that

$$(1-\delta_-)(E+2\delta)^{1/2} < -(1-\delta_+)(E-2\delta)^{1/2}.$$

We can then apply Corollary 3.6 and (3.29) to obtain that

$$\|\langle x\rangle^r b_\mp(x,hD)\chi(P(h))R(E\pm i\kappa,h)b_\pm(x,hD)\langle x\rangle^s\| \leq Ch^{-1}.$$

Since $b_-$ and $b_+$ are of disjoint support and $(1-\chi(P(h)))R(E\pm i\kappa,h)$ is an $h$-pseudo-differential operator uniformly bounded for $\kappa\in[0,1]$. One has

$$\|\langle x\rangle^r b_\mp(x,hD)(1-\chi(P(h)))R(E\pm i\kappa,h)b_\pm(x,hD)\langle x\rangle^s\| \leq C_N h^N$$

for any $N\in\mathbb{N}$ and $r,s\in\mathbb{R}$. (3.30) is proved. □

From Theorems 3.1 and 3.2, one can use appropriate partition of unity of the form $b_+(x,\xi)+b_-(x,\xi)=1$ on $p^{-1}(]E-\delta,E+\delta[)$, one can deduce from Theorems 3.1 and (3.2) the high order resolvent estimates. Let $\ell\in\mathbb{N}$, $\ell\geq 2$. Then one has for $b_\pm\in S_\pm$. For any $s>\ell-1/2$,

$$\|\langle x\rangle^{s-\ell} b_\mp(x,hD)(R(E\pm i\kappa,h))^\ell\langle x\rangle^{-s}\| \leq Ch^{-\ell} \tag{3.53}$$

If $b_\pm\in S_\pm$ for some $\delta_\pm>0$ such that $\delta_-+\delta_+>2$, then for any $s,r\in\mathbb{R}$, there exists $C>0$ such that

$$\|\langle x\rangle^{s} b_\mp(x,hD)(R(E\pm i\kappa,h))^\ell b_\pm(x,hD)\langle x\rangle^{r}\| \leq Ch^{-\ell} \tag{3.54}$$

Uniform propagation estimates of the time-dependent Schrödinger equation

$$ih\partial_t u_h(t) = P(h)u_h(t), \quad u_h(0)=u_0.$$

can be deduced from the high order resolvent estimates. Let $U(t,h)=e^{-itP(h)/h}$ be the associated unitary group. A direct application of (3.53) only gives that for $\chi\in C_0^\infty(]E-\delta,E+\delta[)$ for some $\delta>0$, one has

$$\|\langle x\rangle^{s-r} b_\mp(x,hD)U(t,h)\chi(P(h))\langle x\rangle^{-s}\| \leq C_\epsilon h^{-\epsilon}\langle t\rangle^{-r+\epsilon}, \quad \pm t>0,$$

for any $\epsilon>0$, which is not satisfactory in semi-classical limit. In this subject, the following results are known ([**35**]).

***Theorem 3.7***. — *Assume the condition (3.22) for $r(x)=\langle x\rangle^{-\rho_0}$ for some $\rho_0>0$. Then (3.23) is a necessary and sufficient condition for the following estimate to hold uniformly in $h>0$:*

$$\|\langle x\rangle^{-s}U(t,h)\chi(P(h))\langle x\rangle^{-s}\| \leq C_s\langle t\rangle^{-s}, \quad \forall t\in\mathbb{R},$$

*for any $s\geq 0$, where $\chi\in C_0^\infty(]E-\delta,E+\delta[)$ for some $\delta>0$.*

*If (3.23) is satisfied, one has*

$$\|\langle x\rangle^{s-r} b_\mp(x,hD)U(t,h)\chi(P(h))\langle x\rangle^{-s}\| \leq C_{r,s}\langle t\rangle^{-r}, \quad \pm t>0,$$

*and for $b_\pm$ satisfying the conditions of (3.54)*

$$\|\langle x\rangle^s b_\mp(x,hD)U(t,h)\chi(P(h))b_\pm(x,hD)\langle x\rangle^r\| \le C_{r,s}\langle t\rangle^{-r}, \quad \pm t>0$$

*for all $s,r\ge 0$, uniformly in $h$.*

Note that the necessity of the non-trapping condition (3.23) in uniform propagation estimates of Theorem 3.7 is proved in [**35**] by the method of coherent states. See [**32**] for other applications of coherent states in semi-classical analysis.

**3.2. Potentials with local singularities.** — In the proof of Theorem 3.2, the smoothness of $V$ is only used in the construction of a uniform conjugate operator and in the functional calculus of $P(h)$ used in the last step. In this Subsection, we want to show that local singularities of $V$ can be included. Let $n\ge 1$. Assume that $(x\cdot\nabla)^j V$ are form-compact perturbations of $-\Delta$ for $0\le j\le n+1$ and there exists $R>0$ such that

$$|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^{-\rho_0-|\alpha|}, \qquad \forall\ \alpha\in\mathbb{N}^d, |\alpha|\le n+1, \tag{3.55}$$

for $|x|>R$. Let $E,\mu_0\in\mathbb{R}_+$. Assume that for each $\mu$ with $|\mu|<\mu_0$, there exists $r_\mu\in C_0^\infty(\mathbb{R}^{2d})$ such that $F(h)=A(h)+\mu s(x)+r^w(x,hD)$ is a uniform conjugate operator of $P(h)=-h^2\Delta+V(x)$ at the energy $E$ with

$$iE_I(P(h))[P(h),F(h)]E_I(P(h)) \ge ChE_I(P(h)), \quad C>0, \quad I=]E-\delta,E+\delta[, \tag{3.56}$$

as form on $D(P(h))$ and satisfies (2.15) for some $n\ge 2$ and for $P_\epsilon=P(h)$, $A_\epsilon=F(h)$.

**Remark.** — It is difficult to construct a uniform conjugate operator in form of pseudo-differential operators without sufficient regularity of $V$. But in some cases, one can construct a uniform conjugate operator in form of differential operators. Suppose, for example, that $d\ge 2$ and $V$ is of the form

$$V(x)=\frac{\gamma}{|x|}+U(x),$$

$\gamma\in\mathbb{R}_+$. Assume that $U$ is smooth on $\mathbb{R}^d$ and satisfies (3.55) for some $0<\rho_0\le 1$. This implies that $V(x)$ has only one singularity at $x=0$ and (3.55) is satisfied by $V$ outside any neighborhood of 0. Assume that

$$U(x)+x\cdot\nabla U(x)\le 0. \tag{3.57}$$

We want to show that for any $E>0$, (3.56) (together with (2.15)) is satisfied for $\mu_0=\sqrt{E}$ and for any $n$. In fact, we just take $r=0$ and $F(h)=A_\mu(h)=A(h)+\mu s(x)$. Then

$$i[P(h),A_\mu(h)] = h\{P(h)-h^2\Delta-U(x)-x\cdot\nabla U(x)\}+i[-h^2\Delta,\mu s(x)].$$

By Lemma 3.3,

$$i[-h^2\Delta,\mu s(x)] \ge -h(\mu^2+\epsilon-h^2\Delta),$$

for any $\epsilon > 0$ provided that $\tau$ is small enough. This gives

$$i[P(h), A_\mu(h)] \geq h(P(h) - \mu^2 - \epsilon).$$

For any $E > 0$, $I = [E - \delta, E + \delta]$, let $E_I$ denote the spectral projector of $P(h)$ onto the interval $I$. Clearly,

$$E_I(P(h))i[P(h), A_\mu(h)]E_I(P(h) \geq h(E - \mu^2 - \delta - \epsilon)E_I(P(h)), \quad h \in ]0, 1].$$

For $|\mu|^2 < E$, we can take $\epsilon$ and $\delta$ small enough such that

$$E_I(P(h))i[P(h), A_\mu(h)]E_I(P(h) \geq c_0 h E_I(P(h)). \tag{3.58}$$

To examine multiple commutators of $P(h)$ with $A_\mu(h)$, we remark that

$$\nabla s(x) \cdot D = -i \frac{r(r^2 + \tau'^2)}{(r^2 + \tau'^2)^{3/2}} \frac{\partial}{\partial r}, \quad r = |x|.$$

Therefore its commutator with the Coulomb potential does not worsen the singularity. Till now, $\gamma \in \mathbb{R}$ can be arbitrary. Since $\gamma > 0$, one has

$$\| - h^2 \Delta (P(h) + i)^{-1} \| \leq C, \quad \| \frac{1}{|x|} (P(h) + i)^{-1} \| \leq C$$

uniformly in $h$. Consequently, $(P(h) + i)^{-1} B_k(h) (P(h) + i)^{-1}$ is uniformly bounded, where

$$B_0(h) = [P(h), A_\mu(h)], \quad B_k(h) = [B_{k-1}(h), A_\mu(h)], \ k = 1, 2, 3, \cdots.$$

This shows that the results below hold for repulsive Coulomb singularity. It is an interesting question to prove the same estimates for attractive Coulomb singularity ($\gamma < 0$).

***Theorem 3.8***. — *Assume the conditions (3.55) and (3.56) for some $E > 0$, $\mu_0 > 0$ and $n \geq 2$. The following estimates hold uniformly in $0 < \kappa < 1$ and $h > 0$ small.*

(i) *For any $s \geq 1/2$, there exists $C > 0$ such that*

$$\|R(\lambda \pm i\kappa, h)\|_{\mathcal{L}(B_s, B_s^*)} \leq Ch^{-1}. \tag{3.59}$$

(ii) *For any $1/2 < s < n$ and $b_\pm \in S_\pm(\mu_\pm)$ with $\pm\mu_\pm > -\mu_0$, one has*

$$\|\langle x\rangle^{s-1} b_\mp(x, hD) R(E \pm i\kappa, h) \langle x\rangle^{-s}\| \leq Ch^{-1}. \tag{3.60}$$

(iii) *For any $s, s' \in \mathbb{R}$ with $(s)_+ + (s')_+ < n - 1$, and $b_\pm \in S_\pm(\mu_\pm)$ with $|\mu_\pm| < \mu_0$ and $\mu_+ > \mu_-$, there exists $C > 0$ such that*

$$\|\langle x\rangle^s b_\mp(x, hD) R(\lambda \pm i\kappa, h) b_\pm(x, hD) \langle x\rangle^s\| \leq Ch^{-1}. \tag{3.61}$$

The proof of Theorem 3.8 is the same as that of Corollary 3.6 and is omitted.

