Séminaires & Congrès
15, 2007, p. 253–263

STABILITY OF QUANTUM HARMONIC OSCILLATOR UNDER TIME QUASI-PERIODIC PERTURBATION

by

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***Abstract.* —** We prove stability of the bound states for the quantum harmonic oscillator under non-resonant, time quasi-periodic perturbations by proving that the associated Floquet Hamiltonian has pure point spectrum.

***Résumé* (Stabilité de l'oscillateur harmonique quantique sous les perturbations quasi-périodiques)**

Nous démontrons la stabilité des états bornés de l'oscillateur harmonique sous les perturbations non-résonantes, quasi-périodiques en temps en démontrant que l'hamiltonien Floquet associé a un spectre purement ponctuel.

The stability of the quantum harmonic oscillator is a long standing problem since the establishment of quantum mechanics. The Schrödinger equation for the harmonic oscillator in $\mathbb{R}^n$ (in appropriate coordinates) is the following:

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}\sum_{i=1}^{n}(-\frac{\partial^2}{\partial x_i^2} + x_i^2)\psi, \tag{1}$$

where we assume

$$\psi \in C^1(\mathbb{R}, L^2(\mathbb{R}^n)) \tag{2}$$

for the moment. We start from the 1 dimensional case, $n = 1$. (1) then reduces to

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi. \tag{3}$$

The Schrödinger operator

$$H = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2) \tag{4}$$

2000 *Mathematics Subject Classification.* **—** 35, 37, 42.
***Key words and phrases.* —** Hermite-Fourier series, Floquet Hamiltonian.

is the 1-d harmonic oscillator. Since H is independent of t, it is amenable to a spectral analysis. It is well known that H has pure point spectrum with eigenvalues

$$\lambda_n = 2n+1, \quad n = 0, 1..., \tag{5}$$

and eigenfunctions (the Hermite functions)

$$h_n(x) = \frac{H_n(x)}{\sqrt{2^n n!}} e^{-x^2/2}, \quad n = 0, 1... \tag{6}$$

where $H_n(x)$ is the n^{th} Hermite polynomial, relative to the weight e^{-x^2} $(H_0(x) = 1)$ and

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \sqrt{\pi}\delta_{mn} \tag{7}$$

Using (5-7), the normalized L^2 solutions to (1) are all of the form

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n h_n(x) e^{i\frac{\lambda_n}{2}t} \quad (\sum |a_n|^2 = 1), \tag{8}$$

corresponding to the initial condition

$$\psi(x,0) = \sum_{n=0}^{\infty} a_n h_n(x) \quad (\sum |a_n|^2 = 1). \tag{9}$$

The functions in (8) are almost-periodic (in fact periodic here) in time with frequencies $\lambda_n/4\pi$, $n = 0, 1...$

Equation (3) generates a unitary propagator $U(t,s) = U(t-s,0)$ on $L^2(\mathbb{R})$. Since the spectrum of H is pure point, $\forall u \in L^2(\mathbb{R})$, $\forall\epsilon$, $\exists R$, such that

$$\inf_t \|U(t,0)u\|_{L^2(|x|\leq R)} \geq (1-\epsilon)\|u\| \tag{10}$$

by using eigenfunction (Hermite function) expansions. The harmonic oscillator (4) is an integrable system. The above results are classical. It is natural to ask how much of the above picture remains under perturbation, when the system is no longer integrable. In this paper, we investigate stability of the 1-d harmonic oscillator under time quasi-periodic, spatially localized perturbations. To simplify the exposition, we study the following "model" equation:

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + \delta|h_0(x)|^2 \sum_{k=1}^{\nu} \cos(\omega_k t + \phi_k)\psi, \tag{11}$$

on $C^1(\mathbb{R}, L^2(\mathbb{R}))$, where

$$0 < \delta \ll 1,\ \omega = \{\omega_k\}_{k=1}^{\nu} \in [0,2\pi)^{\nu},\ \phi = \{\phi_k\}_{k=1}^{\nu} \in [0,2\pi)^{\nu},\ h_0(x) = e^{-x^2/2}. \tag{12}$$

In particular, we shall study the validity of (10) for solutions to (11), when U is the propagator for (11). The method used here can be generalized to treat the equation

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + \delta V(t,x), \tag{13}$$

where V is C_0^{∞} in x and analytic, quasi-periodic in t.

