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GALOIS THEORY AND PAINLEVÉ EQUATIONS

by

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Abstract. — The paper consists of two parts. In the first part, we explain an excellent idea, due to mathematicians of the 19-th century, of naturally developing classical Galois theory of algebraic equations to an infinite dimensional Galois theory of non-linear differential equations. We show with an instructive example how we can realize the idea of the 19-th century in a rigorous framework. In the second part, we ask questions arising from general Galois theory and Galois theoretic study of Painlevé equations. We also propose an infinite dimensional Galois theory of difference equations.

***Résumé* (Théorie de Galois et Équations de Painlevé).** — Dans une première partie, nous rappelons une excellente idée de mathématiciens du $19^{\text{ème}}$ siècle en vue d'étendre la théorie de Galois classique pour les équations algébriques en une théorie de Galois de dimension infinie pour les équations différentielles non-linéaires. Nous illustrons par un exemple instructif comment concrétiser cette idée de façon rigoureuse.

Dans une deuxième partie, nous formulons des questions liées à la théorie de Galois générale et aux aspects galoisiens des équations de Painlevé. Nous esquissons, en outre, une théorie de Galois de dimension infinie pour les équations aux différences.

1. Introduction

Since Lie tried to apply the rich idea of Galois and Abel in algebraic equations to analysis, Galois theory of differential equations has been attracting mathematicians. Finite dimensional differential Galois theory was developed by Picard, Vessiot and Kolchin and is widely accepted. As Lie already noticed it, the most important part of differential Galois theory is, however, infinite dimensional. After a few trails have been done about 100 years ago, the subject was almost forgotten. We proposed a differential Galois theory of infinite dimension [**14**] in 1996 which is a Galois theory

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of differential field extension. On the other hand, a Galois theory of foliation by B. Malgrange [**11**] that is also infinite dimensional, appeared in 2001. We do not feel that they are well understood.

Our aim in Part I, Invitation to Galois theory, is to explain with examples that our theory is a consequence of natural development of Galois theory of algebraic equations. We recall how mathematicians of the 19-th century understood Galois theory of algebraic equations and extend it to linear ordinary differential equations in §§2 and 3. §4 is the most substantial section of the first part. We show a marvelous idea of mathematicians of the 19-th century in Subsection 4.1 and realize it in the framework of algebraic geometry. Since the reader can find rigorous reasonings in [**14**], we repeatedly use a concrete and yet sufficiently general case, Instructive Case (IC) in Subsection 4.4, to illustrate clearly what is going on.

In Part II, we ask questions about (1) general Galois theory and (2) Galois theoretic study of Painlevé equations. Among the questions about general Galois theory, we cite descent of the field of definition of our Galois group Infgal(L/K) (Questions 1, 2 and 3) and comparison of Malgrange's theory and ours (Question 4), while calculation of Galois group of Painlevé equations (Question 6), understanding of a remarkable paper of Drach on the sixth Painlevé equation (Questions 7, 8, ..., 11) and arithmetic property of the sixth Painevé equation (Questions 17 and 18) belong to the questions about Galois theoretic study of Painlevé equations. We also propose a Galois theory of difference equation of infinite dimension and calculation of Galois group for qP6 of Jimbo and Sakai (Question 12). We added a star to those questions that seem to require a new idea. The mark is, however, nothing more than a personal impression of the author.

PART I
INVITATION TO GALOIS THEORY

2. Galois theory of algebraic equations

The aim of the first part is to explain how an intuitive idea of Galois theory of algebraic equations develops to infinite dimensional differential Galois theory of non-linear differential equations. We described the latter rigorously in a general framework [**14**]. In this note we try to be more intuitive than formal so that the reader can realize how natural the basic idea of our theory is.

Principal homogeneous space is one of the main ingredients of Galois theory. Let us start by recalling the definition.

Definition 2.1. — *Let G be a group operating on a set S. Then we say that the operation (G, S) is a principal homogeneous space if for an element $s \in S$, the map*

$$G \longrightarrow S, \quad g \longmapsto gs$$

is bijective.

Inspired by Galois theory for algebraic equations, S. Lie had a plan to apply the rich idea of Galois and Abel to differential equations. Galois theory of algebraic equations is an ideal theory and it has been the model of generalizations. Let us go back to the 19-th century and see how the mathematicians of that time understood Galois theory and how they tried to generalize it.