***Lemma 3.9***. — *Let $f$ be a cut-off around $E$. Let (3.55) be satisfied for some $n \geq 1$. Let $P'(h) = -h^2\Delta + \chi(x/R)V(x)$ with $\chi$ a cut-off which is equal to 1 for $|x| > 2$ and to 0 for $|x| < 1$. $R$ is chosen large enough so that $\chi(x/R)V(x)$ is smooth on $\mathbb{R}^d$. The following estimates hold.*

(a) *One has:*

$$\|\langle x\rangle^s (f(P(h)) - f(P'(h)))\langle x\rangle^{s'}\| \le C. \tag{3.62}$$

*for any $s + s' \le n+1$. In particular, for $|s| \le n+1$, one has*

$$\|\langle x\rangle^s f(P(h))\langle x\rangle^{-s}\| \le C. \tag{3.63}$$

*and $f(P(h)) = f(-h^2\Delta) + R(h)$ with $R(h)$ satisfying: $\exists \rho_0 > 0$ such that*

$$\|\langle x\rangle^{s+\rho_0} R(h)\langle x\rangle^{-s}\| \le C, \tag{3.64}$$

*uniformly in $h$.*

(b) *For any $s \in \mathbb{R}$ with $|s| \le n+1$, one has:*

$$\|\langle x\rangle^s (1 - f(P(h)))R(E \pm i\kappa, h)\langle x\rangle^{-s}\| \le C \tag{3.65}$$

*uniformly in $\kappa \in ]0,1]$ and $h > 0$.*

(c) *Let $b_1, b_2 \in S_\pm$ be two bounded symbols with disjoint support. Then for $s_1 + s_2 \le n+1$, one has:*

$$\|\langle x\rangle^{s_1} b_1(x, hD)(1 - f(P(h)))R(E \pm i\kappa, h) b_2(x, hD)\langle x\rangle^{s_2}\| \le C, \tag{3.66}$$

*uniformly in $h > 0$ and $\kappa \in ]0,1]$.*

*Proof.* — The proof is based on the formula of functional calculus (3.47). For (a), we compare $R(z,h)$ with $(P'(h) - z)^{-1}$ and commute repeatedly $\langle x\rangle$ with the resolvent. (b) and (c) are deduced similarly. The details are omitted here. □

***Theorem 3.10***. — *If $\mu_0$ is equal to $E$ in the conditions of Theorem 3.8, the following estimates hold.*

(i) *For any $1/2 < s < n$ and $b_\pm \in S_\pm$*

$$\|\langle x\rangle^{s-1} b_\mp(x, hD)R(E \pm i\kappa, h)\langle x\rangle^{-s}\| \le Ch^{-1}. \tag{3.67}$$

(ii) *For any $s, s' \in \mathbb{R}$ with $(s)_+ + (s')_+ < n-1$ and $b_\pm \in S_\pm$ for some $\delta_\pm$ with $\delta_+ + \delta_- > 2$, there exists $C > 0$ such that*

$$\|\langle x\rangle^s b_\mp(x, hD)R(\lambda \pm i\kappa, h) b_\pm(x, hD)\langle x\rangle^{s'}\| \le Ch^{-1}. \tag{3.68}$$

*Proof.* — We only show that the proofs of Subsection 3.1 go through in presence of local singularities. Consider (3.67) for $b_-$. Let $\epsilon_0 > 0$ be chosen so that supp $b_- \subset \{x\cdot\xi < (1-\epsilon_0)|x||\xi|\}$. Take $\chi_1 \in C_0^\infty(]E - \delta, E + \delta[)$ with $0 \le \chi_1 \le 1$ and $\chi_1 = 1$ on $[E - \delta/2, E + \delta/2]$. Lemma 3.9 (c),

$$\langle x\rangle^{s-1} b_\mp(x, hD)(1 - \chi_1(P(h))R(E \pm i\kappa, h)\langle x\rangle^{-s}$$

is uniformly bounded. From Lemma 3.9 (a), il follows that for $0 < s \le n$,

$$\begin{aligned} &\|\langle x\rangle^{s-1} b_-(x, hD)\chi_1(P(h))R(E + i\kappa)\langle x\rangle^{-s}\| \\ \le\ & C\{\|\langle x\rangle^{-1-n} R(E + i\kappa, h)\langle x\rangle^{-s}\| \\ &+\|\langle x\rangle^{s-1} b_-(x, hD)\chi_1(P')(R(E + i\kappa, h)\langle x\rangle^{-s}\|. \end{aligned} \tag{3.69}$$

(i) of Theorem 3.8 implies that

$$\|\langle x\rangle^{-1-n}R(E+i\kappa,h)\langle x\rangle^{-s}\| \leq Ch^{-1}$$

for $s > 1/2$. Since $\mu_0 = E^{1/2}$, by taking $\delta > 0$ small enough, we can apply Theorem 3.8 and the method of used in the proof of Theorem 3.2 to show that

$$\|\langle x\rangle^{s-1}b_-(x,hD)\chi_1(P')(R(E+i\kappa,h)\langle x\rangle^{-s}\| \leq Ch^{-1}.$$

(3.67) for $b_-$ is proved for any $1/2 < s < n$. □

**3.3. Potentials depending on a parameter.** — In the study of the semi-classical Schrödinger equation with a source term concentrated near one point, one needs uniform resolvent estimates for the Schrödinger operator $P_\epsilon = -\Delta + V(\epsilon x)$. Although this operator is unitarily equivalent with $P(h) = -h^2\Delta + V(x)$ in $L^2$, we can not derive simply the resolvent estimates of $P_\epsilon$ from those of $P(h)$, since the spaces used above are not homogeneous under dilation. We want to explain how the ideas used before can be applied to $P_\epsilon$ to establish uniform resolvent estimates.

Consider the Schrödinger operator $P_\epsilon = -\Delta + V_\epsilon(x)$ on $\mathbb{R}^d$ with potential depending on a parameter $\epsilon \in ]0,1]$. Assume that the multiplication operators

(3.70) $\quad (x\cdot\nabla_x)^j V_\epsilon$, $0 \leq j \leq n+1$, are $-\Delta$-bounded

uniformly in $\epsilon$ for some $n \geq 1$. Let $E > 0$. Let $A_\mu = A_\mu(1)$ be the defined as before with $h = 1$. Assume further that there exists a bounded family of bounded symbols, $\{r_\epsilon, \epsilon \in ]0,1]\}$ such that for some $\mu_0 > 0$,

(3.71) for any $|\mu| < \mu_0$, $F_\mu(\epsilon) = A_\mu + r_\epsilon^w(x,D)$ is a uniform conjugate operator of $P_\epsilon$ at $E$.

Then one has the following

***Theorem 3.11***. — *Let $R_\epsilon(z) = (P_\epsilon - z)^{-1}$. Under the conditions (3.70), (3.71), the following estimates hold uniformly in $\epsilon, \kappa \in ]0,1]$.*

(i) *Let $n = 1$. For $s \geq 1/2$, one has*

(3.72) $$\|R_\epsilon(E \pm i\kappa)\|_{\mathcal{L}(B_s, B_s^*)} \leq C.$$

(ii) *Let $n \geq 2$, $1/2 < s < n$ and $b_\pm \in S_\pm(\mp\mu)$ with $|\mu_\pm| < \mu_0$, there exists $C > 0$ such that*

(3.73) $$\|\langle x\rangle^{s-1}b_\mp(x,D)R_\epsilon(E\pm i\kappa)\langle x\rangle^{-s}\| \leq C$$

(iii) *Let $n \geq 2$, $s, r \in \mathbb{R}$ with $(s)_+ + (r)_+ < n-1$ and $b_\pm \in S_\pm(\mp\mu_\pm)$ with $|\mu_\pm| < \mu_0$ and $\mu_- < \mu_+$, one has*

(3.74) $$\|\langle x\rangle^s b_\mp(x,D)R_\epsilon(E\pm i\kappa)b_\pm(x,D)\langle x\rangle^r\| \leq C.$$

The proof of Theorem 3.11 is the same as Corollary 3.6 with $h = 1$ fixed.

**Example.** — Assume that $(x \cdot \nabla_x)^j V_\epsilon$, $0 \leq j \leq n+1$, are $-\Delta$-bounded uniformly in $\epsilon$ for some $n \geq 1$. Let $E > 0$. Suppose that there exists some $\nu_0 \in ]0, 2]$ and $c_0 > 0$ such that

$$\nu_0(E - V_\epsilon(x)) - x \cdot \nabla V_\epsilon(x) \geq c_0 \tag{3.75}$$

uniformly in $x$ and $\epsilon$. Then the assumption (3.71) is verified for some $\mu_0 > 0$ and $F_\mu(\epsilon) = A_\mu$, $|\mu| \leq \mu_0$. Here $h$ is fixed to be 1 in the definition of the function $s(x) = s_{\tau,h}(x)$. Let $I = [E - \delta, E + \delta]$. Then

$$\|E_I(P_\epsilon) i[-\Delta, s(x)]\| \leq C$$

uniformly in $\epsilon$. Since

$$iE_I(P_\epsilon)[P_\epsilon, A_\mu]E_I(P_\epsilon) \geq E_I(P_\epsilon)(\nu_0(E - \delta) - \nu_0 V_\epsilon(x) - x \cdot \nabla V_\epsilon(x) - |\mu| C) E_I(P_\epsilon),$$

under the condition (3.75), we can take $\delta$ and $\mu_0$ small enough such that

$$iE_I(P_\epsilon)[P_\epsilon, A_\mu]E_I(P_\epsilon) \geq c_0/2 E_I(P_\epsilon)$$

for all $|\mu| \leq \mu_0$. Note that (3.75) is a kind of virial condition and the case $\nu_0 = 2$ is mostly used.

Using the inequality of Hardy (3.34), we can deduce the Morrey-Campanato estimates from the resolvent estimates obtained above. See [**30**], and also [**10**] for discontinuous refraction index. Denote the Morrey-Companato norm

$$|||u|||^2 = \sup_{R>0} \frac{1}{R} \int_{|x|<R} |u|^2 dx$$

and $N(f)$ the dual norm

$$N(f) = \sum_{j \in \mathbb{Z}} \left( 2^{j+1} \int_{C(j)} |f|^2 dx \right)^{\frac{1}{2}}$$

where $C_j = \{x \in \mathbb{R}^d; 2^j \leq |x| \leq 2^{j+1}\}$. These norms are homogeneous in dilation and are useful in the study of concentration phenomenon of the high frequency Helmholtz equation. A consequence of Theorem 3.11 is the following

***Corollary 3.12***. — *Assume $d \geq 2$. Under the assumptions of Theorem 3.11 (i), one has*

$$|||(P_\epsilon - (E \pm i\kappa))^{-1} u||| \leq CN(u), \tag{3.76}$$

*for all $u \in L^2_{\rm loc}$ with $N(u) < \infty$, uniformly in $\epsilon, \kappa$.*

Corollary 3.12 follows from (i) of Theorem 3.11 and the inequality of Hardy (3.34) for appropriate $s > 1/2$. See [**40**] for more details in the case $d \geq 3$.