The perturbation term, $O(\delta)$ term in (11) is motivated by the nonlinear equation:

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + M\psi + \delta|\psi|^2\psi \quad (0 < \delta \ll 1), \tag{14}$$

where M is a Hermite multiplier, *i.e.*, in the Hermite function basis,

$$M = \mathrm{diag}\,(M_n), \quad M_n \in \mathbb{R}, \tag{15}$$

$$Mu = \sum_{n=0}^{\infty} M_n(h_n, u)h_n, \text{ for all } u \in L^2(\mathbb{R}). \tag{16}$$

Specifically, (11) is motivated by the construction of time quasi-periodic solutions to (14) for appropriate initial conditions such as

$$\psi(x, 0) = \sum_{i=1}^{\nu} c_{k_i} h_{k_i}(x). \tag{17}$$

In (11), for computational simplicity, we take the spatial dependence to be $|h_0(x)|^2$ as it already captures the essence of the perturbation in view of (14, 17, 6). The various computations and the Theorem extend immediately to more general finite combinations of $h_k(x)$.

The Floquet Hamiltonian and formulation of stability. — It follows from [**32, 33**] that (11) generates a unique unitary propagator $U(t, s)$, t, $s \in \mathbb{R}$ on $L^2(\mathbb{R})$, so that for every $s \in \mathbb{R}$ and

$$u_0 \in H^2 = \{f \in L^2(\mathbb{R}) | \|f\|^2_{H^2} = \sum_{|\alpha+\beta|\leq 2} \|x^\alpha \partial_x^\beta f\|^2_{L^2} < \infty\}, \tag{18}$$

$$u(\cdot) = U(\cdot, s)u_0 \in C^1(\mathbb{R}, L^2(\mathbb{R})) \cap C^0(\mathbb{R}, H^2) \tag{19}$$

is a unique solution of (11) in $L^2(\mathbb{R})$ satisfying $u(s) = u_0$.

When $\nu = 1$, (11) is time periodic with period $T = 2\pi/\omega$. The 1-period propagator $U(T + s, s)$ is called the Floquet operator. The long time behavior of the solutions to (11) can be characterized by means of the spectral properties of $U(T + s, s)$ [**14, 21, 34**]. Furthermore the nature of the spectrum of U is the same (apart from multiplicity) as that of the Floquet Hamiltonian K [**31**]:

$$K = i\omega\frac{\partial}{\partial\phi} + \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + \delta|h_0(x)|^2\cos\phi \tag{20}$$

on $L^2(\mathbb{R}) \otimes L^2(\mathbb{T})$, where $L^2(\mathbb{T})$ is $L^2[0, 2\pi)$ with periodic boundary conditions.

Decompose $L^2(\mathbb{R})$ into the pure point H_{pp} and continuous H_c spectral subspaces of the Floquet operator $U(T + s, s)$:

$$L^2(\mathbb{R}) = H_{pp} \oplus H_c. \tag{21}$$

We have the following equivalence relations [**14, 34**]: $u \in H_{pp}(U(T + s, s))$ if and only if $\forall\epsilon > 0$, $\exists R > 0$, such that

$$\inf_t \|U(t, s)u\|_{L^2(|x|\leq R)} \geq (1 - \epsilon)\|u\|; \tag{22}$$

and $u \in H_c(U(T+s,s))$ if and only if $\forall R > 0$,

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t dt' \|U(t',s)u\|^2_{L^2(|x|\leq R)} = 0. \tag{23}$$

(Needless to say, the above statements hold for general time periodic Schrödinger equations.)

When $\nu \geq 2$, (10) is time quasi-periodic. The above constructions extend for small δ, cf. [**1, 12, 22**] leading to the Floquet Hamiltonian K:

$$K = i\sum_{k=1}^{\nu} \omega_k \frac{\partial}{\partial \phi_k} + \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + \delta |h_0(x)|^2 \sum_{k=1}^{\nu} \cos\phi_k \tag{24}$$

on $L^2(\mathbb{R}) \otimes L^2(\mathbb{T}^\nu)$, cf. [**7**]. This is related to the so called reducibility of skew product flows in dynamical systems, cf. [**12**]. We note that the Hermite-Fourier functions:

$$e^{-in\cdot\phi} h_j(x), \quad n \in \mathbb{Z}^\nu, \quad \phi \in \mathbb{T}^\nu, \quad j \in \{0, 1...\} \tag{25}$$

provide a basis for $L^2(\mathbb{R}) \otimes L^2(\mathbb{T}^\nu)$.