Let K be a field and let

$$F(x) := a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad a_i \in K, \text{ for } 0 \leq i \leq n \tag{1}$$

$a_0 \neq 0$, be an algebraic equation with coefficients in K. We suppose for simplicity the field K is of characteristic 0. We assume that the roots of the algebraic equation (1) are distinct. Then the symmetric group S_n of degree n on the n letters

$$\{1, 2, \cdots, n\}$$

operates on the set

$$S := \{(x_1, x_2, \cdots, x_n) \,|\, F(x_i) = 0, \text{ for } 1 \leq i \leq n, \, x_i \neq x_j \text{ if } i \neq j\}$$

of ordered sets $(x_1, x_2, \cdots x_n)$ of roots as permutations of the roots and

$$(S_n, S)$$

is a principal homogeneous space.

The basic symmetric functions are expressed by coefficients.

$$\sum_{i=1}^{n} x_i = -\frac{a_1}{a_0},$$
$$\sum_{1\leq i<j\leq n} x_i x_j = \frac{a_2}{a_0},$$
$$\cdots$$
$$x_1 x_2 \cdots x_n = (-1)^n \frac{a_n}{a_0}.$$

If there is no constraints among the roots

$$x_1, x_2, \cdots, x_n$$

with coefficients in K other than those that are a consequence of the relations above, then the Galois group of equation (1) is the full symmetric group S_n. If there are constraints, they determine a subgroup G of S_n, consisting of those elements leaving all the constraints invariant, as Galois group of the algebraic equation (1). To be more precise, let us consider all rational functions

$$A_\alpha(X_1,\, X_2, \cdots, X_n) \in K(X_1,\, X_2, \cdots, X_n)$$

of variables $X_1,\, X_2, \cdots, X_n$ with coefficients in K indexed by an appropriate set I such that

$$A_\alpha(x_1,\, x_2, \cdots, x_n) \in K,$$

The constraints $A_\alpha(x)$ determine the Galois group G as a subgroup of the symmetric group S_n consisting of elements of S_n leaving all the constraints $A_\alpha(x)$ invariant. Namely

$$G := \{g \in S_n \mid A_\alpha(x_{g(1)},\, x_{g(2)}, \cdots, x_{g(n)}) = A_\alpha(x_1,\, x_2, \cdots, x_n) \text{ for all } \alpha \in I\}$$

Let us illustrate this by an example. Let us consider the following algebraic equation over $\mathbf{Q}$.

$$x^3 - 7x + 7 = (x - x_1)(x - x_2)(x - x_3) = 0. \tag{2}$$

Upon setting

$$\mathbf{x} = (x_1, x_2, x_3),$$

we have a constraint

$$D(\mathbf{x}) := (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = \pm 7 \in \mathbb{Q}.$$

$D(\mathbf{x})$ takes value $+7$ or -7 according as the order of the roots. In fact, $D(\mathbf{x})^2$ is, by definition, the discriminant of the cubic equation (2) so that

$$D(\mathbf{x})^2 = -4 \times (-7)^3 - 27 \times 7^2 = 49.$$

Indeed, the discriminant of a cubic equation

$$x^3 + ax + b = 0$$

is equal to

$$-4a^3 - 27b^2.$$

The Galois group must leave the constraint

$$D(\mathbf{x}) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

invariant so that the Galois group is a subgroup of the alternating group $A_3 \subset S_3$. We can moreover show that the Galois group coincides with the alternating group A_3.

We see how principal homogeneous spaces appear in this context. To this end, let us set

$$S := \{(x_1, x_2, x_3) \mid F(x_i) = 0\},$$
$$S_+ := \{\mathbf{x} \in S \mid D(\mathbf{x}) = 7\},$$
$$S_- := \{\mathbf{x} \in S \mid D(\mathbf{x}) = -7\}$$

so that we have

$$S = S_+ \amalg S_-.$$

The alternating group A_3 operates on both sets S_+, S_- and

$$(A_3, S_+), \qquad (A_3, S_-)$$

are principal homogeneous spaces. We started from the principal homogeneous space