In [**7**], the authors proved (3.72) and (3.76) under the general non-trapping assumption, using the approach of semi-classical measures. It is an interesting open question to see if (ii) and (iii) of Theorem 3.11 remain true under this condition.

## 4. Semi-classical measures of the stationary Schrödinger equation

The purpose of this Section is to apply the uniform resolvent estimates to the study of the semi-classical measures of the stationary Schrödinger equation

$$(-h^2\Delta + V(x) - E - i\kappa)u_h = S^h(x), \tag{4.77}$$

where $E > 0$, $\kappa = \kappa(h) \geq 0$ and $\kappa \to 0$ as $h \to 0$. Note that here $\kappa$ can be identically zero: when $\kappa = 0$, $u_h$ is taken as the unique outgoing solution of (4.77) in the sense that $u_h$ is defined as

$$u_h = \lim_{\epsilon\to 0_+} (P(h) - E - i\epsilon)^{-1} S^h,$$

where $P(h) = -h^2\Delta + V(x)$. The high frequency Helmholtz equation (1.9) can be written in the form of (4.77) with $h = \epsilon, u_h = u_\epsilon, S^h = -\epsilon^2\mathcal{S}_\epsilon, \kappa = \epsilon\alpha_\epsilon$, and where $V(x) = E - n^2(x)$. The precise conditions on $V$ and $S^h$ will be stated below. To be simple, we study only the case where the source term is concentrated near one point. See also [**4, 9**]. The case of the source term with concentration-oscillation effect is more difficult and is studied in [**8**] for constant refraction index and in [**40**] for variable refraction index under some conditions. When the refraction index $n(x)^2 = E - V(x)$ presents discontinuity, the propagation of semi-classical measures is studied by E. Fouassier.

To begin with, we recall in Subsections 4.1 and 4.2 some basic properties of Wigner transform and semi-classical measures. See [**5, 15, 14, 16, 25, 26**] for more details. In Subsection 4.3, we apply the results of Section 3 to study (4.77) for source term concentrated near one point.

**4.1. Basic properties of Wigner transform.** — Semi-classical measures or Wigner measures were introduced by Wigner in 1932 in the study of semi-classical limit of quantum mechanics from the point of views of thermodynamic equilibrium. See [**41**]. For $\psi \in L^2(\mathbb{R}^d)$, the Wigner transform of $\psi$ is defined by

$$W(\psi)(x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-iy\cdot\xi}\psi\Big(x+\frac{y}{2}\Big)\overline{\psi\Big(x-\frac{y}{2}\Big)}\, dy, \tag{4.78}$$

for $(x,\xi) \in \mathbb{R}^{2d}$. $W(\psi)$ is quadratic in $\psi$, but is linear with respect to the density function $\rho(x,y) = \psi(x)\overline{\psi(y)}$, *a.e.* in $x, y$. A remarkable property of Wigner transform is that if $\psi = \psi_h(t)$ is solution to the Schrödinger equation

$$\begin{cases} ih\frac{\partial\psi}{\partial t} &= -\frac{h^2}{2}\Delta\psi \\ \psi_{|t=0} &= \psi_0, \end{cases} \tag{4.79}$$

then the scaled Wigner transform, $W_h(x,\xi;t)$ of $\psi$:

$$W_h(x,\xi;t) = \frac{1}{h^d} W(\psi)\Big(x, \frac{\xi}{h}\Big)$$

is solution to the Liouville equation

$$\left\{ \begin{array}{rcl} \frac{\partial W_h}{\partial t} + \xi \cdot \nabla_x W_h & = & 0 \\ W_{h|t=0} & = & \frac{1}{h^d} W(\psi_0)(x, \frac{\xi}{h}) \end{array} \right. \tag{4.80}$$

More generally, if there is an appropriate potential $V(x)$, it was expected that the Wigner transform, $W_h(t)$, of the solution $\psi_h(t)$ to the Scrödinger equation

$$ih \frac{\partial \psi_h(t)}{\partial t} = \Big( - \frac{h^2}{2} \Delta + V(x) \Big) \psi_h(t) \tag{4.81}$$

converges to some limit $f$ as $h \to 0$, which satisfies the associated Liouville equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f - \nabla V(x) \cdot \nabla_\xi f = 0 \quad \text{in} \quad \mathbb{R}^d_x \times \mathbb{R}^d_\xi \times \mathbb{R}_t. \tag{4.82}$$

It is worth to notice that the solution of (4.82) can be written down explicitly in terms of solution of the Hamiltonian system of $p(x,\xi) = \frac{\xi^2}{2} + V(x)$. The approach of E. Wigner allows to relate formally the quantum mechanics to classical mechanics. However, the limit $f$ is, in general, not a function, but only a measure. Rigorous justification of Wigner's approach requires the study of measures obtained as week limit of the Wigner transform of a family of wave functions. This approach was justified for many linear and nonlinear evolution equations. See [**26, 15, 16, 43, 44**].

Let $\psi \in L^2(\mathbb{R}^d)$. Denote

$$\rho(x,y) = \psi(x)\overline{\psi(y)}, \quad \tilde\rho(x,y) = \rho\Big(x + \frac{y}{2}, x - \frac{y}{2}\Big), \quad a.\ e. \text{ in } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

It is clear that

$$\tilde\rho \in L^2(\mathbb{R}^{2d}) \cap C_\infty(\mathbb{R}^d_y; L^1(\mathbb{R}^d_x)) \cap C_\infty(\mathbb{R}^d_x; L^1(\mathbb{R}^d_y))$$

where $C_\infty(\mathbb{R}^d_y; L^1(\mathbb{R}^d_x))$ denotes the space of $L^1_x$-valued functions on $\mathbb{R}^d_y$ which tend to 0 as $y \to \infty$. $C_\infty(\mathbb{R}^d_y; L^1(\mathbb{R}^d_x))$ is equipped with the natural norm. The Wigner transform, $W_h(\psi)$, of $\psi$ depending on a small parameter $h > 0$, is defined by

$$\begin{aligned} W_h(\psi)(x,\xi) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-iy\cdot\xi} \psi\Big(x + \frac{hy}{2}\Big) \overline{\psi\Big(x - \frac{hy}{2}\Big)}\, dy \\ &= (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{-iy\cdot\xi/h} \tilde\rho(x,y)\, dy \end{aligned} \tag{4.83}$$

***Proposition 4.1***. — *One has*

$$\|W_h(\psi)\|^2_{L^2} = (2\pi h)^{-d} \|\tilde\rho\|^2_{L^2} = (4\pi h)^{-d} \|\rho\|^2_{L^2} = (4\pi h)^{-d} \|\psi\|^4_{L^2}, \tag{4.84}$$

$$\int_{\mathbb{R}^d_\xi} W_h(\psi)(x,\xi)\, d\xi = \rho(x,x), \quad \textit{a.e. in } x, \tag{4.85}$$

$$\int_{\mathbb{R}^d_\xi} W_h(\psi)(x,\xi) e^{-h\xi^2/2}\, d\xi = (2\pi h)^{-d/2} \int_{\mathbb{R}^d_y} \tilde\rho(x,hy) e^{-y^2/(2h)}\, dy. \tag{4.86}$$

Remark that

$$W_h(\psi)(x,\xi) = (2\pi h)^{-d}\mathcal{F}_{y\to\xi}\tilde{\rho}(x,\xi/h)$$

where $\mathcal{F}_{y\to\xi}$ is Fourier transform

$$\mathcal{F}_{y\to\xi}u(\xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\xi}u(y)\, dy.$$

(4.84) follows from Plancherel formula for Fourier transform. (4.85) is trivial. (4.86) follows from the same calculation and the inverse Fourier transform of $\xi \to e^{-h\xi^2/2}$.

It is useful to introduce the bilinear mapping associated with Wigner transform which is quadratic in $\psi$. Define

$$w_h(f,g)(x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-iy\cdot\xi} f\Big(x+h\frac{y}{2}\Big)\overline{g\Big(x-h\frac{y}{2}\Big)}\, dy.$$

Clearly, $w_h(f,f) = W_h(f)$. By the properties of Fourier transform on temperate distributions, $w_h$ extends to a continuous bilinear mapping from $\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^{2d})$. One has, for $f$ and $g$ in $L^2$,

$$\int_{\mathbb{R}^d_\xi} w_h(f,g)(x,\xi)\, d\xi = f(x)\overline{g(x)} \tag{4.87}$$

$$\int_{\mathbb{R}^d_\xi} w_h(f,g)(x,\xi)\, dx = \frac{1}{(2\pi h)^d}\hat{f}(\xi/h)\overline{\hat{g}}(\xi/h) \tag{4.88}$$

$$\mathcal{F}_{\xi\to v}(w_h(f,g))(x,v) = f(x-hv/2)\overline{g(x+hv/2)} \tag{4.89}$$

*a.e.* in $x,\xi$ and $v$. For $f,\ g\in\mathcal{S}'$, one has

$$< w_h(f,g), a > \; = \; < a^w(x,hD)f, \overline{g} >, \quad \forall a\in\mathcal{S}(\mathbb{R}^{2d}), \tag{4.90}$$

$$w_h(f,g) = \overline{w_h(g,f)}, \text{ in } \mathcal{S}'(\mathbb{R}^{2d}) \tag{4.91}$$

where $< \cdot,\cdot >$ denotes the dual product between $\mathcal{S}'$ and $\mathcal{S}$.