We say that the harmonic oscillator H is *stable* if K has pure point spectrum. Let $s \in \mathbb{R}$. This implies (by expansion using eigenfunctions of K) that given any $u \in L^2(\mathbb{R})$, $\forall \epsilon > 0$, $\exists R > 0$, such that

$$\inf_t \|U(t,s)u\|_{L^2(|x|\leq R)} \geq (1-\epsilon)\|u\|, \text{a.e.}\, \phi, \tag{26}$$

cf. [**7, 22**]. So (10) remains valid and we have dynamical stability. We now state the main results pertaining to (11).

Theorem. — *There exists $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$, there exists $\Omega \subset [0, 2\pi)^\nu$ of positive measure, asymptotically full measure:*

$$\text{mes}\,\Omega \to (2\pi)^\nu \quad \textit{as } \delta \to 0, \tag{27}$$

such that for all $\omega \in \Omega$, the Floquet Hamiltonian K defined in (24) has pure point spectrum: $\sigma(K) = \sigma_{pp}$. Moreover the Fourier-Hermite coefficients of the eigenfunctions of K have subexponential decay.

As an immediate consequence, we have

Corollary. — *Assume that Ω is as in the Theorem. Let $s \in \mathbb{R}$. For all $\omega \in \Omega$, all $u \in L^2(\mathbb{R})$, all $\epsilon > 0$, there exists $R > 0$, such that*

$$\inf_t \|U(t,s,\phi)u\|_{L^2(|x|\leq R)} \geq (1-\epsilon)\|u\|, \textit{a.e.}\, \phi, \tag{28}$$

where U is the unitary propagator for (11).

We note that this good set Ω of ω is a subset of Diophantine frequencies. This is typical for KAM type of persistence theorem. Stability under time quasi-periodic perturbations as in (11) is, generally speaking a precursor for stability under nonlinear perturbation as in (14) (cf. [**7, 6**]), where M plays the role of ω and varies the tangential frequencies. The above Theorem resolves the Enss-Veselic conjecture dated from their 1983 paper [**14**] in a general quasi-periodic setting.

A sketch of the proof of the Theorem. — Instead of working with K defined on $L^2(\mathbb{R})\otimes L^2(\mathbb{T}^\nu)$ directly, it is more convenient to work with its unitary equivalent H on $\ell^2(\mathbb{Z}^\nu\times\{0,\ 1...\})$, using the Hermite-Fourier basis in (25). We have

$$H = \text{diag }(n\cdot\omega + j + \frac{1}{2}) + \frac{\delta}{2} W\otimes\Delta \tag{29}$$

on $\ell^2(\mathbb{Z}^\nu\times\{0,\ 1...\})$, where W acts on the j indices, $j = 0,\ 1,\ 2...,$

$$W_{jj'} \sim \frac{1}{\sqrt{j+j'}} e^{-\frac{(j-j')^2}{2(j+j')}} \quad \text{for } j+j' \gg 1; \tag{30}$$

Δ acts on the n indices, $n\in\mathbb{Z}^\nu$,

$$\Delta_{nn'} = 1, \quad |n-n'|_{\ell^1} = 1,\ \Delta_{nn'} = 0, \quad \text{otherwise.} \tag{31}$$

The computation of W involves integrals of products of Hermite functions. We will explain shortly this computation, which is independent from the main thread of construction.

The principal new feature here is that W is long range. The j^{th} row has width $O(\sqrt{j})$ about the diagonal element W_{jj}. It is *not* and *cannot* be approximated by a convolution matrix. The potential x^2 breaks translational invariance. The annihilation and creation operators of the harmonic oscillator $a = \frac{1}{\sqrt{2}}(\frac{d}{dx}+x)$, $a^* = \frac{1}{\sqrt{2}}(-\frac{d}{dx}+x)$, satisfying $[a, a^*] = 1$, are generators of the Heisenberg group. So (19) presents a new class of problems distinct from that considered in [**2, 3, 4, 7, 6, 13, 24, 26**].

The proof of pure point spectrum of H is via proving pointwise decay of the finite volume Green's functions: $(H_\Lambda - E)^{-1}$, where Λ are finite subsets of $\mathbb{Z}^\nu\times\{0,\ 1...\}$ and $\Lambda \nearrow \mathbb{Z}^\nu\times\{0,\ 1...\}$. We need decay of the Green's functions at all scales, as assuming E an eigenvalue, *a priori* we do not have information on the center and support of its eigenfunction ψ. The regions Λ where $(H_\Lambda - E)^{-1}$ has pointwise decay is precisely where we establish later that ψ is small there.