$$(S_3, S)$$

and we decompose it to get two principal homogeneous spaces

$$(A_3, S_+), \qquad (A_3, S_-).$$

What makes Galois theory of algebraic equation useful is the fact that we have the Galois correspondence. Let us come back to the algebraic equation (1). We denote by $\bar{K}$ an algebraic closure of K. Let L be a subfield of $\bar{K}$ generated over K by all the roots x_i's for $1 \le i \le n$ of the algebraic equation (1). Namely

$$L := K(x_1,\, x_2, \cdots, x_n) \subset \bar{K}.$$

This type of field extension, a field extension generated over a field K by all the roots of an algebraic equation with coefficients in K, is called a Galois extension. Let us denote the Galois group of the equation (1) by $G(L/K)$. We can show that the group $G\,(L/K)$ is isomorphic to the group $\mathrm{Aut}(L/K)$ of K-automorphisms of the field L so that the group G depends only on the field extension L/K that the algebraic equation (1) determines! We owe this eminent idea to Dedekind. Let M be an intermediate field of the field extension L/K. Then since the coefficients of the algebraic equation (1) are in K and hence in M and since

$$L = K(x_1,\, x_2, \cdots, x_n) = M(x_1,\, x_2, \cdots, x_n),$$

the field extension L/M is also Galois. Hence we can speak of the Galois group $G\,(L/M)$ of the field extension L/M, which is a subgroup of the Galois group $G\,(L/K)$. We have thus defined a map φ from the set

$$\mathit{Field}\,(L/K)$$

of intermediate fields of the field extension L/K to the set of subgroups

$$\mathit{Group}\,(G)$$

of the Galois group $G = G\,(L/K)$ sending an intermediate subfield M to the subgroup $G\,(L/M)$:

$$\varphi : \mathit{Field}\,(L/K) \to \mathit{Group}\,(G).$$

Conversely let H be a subgroup of the Galois group $G = G\,(L/K)$. Then H determines an intermediate field

$$L^H := \{z \in L \mid g(z) = z \text{ for every element } g \in H \subset G = \mathrm{Aut}(L/K)\ \}$$

consisting of those elements of the field L that are left invariant by all the element of H that is a subgroup of the field automorphism group $\mathrm{Aut}(L/K)$.

Theorem 2.2. — *The mappings*

$$\varphi : Field(L/K) \to Group\,(G), \qquad \psi : Group\,(G) \to Field\,(L/K)$$

give a 1:1 correspondence between the elements of the two sets

$$Field\,(L/K), \qquad Group\,(G).$$

Namely

$$\varphi \circ \psi = id, \qquad \psi \circ \phi = id.$$

For an intermediate field M, the following two conditions are equivalent.

1. *The extension M/K is Galois.*

2. *The corresponding subgroup $N := G\,(L/M)$ is a normal subgroup of the Galois group $G = G\,(L/K)$.*

When these equivalent conditions are satisfied, we have a natural group isomorphism

$$G/N \simeq G\,(M/K).$$

3. Picard-Vessiot Theory

An ordinary differential field (F, d) consists of a field F and a derivation $d : F \to F$ on F. For an element $a \in F$ we denote often $d(a)$ by a' and we use the following notation $a^{(2)} = d(d(a))$, $a^{(3)} = d(a^{(2)}), \cdots$. An element $a \in F$ is said to be a constant if $d(a) = 0$. The set C_F of constants of F forms a subfield of F called the field of constants of F. When there is no danger of confusion, we do not make the derivation d explicit. Now let K be an ordinary differential field of characteristic 0 and $\mathbb{K}$ a differential overfield such that the field of constants $C_{\mathbb{K}}$ coincides with C_K. Given a matrix $A \in M_n(K)$, we consider a system of linear differential equations

$$Y' = AY, \tag{3}$$

where $Y \in GL_n(\mathbb{K})$. We denote by S the set of all the solutions of (3) in $\mathbb{K}$. Namely,

$$S := \{Y \in GL_n(\mathbb{K}) \,|\, Y' = AY\}.$$

Lemma 3.1. — *The following assertion holds.*

1. *For $Y \in S$, $g \in GL_n(C_{\mathbb{K}})$, $Yg \in S$.*
2. *If $Y_1, Y_2 \in S$, then*

$$Y_1 Y_2^{-1} \in GL_n(C_{\mathbb{K}})$$

.