***Proposition 4.2*** **([16])**

*(a) For $f,g\in L^2(\mathbb{R}^d_x)$, one has*

$$\mathcal{F}_{\xi\to v}w_h(f,g)(x,v) \in C_0(\mathbb{R}^d_v; L^1(\mathbb{R}^d_x)) \tag{4.92}$$

$$\mathcal{F}_{x\to\eta}w_h(f,g)(\eta,\xi) \in C_0(\mathbb{R}^d_\eta; L^1(\mathbb{R}^d_\xi)) \tag{4.93}$$

*and their respective norms are uniformly bounded by $\|f\|\|g\|$.*

*(b) Let $a,b\in\mathcal{S}(\mathbb{R}^{2d})$. Then,*

$$< w_h(f,g), ab >_{\mathcal{S}',\mathcal{S}} = < a^w(x,hD)f, b^w(x,hD)\overline{g} >_{\mathcal{S}',\mathcal{S}} + r_h \tag{4.94}$$

*where $|r_h|\le hC(a,b)\|f\|\,\|g\|$ for some $C(a,b)$ independent of $f,g$ and $h$.*

*Proof*

(a) (4.92) follows from (4.89) and

$$\sup_{v\in\mathbb{R}^d}\|f(\cdot-hv/2)\overline{g(\cdot+hv/2)}\|_{L^1(\mathbb{R}^d_x)} \le \|f\|_{L^2}\|g\|_{L^2}.$$

(4.93) can be deduced from the following relation

$$\mathcal{F}_{x\to\eta} w_h(f,g)(\eta,\xi) = \frac{1}{(2\pi h)^d}\hat{f}\Big(\frac{\xi}{h}+\frac{\eta}{2}\Big)\overline{\hat{g}\Big(\frac{\xi}{h}-\frac{\eta}{2}\Big)}$$

and the Parseval formula.

(b) By (4.90), $< w_h(f,g), ab >=< \overline{g}, (ab)^w(x,hD)f >$. By the calculus of semi-classical pseudo-differential operators, $(ab)^w(x,hD) = b^w(x,hD)a^w(x,hD) + hR^w(x,hD;h)$, where $R(h)$ is a bounded family in $\mathcal{S}(\mathbb{R}^{2d})$. Since $b^w(x,hD)$ is invariant by transposition, we obtain

$$< w_h(f,g), ab >=< a^w(x,hD)f, b^w(x,hD)\overline{g} > +r_h$$

where $r_h = h < \overline{g}, R^w(x,hD;h)f >$ satisfies the desired estimate, due to the uniform $L^2$-boundedness for semi-classical pseudo-differential operators with bounded symbol. □

### 4.2. Semi-classical measures.

— Let $\mathbf{X}$ denote the space

$$\mathbf{X} = \{\varphi \in C_\infty(\mathbb{R}^{2d}_{x,\xi}); \mathcal{F}_{\xi\to z}\varphi(x,z) \in L^1(\mathbb{R}^d_z; C_\infty(\mathbb{R}^d_x))\}$$

equipped with the norm

$$\|\varphi\|_{\mathbf{X}} = \int_{\mathbb{R}^d_z} \sup_x |\mathcal{F}_{\xi\to z}\varphi(x,z)|dz,$$

where $C_\infty$ is the space of continuous functions tending to 0 at the infinity. $\mathbf{X}$ is a Banach algebra and $\mathcal{S}(\mathbb{R}^{2d})$, $C_0^\infty(\mathbb{R}^{2d})$ are dense in $\mathbf{X}$.

Let $\{u_n\}$ be a sequence in $L^2(\mathbb{R}^d)$. Denote

$$U_{h,n}(x,\xi) \quad = \quad W_h(u_n)(x,\xi).$$

***Theorem 4.3* ([14]).** — *Let $\{u_n\}$ be bounded sequence in $L^2$. There exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, a sequence $\{h_k\}$ with $h_k \to 0$ and a positive bounded Radon measure $\mu$ on $\mathbb{R}^d$ such that for any $a \in C_0^\infty(\mathbb{R}^{2d})$*

$$\lim_{k\to\infty} < a^w(x,h_kD)u_{n_k}, u_{n_k} >= \iint a(x,\xi)\mu(dxd\xi). \tag{4.95}$$

*$\mu$ is called the semi-classical measure (or Wigner measure) associated with $\{u_{n_k}\}$.*

*Proof.* — Let $U_{h,n}$ be defined as above. For any $f \in \mathbf{X}$, one has

$$\int_{\mathbb{R}^{2d}} U_{h,n}(x,\xi)f(x,\xi)dxd\xi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} \mathcal{F}_{\xi\to z}f(x,z)u_n\Big(x+\frac{hz}{2}\Big)\overline{u_n\Big(x-\frac{hz}{2}\Big)}\,dxdz.$$

It follows that

$$\Big|\int_{\mathbb{R}^{2d}} U_{h,n}(x,\xi)f(x,\xi)dxd\xi\Big| \le \frac{1}{(2\pi)^d}\|f\|_{\mathbf{X}}\|u_n\|^2 \le C\|f\|_{\mathbf{X}}.$$

This proves that $\{U_{h,n}\}$ is bounded in $\mathbf{X}^*$. Since $\mathbf{X}$ is separable, there exists a subsequence $\{U_{h_k,n_k}\}$ of $\{U_{h,n}\}$ and $\mu \in \mathbf{X}^*$ such that $h_k \to 0$ and $\{U_{h_k,n_k}\}$ converges $*$-weakly to $\mu$:

$$\lim_{k\to\infty} \int U_{h_k,n_k} f dx d\xi = \int_{\mathbb{R}^{2d}} f(x,\xi)\mu(dxd\xi), \quad \forall f \in \mathbf{X}.$$

By (4.90), for $a \in C_0^\infty(\mathbb{R}^{2d})$,

$$< a^w(x, h_k D)u_{n_k}, u_{n_k} >_{L^2} = < U_{h_k,n_k}, a >_{\mathcal{S}',\mathcal{S}} .$$

It follows that

$$< a^w(x, h_k D)u_{n_k}, u_{n_k} >_{L^2} \to \int_{\mathbb{R}^{2d}} a(x,\xi)\mu(dxd\xi), \quad k \to \infty.$$

It remains to prove that $\mu$ is a measure. For any $a \in C_0^\infty(\mathbb{R}^{2d})$, take $\phi \in C_0^\infty$ with $0 \le \phi \le 1$ and $\phi a = a$. For $\eta > 0$, put $b_\eta = \phi\sqrt{a+\eta}$. Then, $b_\eta \in C_0^\infty$ and $b_\eta^2 = a + \eta\phi^2$. Making use of symbolic calculus of semi-classical pseudo-differential operators, we have

$$a^w(x,hD) = b_\eta^w(x,hD)^2 - \eta\phi^w(x,hD)^2 + O_\eta(h), \quad \text{in} \quad \mathcal{L}(L^2).$$

From this decomposition and the boundedness of $\{u_n\}$, one obtains that there exists $C > 0$ independent of $\eta$ such that

$$\liminf_{h\to 0} < a^w(x,hD)u_n, u_n > \ge -C\eta.$$

Since $\eta > 0$ is arbitrary, we get

$$\int a(x,\xi)\mu(dxd\xi) = \lim_{k\to\infty} < a^w(x,h_k D)u_{n_k}, u_{n_k} > \ge 0.$$

Therefore, $\mu$ is a positive distribution, thus a measure on $\mathbb{R}^{2d}$. See [**18**]. It is clear that $\mu(\mathbb{R}^{2d}) \le \sup_k \|u_{n_k}\|^2 < \infty$. ☐

**Remark** When $\{u_n\}$ is only bounded in $L^2_{\rm loc}$, using the properties of Wigner transform in $\mathcal{S}'$, one can still show that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a locally bounded positive Radon measure $\mu$ on $\mathbb{R}^{2d}$ such that

$$\lim_{k\to\infty} < a^w(x,h_k D)u_{n_k}, u_{n_k} > = \iint a(x,\xi)\mu(dxd\xi), \forall a \in C_0^\infty.$$

See [**5, 14**]. We will use this remark in the following Subsection.

Let $\{u_\epsilon\}$ be a bounded sequence in $L^2$ with $\epsilon \in I$ where $I$ is a countable set with 0 as the only accumulating point. Let

$$U_\epsilon = W_\epsilon(u_\epsilon), \quad \tilde{U}_\epsilon = U_\epsilon * \left(\frac{1}{(\epsilon\pi)^d} e^{-(x^2+\xi^2)/(4\epsilon)}\right).$$

By extracting successively subsequences, we can assume, by an abuse of notation, that

$$\begin{aligned} u_\epsilon &\rightharpoonup u \in L^2 \\ U_\epsilon &\overset{*}{\rightharpoonup} \mu \in \mathbf{X}^* \\ \tilde{U}_\epsilon &\overset{*}{\rightharpoonup} \tilde{\mu} \in \mathbf{X}^*. \end{aligned}$$

A sequence $\{v_\epsilon\} \subset L^2(\mathbb{R}^d)$ will be said compact at infinity if

$$\sup_\epsilon \int_{|x|>R} |v_\epsilon(x)|^2 \, dx \to 0, \quad \text{as} \quad R \to \infty. \tag{4.96}$$

The basic properties of semi-classical measures can be resumed in the following

***Theorem 4.4*** **([26])**
*(a) One has $\mu = \tilde{\mu}$.*
*(b) $\mu \geq |u(x)|^2 \delta_0(\xi)$ and*

$$\|u\|^2 \leq \mu(\mathbb{R}^{2d}) \leq \liminf_{\epsilon \to 0} \|u_\epsilon\|^2.$$

*(c) $|u_\epsilon(x)|^2$ converges weakly in sense of measures to $\int_{\mathbb{R}^d_\xi} d\mu(\cdot, \xi)$ if and only if the family $\{\epsilon^{-d} |\hat{u}(\xi/\epsilon)|^2\}$ is compact at infinity.*

*(d) The equality $\mu(\mathbb{R}^{2d}) = \lim_{\epsilon\to 0} \|u_\epsilon\|^2$ hold if and only if both $\{u_\epsilon(x)\}$ and $\{\epsilon^{-d}|\hat{u}(\xi/\epsilon)|^2\}$ are compact at infinity. In this case, $\{u_\epsilon\}$ converges strongly to $u$ in $L^2$ if and only if $\mu = |u(x)|^2\delta_0(\xi)$.*

*(e) Let $\mu$ be a positive finite Radon measure. Let $u \in L^2$ such that $\mu \geq |u(x)|^2 \delta_0(\xi)$. Then there exists a sequence $\{u_\epsilon\}$ in $L^2$ such that $u_\epsilon \rightharpoonup u$ in $L^2$, $U_\epsilon \overset{*}{\rightharpoonup} \mu$ in $\mathbf{X}^*$ and $\mu(\mathbb{R}^{2d}) = \lim_{\epsilon\to 0} \|u_\epsilon\|^2$.*

This result shows that semi-classical measures contain information about the lack of compactness for a bounded sequence in $L^2$.