For the initial scales, the estimates on $G_\Lambda(E) = (H_\Lambda - E)^{-1}$ are obtained by direct perturbation theory in δ for $0 < \delta \ll 1$. For subsequent scales, the proof is a multiscale induction process using the resolvent equation. Assume we have estimates on $G_{\Lambda'}$ for cubes Λ' at scale L'. Assume Λ is a cube at a larger scale L, $L \gg L'$. Intuitively, if we could establish that for most of $\Lambda' \subset \Lambda$, $G_{\Lambda'}(E)$ has pointwise decay, then assuming we have some *a priori* estimates on $G_\Lambda(E)$, we should be able to prove that $G_\Lambda(E)$ also has pointwise decay.

There are "two" directions in the problem, the higher harmonics direction n and the spatial direction j. The off-diagonal part of H is Toeplitz in the n direction, corresponding to the discrete Laplacian Δ. Since the frequency ω is in general a vector (if $\nu \geq 2$), $n\cdot\omega$ does not necessarily $\to\infty$ as $|n|\to\infty$. So the n direction is non-perturbative. We use estimates on $G_{\Lambda'}$ and semi-algebraic techniques as in [**5, 7**] to control the number of resonant Λ', where $G_{\Lambda'}$ is large, in Λ.

In the j direction, we do analysis, *i.e.*, perturbation theory. This is the new feature. From (29) and Schur's lemma, $\|W \otimes \Delta\| = O(1)$. So the ℓ^2 norm of the perturbation does not decay (relative to eigenvalue spacing) in j. However when $\delta = 0$, H is diagonal with eigenvalues $n \cdot \omega + j$ and eigenfunctions $\delta_{n,j}$, the canonical basis for $\ell^2(\mathbb{Z}^\nu \times \{0, 1...\})$. We have

$$\|[W \otimes \Delta]\delta_{n,j}\| = O(\frac{1}{j^{1/4}}) \quad (j \geq 1), \tag{32}$$

which decays in j.

This is intuitively reasonable, as W stems from a spatially localized perturbation from (11). As j increases, The Hermite functions h_j become more extended, cf. (6). So the effect of the spatial perturbation should decrease as j increases.

Assuming ω is Diophantine:

$$\|n \cdot \omega\|_{\mathbb{T}} \geq \frac{c}{|n|^\alpha} \quad (c > 0,\, n \neq 0,\, \alpha > 2\nu), \tag{33}$$

where $\|\cdot\|_{\mathbb{T}}$ is the distance to the nearest integer, this enables us to preserve local eigenvalue spacing for Λ which are appropriately proportioned in n, j. This in turn leads to decay of Green's functions. Combining the estimates in the n and j directions, we obtain estimates on the Green's function at the larger scale L.

Integrals of products of Hermite functions. — From (24, 29), computation of W involves computing the following integrals:

$$\int_{-\infty}^{\infty} h_0^2(x) h_m(x) h_n(x) dx \tag{34}$$

$$= \frac{1}{\sqrt{2^{n+m} m! n!}} \int_{-\infty}^{\infty} e^{-2x^2} H_0^2(x) H_m(x) H_n(x) dx, \quad m,\, n = 0,\, 1..., \tag{35}$$

where H_m, H_n are respectively the m^{th}, n^{th} Hermite polynomial, $H_0(x) = 1$.

Let

$$I = \int_{-\infty}^{\infty} e^{-2x^2} H_0^2(x) H_m(x) H_n(x) dx. \tag{36}$$

The idea is to view $e^{-x^2} H_0^2(x)$ as $e^{-x^2} H_0(\sqrt{2}x)$, *i.e.*, the 0^{th} Hermite function relative to the weight e^{-2x^2} and to use the generating function of Hermite polynomials to reexpress

$$H_m(x) H_n(x) = \sum_{\ell=0}^{m+n} a_\ell H_\ell(\sqrt{2}x). \tag{37}$$

We then have

$$I = a_0 \int [H_0(\sqrt{2}x)]^2 e^{-2x^2} dx = a_0 \sqrt{\pi/2} \tag{38}$$

using (7), recovering an apparently classical result, which could be found in *e.g.*, [**18, 28**]. More generally, we are interested in computing

$$I=\int_{-\infty}^{\infty}e^{-2x^2}H_p(x)H_q(x)H_m(x)H_n(x)dx,\quad p,\,q,\,m,\,n=0,\,1... \tag{39}$$

which are needed for the nonlinear equation or if we consider more general perturbations of the harmonic oscillator. Following the same line of arguments, we decompose $H_p(x)H_q(x)$ into

$$H_p(x)H_q(x)=\sum_{\ell=0}^{p+q}b_\ell H_\ell(\sqrt{2}x). \tag{40}$$

Combining (23) with (25), and assuming (without loss of generality), $p+q\leq m+n$, we then have

$$I=\sum_{\ell=0}^{p+q}a_\ell b_\ell c_\ell, \tag{41}$$

where $c_\ell=\int_{-\infty}^{\infty}[H_\ell(\sqrt{2}x)]^2e^{-2x^2}dx$.