**4.3. The Schrödinger equation with concentration effect.** — For the high frequency Helmholtz equation with the source term concentrated near one point $x = 0$, [**4**] shows that the correct normalization of the source is given by

$$S_\epsilon(x) = \epsilon^{-\frac{3+d}{2}} S(\frac{x}{\epsilon})$$

for some $S$ independent of $\epsilon$. By a change of notation, the equation can be put into the form

$$(-h^2\Delta + V(x) - E - i\kappa)u_h(x) = h^{\frac{1-d}{2}} S(\frac{x}{h}) \tag{4.97}$$

where

$$h = \epsilon \to 0, \quad \kappa = \kappa(h) \geq 0.$$

We assume that $E > 0$ and $V$ satisfies

$$(x \cdot \nabla)^j V(x) \text{ is bounded on } \mathbb{R}^d \text{ for } 0 \leq j \leq 3. \tag{4.98}$$

Assume also that for some $\nu_0 \in ]0,2]$, there exists $c_0 > 0$ such that

$$\nu_0(E - V(x)) - x \cdot \nabla V(x) \geq c_0. \tag{4.99}$$

Put $w_h(x) = h^{(d-1)/2} u_h(hx)$. Then $w_h$ is the solution of

$$(-\Delta + V(hx) - E - i\kappa) w_h(x) = S(x) \tag{4.100}$$

***Theorem 4.5***. — *Assume (4.98) and (4.99).*

*(a) Let $S \in B_{\frac{1}{2}}$. One has $w_h \in B^*_{\frac{1}{2}}$ and*

$$\|w_h\|_{B^*_{\frac{1}{2}}} \leq C \|S\|_{B_{\frac{1}{2}}} \tag{4.101}$$

*(b) Assume that $\langle x\rangle^{r_0} S \in L^2$ for some $r_0 > 3/2$, $E - V(0) > 0$ and*

$$(x \cdot \nabla)^j V(x) \text{ is uniformly continuous on } \mathbb{R}^d \text{ for } j = 0, 1. \tag{4.102}$$

*Let $w_0$ is the outgoing solution of the equation*

$$(-\Delta + V(0) - E - i0) w_0(x) = S(x). \tag{4.103}$$

*For any $s > 3/2$, one has*

$$\lim_{h\to 0} \|w_h - w_0\|_{B^*_s} = 0. \tag{4.104}$$

*In particular, $w_h$ converges $*$-weakly to $w_0$ in $B^*_{\frac{1}{2}}$.*

**Remark.** — The $*$-weak convergence of $w_h$ to $w_0$ is conjectured in [**4**] and is proved in [**9**] under the general non-trapping assumption (1.5) and a condition on the geometry of self intersection set near zero of the Hamilton flow. Under some additional decay assumptions, the results of [**40**] for source having concentration-oscillation near a subspace, when simplified to the case of point source, proved that there exists a subsequence of $\{w_h\}$ converging $*$-weakly to $w_0$ in $B^*_{\frac{1}{2}}$. For smooth potentials, a proof of (b) is given in [**39**], using microlocal resolvent estimates. Since no decay of $V(x)$ is needed, Theorem 4.5 holds for $N$-body Schrödinger operators.

To prove Theorem 4.5 (b), we need the following

***Lemma 4.6***. — *Let $A = (x \cdot D + D \cdot x)/2$.*

*(a) Let $V$ be bounded and uniformly continuous on $\mathbb{R}^d$. For any $\delta > 0$, $f \in C_0^\infty(\mathbb{R}_+)$, one has*

$$\lim_{h\to 0} \|\langle A\rangle^{-\delta}(V(hx) - V(0)) f(-\Delta)\| = 0. \tag{4.105}$$

*and*

$$\lim_{h\to 0} \|(V(hx) - V(0)) f(-\Delta)\langle A\rangle^{-\delta}\| = 0. \tag{4.106}$$

*(b) Suppose that $x \cdot \nabla V$ is uniformly continuous on $\mathbb{R}^d$. Let $\chi_+$ be a cut-off of $[0, \infty[$.*

$$\lim_{h\to} \|\langle A\rangle^{s'} [\chi_+(A), V(hx)] \langle A\rangle^s f(-\Delta)\| = 0 \leq C \tag{4.107}$$

*for any $s, s' \in [0, 1[$.*

*Proof.* — To prove (4.105), it is sufficient to show that for any $g \in C_0^\infty(\mathbb{R})$

$$\lim_{h\to 0} \|g(A)(V(hx) - V(0))f(-\Delta)\| = 0. \tag{4.108}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(s) = 1$ for $s \le 1$ and $\chi(s) = 0$ for $s \ge 2$. Let $\chi_R(x) = \chi(\frac{|x|}{R})$ and

$$K_R = \mathcal{M}(g(A)(1-\chi_R)f(-\Delta))\mathcal{F}^* : L^2(\mathbb{R}^d; d\xi) \to L^2(\mathbb{R}, L^2(\mathbf{S}^{d-1}); d\lambda d\omega),$$

where $\mathcal{M}$ is the Mellin transform defined by (3.49) with $h = 1$. The kernel of $K_R$ is given by

$$K_R(\lambda,\omega;\xi) = \frac{R}{(2\pi)^{(d+1)/2}} g(\lambda) \int_0^\infty e^{(d/2-1+i\lambda)(\log r + \log R) + iR r\omega\cdot\xi)} (1-\chi(r))f(\xi^2).$$

For $\lambda \in$ supp $g$ and $\xi$ in the support of $f(\xi^2)$, the derivative of the phase verifies

$$|\partial_r(\lambda \log r + Rr\omega\cdot\xi)| \ge (\delta_0 R - C)r > 0,$$

for $r > 1$ and $R$ large enough. We can use the method of non-stationary phase to show that

$$\|g(A)(1-\chi_R(x))f(-\Delta)\| = \|K_R\| \to 0$$

as $R \to \infty$. For $U \in C_b^1(\mathbb{R}^d)$, space of bounded $C^1$ function with bounded derivatives, we can use (3.47) to show that $\|[f(-\Delta), U(hx)]\| = O(h)$. For $V$ bounded and uniformly continuous on $\mathbb{R}^d$, we can approximate it by a sequence $\{V_n\}$ of $C_b^1(\mathbb{R}^d)$ such that $\|V - V_n\|_{L^\infty} \to 0$, as $n \to 0$. It can be derived that $\|[f(-\Delta), V(hx)]\| \to 0$ as $h \to 0$. This proves

$$\lim_{h\to 0, R\to\infty} \|g(A)(1-\chi_R(x))V(hx)f(-\Delta)\| = 0. \tag{4.109}$$

For each fixed $R$, one has $\|\chi_R(x)(V(hx) - V(0)\|_{L^\infty} \to 0$, as $h \to 0$ (4.105) follows by an elementary argument. (4.106) results from(4.105) and the limit

$$\lim_{h\to 0} \|[f(-\Delta), V(hx)]\| = 0.$$

To show (4.107), using the formula of functional calculus (3.47) and commuting $V(hx)$ with $(A-z)^{-1}$, $[\chi_+(A), V(hx)]$ can be expressed as

$$[\chi_+(A), V(hx)] = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\chi}_+(z)}{\partial\overline{z}}(A-z)^{-1} hx\cdot\nabla V(hx)(A-z)^{-1} L(dz)$$

Since $\chi_+(r)$ is constant for $|r|$ large, we can find an almost-holomorphic extension $\tilde{\chi}_+(z)$ of $\chi_+$ on $\mathbb{C}$ such that $\frac{\partial\tilde{\chi}_+(z)}{\partial\overline{z}}$ is supported in a region $\{z; |\Im z| \ge \delta|\Re z|\}$ for some $\delta > 0$ and

$$\left|\frac{\partial\tilde{\chi}_+(z)}{\partial\overline{z}}\right| \le \frac{C}{|z|}$$

for $|z|$ large enough. Since $x\cdot\nabla V(x)$ is vanishing at $x = 0$, one can apply the part (a) to $x\cdot\nabla V(x)$ to show (4.107). □

**Proof of Theorem 4.5**. — (4.98) and (4.99) show that $A$ is a uniform conjugate operator of $P_h = -\Delta + V(hx)$ at $E$. (i) of Theorem 3.11 gives

$$\|(P_h - E - i\kappa)^{-1}\|_{\mathcal{L}(B_{\frac{1}{2}}, B^*_{\frac{1}{2}})} \leq C \tag{4.110}$$

uniformly in $h$ and $\kappa$. (4.101) follows.

To prove (b), put

$$R_h(E + i\kappa) = (-\Delta + V(hx) - E - i\kappa)^{-1}$$

and

$$R_0(E + i\kappa) = (-\Delta + V(0) - E - i\kappa)^{-1}.$$

$R_0(E + i\kappa)S$ converges to $w_0$ in $B^*_{\frac{1}{2}}$, as $\kappa \to 0$. Write $v_h = w_h - R_0(E + i\kappa)S$ as

$$v_h = R_h(E + i\kappa)(V(0) - V(hx))R_0(E + i\kappa)S$$

Let $\rho \in C_0^\infty(]E - 2\delta, E + 2\delta[)$ with $\rho(\lambda) = 1$ on $[E - \delta, E + \delta]$ and

$$r_h = R_h(E + i\kappa)(V(0) - V(hx))R_0(E + i\kappa)\rho(-\Delta)S$$

One can check that for any $r \geq 0$

$$\|\langle x\rangle^r R_0(E + i\kappa)\rho(-\Delta)S\| \leq C\|\langle x\rangle^r S\|$$

and consequently for any $f \in C_0^\infty(\mathbb{R}^d)$

$$| < v_h - r_h, f > | \leq C\delta_1(h)\|\langle x\rangle^r S\|\|\langle x\rangle^s f\| \tag{4.111}$$

for some $1/2 < s < r \leq r_0$. Here

$$\delta_1(h) = \|\langle x\rangle^{-r+s}(V(hx) - V(0))\| \to 0, \quad \text{as } h \to 0.$$

To show that $< r_h, f > \to 0$, take $\chi_\pm \in C^\infty(\mathbb{R})$ such that $\chi_+ + \chi_- = 1$ on $\mathbb{R}$, $\chi_+ = 1$ on $[\frac{1}{2}, \infty[$, and 0 on $]\infty, -\frac{1}{2}]$. Decompose $\rho(-\Delta)R_0(E + i\kappa)$ as

$$\rho(-\Delta)R_0(E + i\kappa) = \rho(-\Delta)(\chi_+(A) + \chi_-(A)\rho(-\Delta)R_0(E + i\kappa)\rho_1(-\Delta)$$

where $\rho_1 \in C_0^\infty$ with $\rho\rho_1 = \rho$. Inserting this decomposition into $r_h$ and applying Theorem 2.2, we obtain for $1/2 < s < s' < r_0 - 1$

$$\begin{aligned} | < r_h, f > | \leq\ & C\delta_2(h)(\|\langle A\rangle^{s'}\chi_-(A)R_0(E + i\kappa)S\|\|\langle A\rangle^s\rho(-\Delta)f\| \\ & + \|\langle A\rangle^s\rho_1(-\Delta)S\|\|\langle x\rangle^{s'}b_+(x, D)R_h(E - i\kappa)f\|) \\ & + | < \rho(-\Delta)R_0(E + i\kappa)S, [\chi_+(A), V(hx)]R_h(E - i\kappa)f > | \\ \leq\ & C\delta_2(h)\|\langle x\rangle^{s'+1}S\|\|\langle x\rangle^{s+1}f\| + C\delta_3(h)\|\langle x\rangle^{s'}S\|\|\langle x\rangle^s f\|, \end{aligned} \tag{4.112}$$

where $C$ is independent of $h$ and $\kappa$ and

$$\delta_2(h) = \|\langle A\rangle^{s-s'}(V(0) - V(hx))\rho(-\Delta)\| + \|(V(0) - V(hx))\rho(-\Delta)\langle A\rangle^{s-s'}\| \to 0$$

and

$$\delta_3(h) = \|\langle A\rangle^s[\chi_+(A), V(hx)]\rho(-\Delta)\langle A\rangle^{s'}\| \to 0$$

for $s, s' > 1/2$ with $s + s' < 2$, according to Lemma 4.6 (a) and (b). From the above estimates on $v_h - r_h$ and on $r_h$, we obtain that for $s, s' > 1/2$ with $1 + s' \leq r_0$

$$|< v_h, f >| \leq C\delta(h)\|\langle x\rangle^{s'+1}S\|\|\langle x\rangle^{s+1}f\|, \forall f \in C_0^\infty \tag{4.113}$$

with $\delta(h) \to 0$, uniformly in $S, f$ and $\kappa$. (4.113) gives

$$\|v_h\|_{B^*_{1+s}} \leq C\delta(h)\|\langle x\rangle^{r_0}S\|. \tag{4.114}$$

Since $v_h$ is bounded in $B^*_{\frac{1}{2}}$, an argument of density shows that $v_h$ tends to 0 $*$-weakly in $B^*_{\frac{1}{2}}$. □

***Theorem 4.7***. — *Let (4.98) and (4.99) be satisfied.*

*(a) Let $S \in B_{\frac{1}{2}}$. Then $u_h \in B^*_{\frac{1}{2}}$ and there exists $C > 0$ such that*

$$\|u_h\|_{B^*_{\frac{1}{2}}} \leq C\|S\|_{B_{\frac{1}{2}}} \tag{4.115}$$

*uniformly in $h$.*

*(b) Assume $\langle x\rangle S \in L^2$. There exists $\mu_0 > 0$ such that for $b_- \in S_-(\mu_0)$*

$$\|b_-(x, hD)u_h\|_{L^2} \leq C\|\langle x\rangle S\| \tag{4.116}$$

*uniformly in $h$.*

*Proof*

(a) By Theorem 4.5,

$$\|w_h\|_{B^*_{\frac{1}{2}}} \leq C\|S\|_{B_{\frac{1}{2}}}.$$

For $0 < h < 1$, one has

$$\begin{aligned}
\|w_h\|_{B^*_{\frac{1}{2}}} &= \sup_{R>1} \frac{1}{R^{\frac{1}{2}}}\Big(\int_{|x|<R} |u_h(hx)|^2 h^{d-1} dx\Big)^{\frac{1}{2}} \\
&= \sup_{R>1} \frac{1}{(hR)^{\frac{1}{2}}}\Big(\int_{|x|<hR} |u_h(x)|^2 dx\Big)^{\frac{1}{2}} \\
&= \sup_{R'>h} \frac{1}{R'^{\frac{1}{2}}}\Big(\int_{|x|<R'} |u_h(x)|^2 dx\Big)^{\frac{1}{2}} \\
&\geq \sup_{R'>1} \frac{1}{R'^{\frac{1}{2}}}\Big(\int_{|x|<R'} |u_h(x)|^2 dx\Big)^{\frac{1}{2}} = \|u_h\|_{B^*_{\frac{1}{2}}}.
\end{aligned}$$

(4.115) follows.

(b) To prove (4.116), remark that $A_\mu$ is a uniform conjugate operator of $P_h$ for all $\mu$ with $|\mu| \leq \mu_0$ for some $\mu_0 > 0$. Theorem 3.11 (ii) can be applied to $R_h(E + i\kappa)$. For $s = 1$, we obtain that for any $b_- \in S_-(\mu_0)$

$$\|b_-(x, D)w_h\| \leq C\|\langle x\rangle S\|$$

uniformly in $h$. It follows that

$$\|b_-(h^{-1}x, hD)u_h\| \leq Ch^{1/2}\|\langle x\rangle S\|.$$

Let $\chi \in C^\infty(\mathbb{R}), \eta \in C_0^\infty(\mathbb{R}^d_\xi)$ with supp $\chi \subset ]-\infty, \mu_0[$ and $\chi(t) = 1$ for $t \leq \mu_0 - \epsilon$. Take $b_{-,0} \in S_-(\mu_0)$ as

$$b_{-,0}(x,\xi) = \chi\Big(\frac{x\cdot\xi}{\langle x\rangle}\Big)\eta(\xi).$$

Then $b_{-,0}(h^{-1}x, hD) = b_0(x, hD; h)$ where

$$b_0(x,\xi;h) = \chi\Big(\frac{x\cdot\xi}{(h^2+|x|^2)^{1/2}}\Big)\eta(\xi).$$

Let $\rho \in C^\infty(\mathbb{R}^d_x)$ with $\rho(x) = 1$ for $|x| \geq 2\delta$ , 0 for $|x| \leq \delta$. For arbitrary $b_- \in S_-(\mu)$ with $|\mu| < \mu_0 - \epsilon$, using localization in energy, we can suppose without loss that $b_-$ is compactly supported in $\xi$. Then we can choose suitable $\chi$ and $\eta$ in $b_0$ such that

$$\rho(x)b_-(x,\xi) = \rho(x)b_-(x,\xi)b_0(x,\xi;h)$$

for all $h > 0$ small enough. Note that $\rho(x)b_0(x,\xi;h)$ is in $S_-(\mu_0)$. Using symbolic calculus and Theorem 4.7 (a), we obtain

$$\|\rho(x)b_-(x,hD)u_h\| \leq C\|b_0(x,hD;h)u_h\| + Ch\|\langle x\rangle^{-1}u_h\| \leq C'h^{1/2}\|\langle x\rangle S\|.$$

Since $1-\rho(x)$ is of compact support, Theorem 4.7 (a) implies

$$\|(1-\rho(x))b_-(x,hD)u_h\| \leq C\|S\|_{B^*_{\frac12}}.$$

This shows that

$$\|b_-(x,hD)u_h\| \leq C\|S\|_{B^*_{\frac12}} + Ch^{1/2}\|\langle x\rangle S\|. \tag{4.117}$$

(b) is proved. □

Theorem 4.7 (a) shows that $\{u_h; h \in ]0,h_0]\}$ is bounded in $L^2_{\rm loc}$. By the remark following Theorem 4.3, there exists a subsequence $\{u_{h_k}\}$ and a locally bounded positive Radon measure $f$ on $\mathbb{R}^{2d}$ such that for any $a(x,\xi) \in C_0^\infty(\mathbb{R}^{2d})$, one has

$$\lim_{j\to\infty}\Big(a(x,h_jD)u_{h_j}, u_{h_j}\Big) = \int_{\mathbb{R}^{2d}} a(x,\xi)f(x,\xi)\,dx\,d\xi. \tag{4.118}$$

Theorem 4.8 (a) below is announced in [**4**], but the proof given there can only give an estimate of the form

$$\sup_{R>1}\frac{1}{R^s}\int_{|x|\leq R}\int_{\xi\in\mathbb{R}^d} f(x,\xi)\,dx\,d\xi \leq C_s$$

for any $s > 1$. See also [**8, 40**]. We give a complete proof here.

***Theorem 4.8***. — *Let the conditions of Theorem 4.5 (a) be satisfied. Let $f$ be a semi-classical measure constructed as above.*

*(a) One has*

$$\sup_{R>1}\frac{1}{R}\int_{|x|\leq R}\int_{\xi\in\mathbb{R}^d} f(x,\xi)\,dx\,d\xi \leq C\|S\|^2_{B_{\frac12}}. \tag{4.119}$$

*(b) Assume in addition that $V$ is bounded and uniformly continuous on $\mathbb{R}^d$. Then, supp $f \subseteq p^{-1}(E)$.*

*Proof*

(a) To prove (4.119), take a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ with $0 \le \chi \le 1$, supp $\chi \subset (-1, 2)$ and $\chi(r) = 1$ on $[0, 1]$. Set

$$a_{R,\delta} = \frac{1}{R}\chi\Big(\frac{|x|}{R}\Big)e^{-\delta|\xi|^2}$$

for $\delta > 0$, and $v_h = e^{-\delta|hD|^2}u_h$. Using the uniform continuity of $e^{-\delta|hD|^2}$ in weighted $L^2$-spaces and an argument of interpolation (see Theorem 14.1.4 of [**18**]), one deduces that

$$e^{-\delta|hD|^2} : B_s^* \to B_s^*$$

is uniformly bounded. Therefore,

$$\|v_h\|_{B^*_{\frac12}} \le C\|u_h\|_{B^*_{\frac12}}.$$

We can then estimate

$$\begin{aligned}
\Big\|\frac{1}{R}\chi\Big(\frac{|x|}{R}\Big)v_h\Big\|_{B_{\frac12}} &= \Big\|\frac{1}{R}\chi\Big(\frac{|x|}{R}\Big)v_h\Big\|_{L^2(|x|\le 1)} + \sum_{1\le 2^k\le R+1}\frac{2^{\frac{k}{2}}}{R}\|v_h\|_{L^2(2^{k-1}\le|x|<2^k)} \\
&= \frac{1}{R}\Big(1 + \sum_{1\le 2^k\le R+1} 2^k\Big)\sup_{k\ge 0}\frac{1}{2^{\frac{k}{2}}}\|v_h\|_{L^2(|x|\le 2^k)} \le \frac{2^{N_0+1}}{R}\|v_h\|_{B^*_{\frac12}} \\
&\le 3\|v_h\|_{B^*_{\frac12}} \le C\|u_h\|_{B^*_{\frac12}},
\end{aligned}$$

where $N_0$ is taken such that $2^{N_0} \le R + 1$. As a consequence, one has

$$\Big|\Big(a_{R,\delta}(x, hD)u_h, u_h\Big)\Big| \le C\|u_h\|^2_{B^*_{\frac12}}, \tag{4.120}$$

uniformly in $h$, $R$ and $\delta$. This, together with (4.118) implies that

$$\frac{1}{R}\int_{|x|\le R}\int_{\mathbb{R}^d}\chi\Big(\frac{|x|}{R}\Big)e^{-\delta|\xi|^2}f(x,\xi)\,dx\,d\xi \le C\sup_{h>0}\|u_h\|^2_{B^*_{\frac12}}.$$

Taking $\delta \to 0$, we obtain (4.119).