The computation for general p, q is technically more involved and is carried out in [**30**]. Unlike the special case $p=q=0$, we did not find the corresponding result for general p, q in existing literature.

The computation of I in (39) is *exact*, reflecting the integrable nature of the quantum harmonic oscillator. The proof of the theorem is, however, general. It is applicable as soon as the kernel W satisfies (30). Following the precedent discussion on I for general p, q, and using properties of Hermite series (cf. [**29**] and references therein), one should be able to extend the Theorem to V, which are C_0^∞in x and analytic quasi-periodic in t, leading to perturbation kernels in the Hermite-Fourier basis satisfying conditions similar to (30) in the j direction and exponential decay condition in the n direction.

When the perturbation V is independent of time and is a 0^{th} order symbol, satisfying

$$|d^\alpha V|\leq C_\alpha(1+|x|)^{-\alpha},\quad \alpha=0,\,1... \tag{42}$$

the corresponding Schrödinger equation has been studied in *e.g.*, [**27, 23, 35**], where it was shown that certain properties of the harmonic oscillator equation extend to the perturbed equation. The spectral property needed for the construction here is more detailed and stringent. Hence it is reasonable to believe that the set of potentials V will be more restrictive than that in (42)

Some perspectives on the Theorem. — The Theorem shows that for small δ, there is a subset $\Omega\subset[0,2\pi)^\nu$ of Diophantine frequencies of positive measure, such that if $\omega\in\Omega$, then (10) is satisfied. Hence spatially localized solutions remain localized for all time. It is natural to ask what happens if the forcing frequencies ω are in the complement set, $\omega \in \Omega^c$.

If ω is rational, the perturbing potential V is bounded and has sufficiently fast decay at infinity, it is known from general compactness argument [**14**] that the Floquet Hamiltonian has pure point spectrum. In our example, this can be seen as follows. In (11) restricting to periodic perturbation ($\nu = 1$), it is easy to see that for

$$\forall\omega,\ A = (n\omega + j + z)^{-1} W \otimes \Delta, \text{ where } \Im z = 1 \text{ is compact.} \tag{43}$$

We remark that (43) holds *for all scalar* ω.

Assume ω is rational: $\omega = p/q$, $(q \neq 0)$. Since $H_0 = n\omega + j$ has pure point spectrum (with infinite degeneracy) and the spacing between different eigenvalues is $1/q$, (43) implies that H has pure point spectrum. When ω is irrational, H_0 typically has dense spectrum. No conclusion can be drawn from (43).

If V is unbounded, we have a different situation. The results in [**20, 19**] combined with [**34**] show that for the following *unbouded* time *periodically* perturbed harmonic oscillator:

$$i\frac{\partial u}{\partial t} = \frac{1}{2}(-\Delta + x^2)u + 2\epsilon(\sin t)x_1 u + \mu V(t,x)u, \quad x = (x_1, ...x_n) \in \mathbb{R}^n \tag{44}$$

where $V(t,x)$ is a real valued smooth function of (t,x), satisfying

$$V(t+2\pi, x) = V(t), \quad |V(t,x)| \lesssim |x| \quad \text{as } x \to \infty, \quad |\partial_x^\alpha V(t,x)| \leq C_\alpha, \quad |\alpha| \geq 1, \tag{45}$$

the solutions diffuse to infinity as $t \to \infty$. More precisely, for all $u_0 \in L^2(\mathbb{R}^n) \cap H^2(\mathbb{R}^n)$, for any $R > 0$, the solution u_t satisfies

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_0^T dt \|u_t\|_{L^2(|x| \leq R)} = 0. \tag{46}$$

In (44), $\nu = 1$ (periodic), $\omega = 1$, $\omega \in \Omega^c$, (46) is an opposite of (10). However the perturbation is unbounded. Moreover the proof in [**19**] uses in an essential way that the potential is linear at infinity, hence positivity of the commutator: $[\frac{d}{dx_1}, x_1] = 1$.