(b) Let $0 \le a_0(x,\xi) \in C_0^\infty(\mathbb{R}^{2d})$ such that supp $a_0 \cap p^{-1}(E) = \varnothing$. Let $\rho \in C_0^\infty(\mathbb{R})$ with support sufficiently near $E$ such that supp $a_0 \cap$supp $\rho(p) = \varnothing$. If $V$ is smooth with bounded derivatives, by the functional calculus of $h$-pseudo-differential operators, one sees that

$$\|a_0(x, hD)u_h\|_{B_{\frac12}} \le C\|(1 - \rho(P(h)))u_h\|_{L^2} + \|a_0(x, hD)\rho(P(h))u_h\|_{B_{\frac12}} \tag{4.121}$$

is bounded by $O(h^{1/2})$. Part (a) of Theorem 4.7 gives

$$\Big|\Big(a_0(x, hD)u_h, u_h\Big)\Big| \le \|a_0(x, hD)u_h\|_{B_{\frac12}}\|u_h\|_{B^*_{\frac12}} \le Ch^{\frac12}.$$

(4.118) implies

$$\int_{\mathbb{R}^{2d}} a_0(x,\xi) f(x,\xi)\, dx\, d\xi = 0,$$

for any $a_0 \in C_0^\infty(\mathbb{R}^{2d})$ with $a_0 = 0$ in a neighborhood of $p^{-1}(E)$. This shows that

$$\text{supp } f \subseteq p^{-1}(E) \tag{4.122}$$

when $V$ is smooth with all derivatives bounded.

For $V$ satisfying the conditions of (b), we can construct a sequence $\{V_n\}$ of smooth functions with bounded derivatives such that

$$\|V_n - V\|_{L^\infty} \to 0, \quad n \to \infty$$

Put $P_n(h) = -h^2\Delta + V_n(x)$. Then using (3.47), one can show that

$$\delta_n = \sup_{h\in]0,1]} \|\langle x\rangle^{-1}(\rho(P_n(h)) - \rho(P(h)))\langle x\rangle\| \to 0, \quad n \to \infty.$$

From Theorem 4.7, it follows that

$$\begin{aligned} &\|a_0(x,hD)\rho(P(h))u_h\|_{B_{\frac12}} \\ &\leq \; C(\|\langle x\rangle^{-1}(\rho(P_n(h)) - \rho(P(h)))\langle x\rangle\| + \|a_0(x,hD)\rho(P_n(h))\|)\|u_h\|_{B^*_{\frac12}} \\ &\leq \; C\delta_n + C_n h \end{aligned}$$

where $C$ is independent of $n$ and $h$. This implies

$$\begin{aligned} \|a_0(x,hD)u_h\|_{B_{\frac12}} &\leq C\|(1-\rho(P(h)))u_h\|_{L^2} + \|a_0(x,hD)\rho(P(h))u_h\|_{B_{\frac12}} \\ &\leq C_n h + C\delta_n \end{aligned}$$

for any $n$. It follows that for $V$ satisfying the conditions of Theorem 4.5 (a),

$$\lim_{h\to 0} \|a_0(x,hD)u_h\|_{B_{\frac12}} = 0. \tag{4.123}$$

The argument used above for $V$ smooth shows that (4.122) still holds in the general case. □

Let $f_h(x,\xi)$ denote the Wigner transform of $u_h(x)$:

$$f_h(x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-iy\cdot\xi} u_h\Big(x + \frac{hy}{2}\Big)\overline{u_h\Big(x - \frac{hy}{2}\Big)}\, dy.$$

Using the equation (4.97), an elementary calculation shows that

$$\alpha_h f_h + \xi\cdot\nabla_x f_h - \Theta_h(f_h) = Q_h \tag{4.124}$$

where $\Theta_h$ is defined by

$$\Theta_h(f_h)(x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}_{y,\eta}} e^{-iy\cdot(\xi-\eta)}\frac{1}{2hi}\Big(V\Big(x+\frac{hy}{2}\Big) - V\Big(x-\frac{hy}{2}\Big)\Big) f_h(x,\eta) dy d\eta$$

and

$$Q_h(x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-iy\cdot\xi}\frac{1}{2hi}\Big[S_h\Big(x+\frac{hy}{2}\Big)\overline{u_h\Big(x-\frac{hy}{2}\Big)} - \overline{S_h\Big(x-\frac{hy}{2}\Big)}u_h\Big(x+\frac{hy}{2}\Big)\Big]dy$$

By (4.118), one has for some sequence $h_k \to 0$

$$< a, f_{h_k} > = < a^w(x, h_k D)u_{h_k}, u_{h_k} > \to \iint a(x,\xi)f(x,\xi)dxd\xi \tag{4.125}$$

for any $a \in C_0^\infty$. Assume that

$$\kappa = h\alpha_h \quad \text{ with } \alpha_h \to \alpha$$

as $h \to 0$. If $\nabla V$ is uniformly continuous on $\mathbb{R}^d$, one can show that

$$\alpha_{h_k} f_{h_k} + \xi\cdot\nabla_x f_{h_k} - \Theta_{h_k}(f_{h_k}) \to \alpha f + \xi\cdot\nabla_x f - \nabla V(x)\cdot\nabla_\xi f \tag{4.126}$$

in $\mathcal{D}'$. We only give the proof of the limit

$$\Theta_{h_k}(f_{h_k}) \to \nabla V(x)\cdot\nabla_\xi f. \tag{4.127}$$

For $a \in C_0^\infty(\mathbb{R}^{2d})$, we have

(4.128)

$$(\Theta_h(f_h), a) = \int_{\mathbb{R}^{2d}_{x,\eta}} f_h(x,\eta)\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{R^d_y} e^{i\eta y}\frac{V(x+\frac{hy}{2}) - V(x-\frac{hy}{2})}{2ih}\widehat{a}(x,y)\,dy\,dx\,d\eta,$$

where $\widehat{a}(x,y) = \mathcal{F}_{\xi\to y}(a(x,\xi))$. Denote

$$G_h(x,\eta) = \frac{1}{2i(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d_y} e^{iy\eta}\int_{-1}^1 y\cdot\nabla V\Big(x+\frac{h\theta y}{2}\Big)\,d\theta\widehat{a}(x,y)\,dy.$$

By Theorem 4.7, to prove (4.127), we only need to show that for any $s > 1$

$$\int_y \sup_x \Big\{\langle x,y\rangle^s\Big|\mathcal{F}_{\eta\to y}(G_h(x,\eta) - \frac{1}{2}\nabla V(x)\cdot\nabla_\eta a(x,\eta))\Big|\Big\} \to 0 \tag{4.129}$$

as $h \to 0$. But

$$\begin{aligned} &\int_{R^d_y}\sup_{x\in R^d}\Big\{\langle x,y\rangle^s\Big|\frac{1}{2}\int_{-1}^1 \nabla V\Big(x+\frac{h\theta y}{2}\Big)\,d\theta y\widehat{a}(x,y) - y\cdot\nabla V(x)\hat{a}(x,y)\Big|\Big\}\,dy \\ \le\; &\int_{R^d_y}\sup_{x\in R^d}\Big\{\langle x,y\rangle^s\Big|\frac{1}{2}\int_{-1}^1 \Big(y\cdot\nabla V\Big(x+\frac{h\theta y}{2}\Big) - y\cdot\nabla V(x)\Big)\,d\theta\widehat{a}(x,y)\Big|\Big\}\,dy \end{aligned}$$

(4.130)

We decompose the last integral into two parts, denoted $I(h)$ and $II(h)$, according to $|y| \ge h^{-r}$ or $|y| < h^{-r}$, $0 < r < 1$. For $|y| < h^{-r}$,

$$\Big|\nabla V\Big(x+\frac{h\theta y}{2} - \nabla V(x)\Big)\Big| \to 0, \quad h \to 0$$

uniformly in $x$, since $\nabla V$ is uniformly continuous on $\mathbb{R}^d$. This shows $I(h) \to 0$. When $|y| \ge h^{-r}$, $\widehat{a}(x,y)$ is rapidly decreasing in $y$. In this case,

$$|\widehat{a}(x,y)| = O(h^N\langle x,y\rangle^{-N})$$

for any $N$, which gives $II(h) \to 0$. This proves (4.127).

A more subtle task is to compute explicitly the limit of the source term $Q_h$ which depends on $u_h$. To do this, we take $\varphi, \psi \in \mathcal{S}$ and write

$$\begin{aligned}
&\int Q_h(x,\xi)\varphi(x)\psi(\xi)\ dxd\xi \\
&= -\frac{1}{(2\pi)^d 2ih^{(d+1)/2}}\int_{\mathbb{R}^{2d}}\Big[S\Big(\frac{x}{h}+\frac{y}{2}\Big)\overline{u_h\Big(x-\frac{hy}{2}\Big)} \\
&\quad -\overline{S\Big(\frac{x}{h}-\frac{y}{2}\Big)}u_h(x+\frac{hy}{2})\Big]\varphi(x)\hat{\psi}(y)dxdy \\
&= -\frac{h^{(d-1)/2}}{(2\pi)^d 2i}\int_{\mathbb{R}^{2d}}\Big[S(x')\overline{u_h(h(x'-y))}\varphi\Big(hx'-\frac{hy}{2}\Big) \\
&\quad -\overline{S(x')}u_h(h(x'+y))\varphi\Big(hx'+\frac{hy}{2}\Big)\Big]\hat{\psi}(y)dx'dy.
\end{aligned}$$

Recall that according to Theorem 4.5 (b), $h^{(d-1)/2}u_h(hx)$ converges $*$-weakly to $w_0$ in $B^*_{\frac{1}{2}}$ and strongly to $w_0$ in $B^*_s$ for any $s > 3/2$. We can estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} S(x+y)(\overline{w_h(x)}-\overline{w_0(x)})\varphi\Big(hx+\frac{hy}{2}\Big)dx\Big| \\
&\leq\ C\delta(h)\Big\|S(\cdot+y)\varphi(h\cdot+\frac{hy}{2})\Big\|_{B_s}
\end{aligned}$$

with $\delta(h) \to 0$ as $h \to 0$. Since $\langle x\rangle^{r_0}S \in L^2$ for some $r_0 > 3/2$, we can take $3/2 < s < r_0$ to prove that

$$\int_{\mathbb{R}^d}\Big\|S(\cdot+y)\varphi(h\cdot+\frac{hy}{2})\Big\|_{B_s}|\hat{\psi}(y)|dy \leq C(\varphi,\psi)$$

uniformly in $h$. It follow that

$$\begin{aligned}
&\lim_{h\to 0}\int Q_h(x,\xi)\varphi(x)\psi(\xi)\ dxd\xi \\
&= -\lim_{h\to 0}\frac{1}{(2\pi)^d 2i}\int_{\mathbb{R}^{2d}}\Big[S(x')\overline{w_0(x'-y)}\varphi\Big(hx'-\frac{hy}{2}\Big) \\
&\quad -\overline{S(x')}w_0(x'+y)\varphi\Big(hx'+\frac{hy}{2}\Big)\Big]\hat{\psi}(y)dx'dy \\
&= -\frac{\varphi(0)}{2i(2\pi)^d}\int\Big[S(x')\overline{w_0(x'-y)}-\overline{S(x')}w_0(x'+y)\Big]\hat{\psi}(y)dx'dy \\
&= \frac{\varphi(0)}{(2\pi)^d}\int_{\mathbb{R}^d}\Im(\xi^2+V(0)-E-i0)^{-1}|\hat{S}(\xi)|^2\psi(\xi)d\xi
\end{aligned}$$

We finally find that $Q_h \to \frac{1}{(2\pi)^d}\delta(x)\Im(\xi^2+V(0)-E-i0)^{-1}|\hat{S}(\xi)|^2$ in sense of distributions. We have proved the following

***Theorem 4.9***. — *Let $\alpha_h \equiv \kappa h^{-1} \to \alpha \geq 0$. Under the assumptions of Theorem 4.5 (b), assume that $\nabla V$ is uniformly continuous on $\mathbb{R}^d$, then, the semi-classical measure $f$ verifies the following Liouville equation*

$$\alpha f + \xi \cdot \nabla_x f - \frac{1}{2}\nabla_x V(x) \cdot \nabla_\xi f = Q(x,\xi), \quad \text{in } \mathcal{D}'(\mathbb{R}^{2d}) \tag{4.131}$$

*with*

$$Q(x,\xi) = \frac{\pi}{(2\pi)^d}|\hat{S}(\xi)|^2 \delta(x)\delta(\xi^2 + V(0) - E) \tag{4.132}$$

Under stronger decay and smoothness conditions, Theorem 4.9 is proved in [**4, 9**] for point source and in [**8, 40**] for source term supported on a subspace. In these works, an additional regularizing condition (1.11) is needed if $\alpha = 0$.

The following result seems to be new. It describes a strong outgoing radiation property of the semi-classical measure $f$.

***Theorem 4.10***. — *Under the conditions of Theorem 4.7 (b), there exists some $c_0 > 0$ such that*

$$\text{supp } f \cap \Omega_- = \varnothing, \tag{4.133}$$

*where $\Omega_- = \{(x,\xi); x \cdot \xi < c_0|x|\}$.*

*Proof*. — Let $\mu_0 > 0$ be given in Theorem 4.7 and let $0 < c_0 < \mu_0$. Then, any $a \in C_0^\infty(\Omega_-)$ belongs to $S_-(\mu_0)$. Since $a$ is equal to zero for $x$ near 0, the proof of Theorem 4.7 (b) shows that

$$\|a(x,hD)u_h\|_{B_{\frac{1}{2}}} \leq Ch^{1/2}.$$

It follows that

$$| < a^w(x,hD)u_h, u_h > | \leq Ch^{1/2}.$$

Using the subsequence defining $f$, one obtains that

$$\iint a(x,\xi)f(x,\xi)dxd\xi = \lim_{h_k \to 0} < a^w(x,h_k D)u_{h_k}, u_{h_k} >= 0$$

for any $a \in C_0^\infty(\Omega_-)$. (4.133) follows. □

The outgoing radiation condition determines uniquely a solution of (4.131).

***Corollary 4.11***. — *With the conditions of Theorem 4.7 (b), the solution $f$ of (4.131) is given by*

$$f(x,\xi) = \int_0^\infty e^{-\alpha s} Q(y(-s;x,\xi), \eta(-s;x,\xi))ds, \quad \text{in } \mathcal{D}'(\mathbb{R}^{2d}) \tag{4.134}$$

*where $(y(s),\eta(s))$ is solution of the Hamiltonian system*

$$\begin{cases} \frac{\partial y}{\partial s} = \eta(s), & y(0) = x, \\ \frac{\partial \eta}{\partial s} = -\frac{1}{2}V(y(s)), & \eta(0) = \xi. \end{cases} \tag{4.135}$$

*Proof.* — Let $\Omega = p^{-1}(]E-\delta, E+\delta[)$ for some $\delta > 0$ small enough. Under the condition (4.99), one can show that for any $R > 0$, there exists $T_0 > 0$ such that

$$(x,\xi) \in \Omega \text{ with } |x| < R \quad \Rightarrow \quad \Phi^t(x,\xi) \in \Omega_-, \quad \forall t < -T_0.$$

where $\Phi^t(x,\xi) = (y(t;x,\xi), \eta(t;x,\xi))$. In fact, the assumption (4.99) implies that for some $b_0 > 0$,

$$y(t;x,\xi) \cdot \eta(t;x,\xi) \leq x \cdot \xi + b_0 t \leq 0$$

for all $t \leq -T_0$ if $T_0 = T_0(R)$ is large enough and $(x,\xi) \in \Omega$ with $|x| < R$. For any $a \in C_0^\infty(\Omega)$, by (4.131) and (4.133), the function

$$G(t) =< e^{\alpha t} f \circ \Phi^t, a >$$

verifies

$$\frac{d}{dt} G(t) =< e^{\alpha t} Q \circ \Phi^t, a >, \quad G(t) = 0, \quad t < -T_0.$$

It follows that $G(t)$ is uniquely determined by

$$G(t) = \int_{-\infty}^t e^{\alpha s} < Q \circ \Phi^s, a > ds,$$

which implies

$$< f, a >= G(0) = \int_0^{+\infty} < e^{-\alpha s} Q \circ \Phi^{-s}, a > ds,$$

for any $a \in C_0^\infty(\Omega)$. Since supp $f \subseteq p^{-1}(E)$, Corollary 4.11 is proved. □

**Remark.** A weak version of radiation condition of the limiting measure $f$ is also proved in [**4, 9, 8, 39, 40**]. In [**4, 8, 40**], it is shown that under some conditions, $f$ verifies

$$\int_{R^{2d}} \overline{R(x,\xi)} f(x,\xi)\, dx\, d\xi = \int_{\mathbb{R}^{2d}} \overline{g(x,\xi)} Q(x,\xi)\, dx\, d\xi, \quad \forall\ R \in \mathcal{D}(\Omega), \tag{4.136}$$

where $g$ is the solution of the equation

$$\alpha g - \xi \cdot \nabla_x g + \frac{1}{2} \nabla_x V(x) \cdot \nabla_\xi g = R. \tag{4.137}$$

given by

$$g(x,\xi) = \int_0^\infty e^{-\alpha s} R(y(s;x,\xi), \eta(s;x,\xi)) ds. \tag{4.138}$$

Note that (4.134) implies that

$$\lim_{t \to -\infty} f \circ \Phi^t = 0, \quad \text{in } \mathcal{D}'(\Omega), \tag{4.139}$$

where $\Phi^t(x,\xi)$ is the solution to the Hamiltonian system. In fact, for $R \in \mathcal{D}(\Omega)$,

$$\int_{R^{2d}} \overline{R(x,\xi)} f(\Phi^t(x,\xi))\, dx\, d\xi = \int_{R^{2d}} \int_0^\infty e^{-\alpha s} \overline{R(\Phi^{s-t}(x,\xi))} ds Q(x,\xi)\, dx\, d\xi.$$

Since supp $Q$ is compact, the non-trapping condition implies that there exists $T_1 > 0$ such that for all $(x,\xi) \in \text{supp } Q$, one has

$$|\Phi^\tau(x,\xi)| > R_1, \quad \forall\ \tau > T_1,$$

where $R_1$ is taken large enough so that supp $R \subset \{|x| + |\xi| < R_1\}$. This shows

$$\int_{R^{2d}} \int_0^\infty e^{-\alpha s} \overline{R(\Phi^{s-t}(x,\xi))} ds Q(x,\xi)\, dx\, d\xi = 0, \quad t < -T_1.$$

This proves (4.139). Clearly, (4.139) also follows more directly from (4.133), as can be seen from the proof of Corollary 4.11.

The results of Theorems 4.9 and 4.10 hold for any subsequence $\{u_{h_{k_j}}\}$ extracted from a subsequence $\{u_{h_k}\}$ of $\{u_h\}$. The uniqueness of the limiting measure $f$ allows to conclude that the whole sequence $\{u_h\}$ satisfies

$$\lim_{h\to 0} < a^w(x,hD)u_h, u_h >= \iint a(x,\xi) f(x,\xi) dx d\xi, \tag{4.140}$$

for any $a \in C_0^\infty(\mathbb{R}^{2d})$. This shows that the sequence $\{u_h\}$ is pure, according to the terminology of [**14**].

The results presented in this Subsection shows that under some conditions, the stationary Schrödinger equation

$$(-h^2\Delta + V(x) - E)u_h = S^h(x)$$

converges, in the sense of semi-classical measures, to the Liouville equation

$$\xi \cdot \nabla_x f - \frac{1}{2}\nabla_x V(x) \cdot \nabla_\xi f = Q(x,\xi)$$

where

$$Q(x,\xi) = \frac{\pi}{(2\pi)^d} |\hat{S}(\xi)|^2 \delta(x)\delta(\xi^2 + V(0) - E)$$

and this convergence preserves the nature of the corresponding radiation conditions.

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