In the exactly solvable case where the time periodic perturbations is quadratic in the spatial coordinates, it is known that the Floquet Hamiltonian exhibits a transition between pure point and continuous spectrum as the frequency is varied [**8**]. The perturbation there is again unbounded.

Some related results. — To our knowledge, when $\omega \in \Omega$ is nonresonant, there were no results in the literature on the perturbed harmonic oscillator equation of type (11), even in the time periodic case, *i.e.*, $\omega \in [0, 2\pi)$. The main difficulties encountered by the traditional KAM method seem to be (i) the eigenvalue spacing for the unperturbed operator does *not* grow, $\lambda_{k+1} - \lambda_k = 1$, (ii) the perturbation W in the Hermite basis has slow decay (30).

When the eigenvalue spacing for the unperturbed operator grows: $|E_{j+1} - E_j| > j^\beta \quad (\beta > 0)$, which corresponds to a potential growing faster than quadratically at infinity, and when the perturbation is *periodic* in time, related stability results were proven in [**11**]. In [**9**], under time *periodic* perturbation and replacing W in (30) by a faster decaying kernel, hence decaying norm in j, which no longer corresponds to the physical case of harmonic oscillator under time periodic, spatially localized perturbation, stability results were also proven. Both papers used some modified KAM method.

Motivation for studying (11). — As mentioned earlier, the motivation for analyzing (11) partly comes from the nonlinear equation (14). In [**2, 3, 4, 13**], time quasi-periodic solutions were constructed for the nonlinear Schrödinger equation in $\mathbb{R}^d$ with Dirichlet or periodic boundary condition

$$i\frac{\partial}{\partial t}\psi = (-\Delta + M)\psi + \delta|\psi|^{2p}\psi, \qquad (p \in \mathbb{N}^+;\ 0 < \delta \ll 1) \tag{47}$$

where M is a Fourier multiplier; see [**24, 26**] for the Dirichlet case in $\mathbb{R}$ with a potential in place of M. In [**6**], time quasi-periodic solutions were constructed for the nonlinear random Schrödinger equation in $\mathbb{Z}^d$

$$i\frac{\partial}{\partial t}\psi = (-\epsilon\Delta + V)\psi + \delta|\psi|^{2p}u, \qquad (p \in \mathbb{N}^+;\epsilon,\ 0 \ll \delta \ll 1), \tag{48}$$

where $V = \{v_j\}_{j\in\mathbb{Z}^d}$ is a family of random variables.

The proofs in [**2, 3, 4, 6**] use operator method, which traces its origin to the study of Anderson localization [**15**]. This method was first applied in the context of Hamiltonian PDE in [**10**]. The proofs in [**13, 24, 26**] use KAM type of method.

In (47) (specializing to 1-d), the eigenvalues of the linear operator are n^2, so $E_{n+1} - E_n \sim n$, the eigenfunctions e^{inx}, however, are extended: $|e^{inx}| = 1$ for all x. Let us call this case A, where there is eigenvalue separation. In (48), the eigenvalues of the linear operator form a *dense* set, the eigenfunctions, on the other hand are not only *localized* but localized about *different* points in $\mathbb{Z}^d$ from Anderson localization theory, see *e.g.*, [**16, 17**]. This is case B, where there is eigenfunction separation. The existence of time quasi-periodic solutions, *i.e.*, KAM type of solutions in A is a consequence of eigenvalue separation; while in B, eigenfunction separation.

Equation (11) and its nonlinear counterpart

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + M\psi + \delta|\psi|^{2p}\psi, \qquad (p \in \mathbb{N}^+;\ 0 < \delta \ll 1), \tag{49}$$

where M is a Hermite multiplier, stand apart from both (47, 48). It is neither A, nor B. There is eigenvalue spacing, but it is a constant: $\lambda_{n+1} - \lambda_n = 1$. In particular, it does not grow with n. The eigenfunctions (Hermite functions) h_n are "localized" about the origin. But they become more extended as n increases because of the presence of the Hermite polynomials, cf. (6). This in turn leads to the long range kernel W in (30) and long range nonlinearity in (49) in the Hermite function basis, cf. [**30**].

From the KAM perspective a la Kuksin, this is a borderline case, where Theorem 1.1 in [**25**] does not apply. The more recent KAM type of theorem in [**13**] does not apply either, because W is long range and not close to a Toeplitz matrix (cf. (30)) for the reasons stated earlier. These are the features which make (11, 49) interesting from a mathematics point of view, aside from its apparent relevance to physics.

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