Proof. — The first assertion is trivial. To prove the second, it is sufficient to notice

$$\begin{aligned}(Y_1^{-1}Y_2)' = (Y_1^{-1})'Y_2 + Y_1^{-1}Y_2' &= (-Y_1^{-1}Y_1'Y_1^{-1})Y_2 + Y_1^{-1}AY_2 \\ &= (-Y_1^{-1}(AY_1)Y_1^{-1})Y_2 + Y_1^{-1}AY_2 = 0.\end{aligned}$$

□

Lemma (3.1) shows that the general linear group $GL_n(C_{\mathbb{K}}) = GL_n(C_K)$ operates on the set S by right multiplication in such a manner that $(GL_n(C_{\mathbb{K}}), S)$ is a principal homogeneous space as in the case of algebraic equations.

From now on, we take a solution $Y \in S$ once for all. The choice of a solution does not affect the argument below. Indeed, the other solutions are expressed as Yg for $g \in GL_n(C_K)$. As in the case of algebraic equations, if there is no constraints among the entries of Y except for trivial constrains given by elements of K, then the Galois group of the linear differential equation (3) is the full general linear group $GL_n(C_K)$. Otherwise, constraints determine a closed subgroup of the algebraic group $GL_n(C_K)$ consisting those elements of the algebraic group $GL_n(C_K)$ leaving all the constraints invariant as the Galois group of the linear differential equation (3). Here, we should make the definition of the constraints clear. A constraint should be a rational function

$$A(Y, Y', Y^{(2)}, \cdots)$$

with coefficients in K of the entries y_{ij}'s and their derivatives for $1 \leq i, j \leq n$ such that the value

$$A(Y, Y', Y^{(2)}, \cdots) \in K.$$

But, thanks to the differential equation (3), we can eliminate the derivatives Y', $Y^{(2)}, \cdots$. So a constraint is a rational function $A(Y)$ of the entries y_{ij}'s for $1 \leq i, j \leq n$ of Y with coefficients in K such that $A(Y) \in K$. In the most general case there is no non-trivial constraint. Indeed, in that case, the entries y_{ij} $(1 \leq i, j \leq n)$ are algebraically independent over the base field K.

Let us consider, for example, the Bessel equation.

$$y'' + x^{-1}y' + (1 - \nu^2 x^{-2})y = 0, \tag{4}$$

$\nu \in \mathbb{C}$ being a complex parameter. We assume $\nu \notin 1/2 + \mathbf{Z}$. If we write the Bessel equation (4) in matrix form,

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 + \nu^2 x^{-2} & -x^{-1} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}.$$

We have to clarify the differential fields. The base field K is the field $\mathbb{C}(x)$ of rational functions with coefficients in $\mathbb{C}$ and the overfield $\mathbb{K}$ is the field of all the meromorphic functions on a small neighborhood of 1 in the complex plane. For example an open disc centered at 1 with radius 1 on the complex plane. The derivation d is the derivation d/dx with respect to the independent variable x.

Lemma 3.2. — *There exists a constant $c \in C_K$ such that*

$$det\, Y = cx^{-1}.$$

Proof. — We know

$$(det\, Y)^{-1}(\det Y)' = tr\, A = -x^{-1}.$$

So $\det Y$ satisfies a linear homogeneous differential equation

$$y' + x^{-1}y = 0 \tag{5}$$

as well as x^{-1}. Hence

$$(det\, Y)x \in L$$

is a constant so that

$$(det\, Y)x = c \in C_L = C_K. \quad \square$$

Now $det\, Y = cx^{-1} \in K$ gives a constraint. The Galois group G is a subgroup of $GL_2(C_K)$ consisting of those elements leaving $\det Y$ invariant. Namely

$$G \subset \{g \in GL_2(C_K) \mid \det(Yg) = \det Y\} = SL_2(C_K).$$

We can show that indeed we have

$$G = SL_2(C_K).$$

See Kolchin [**10**], Appendix.

Now we look at principal homogeneous spaces appearing here. Let us set

$$S_c = \{Y \in S \mid \det Y = cx^{-1}\}.$$

for $c \in C_K$. Then the Galois group $G = SL_2(C_K)$ operates on the set S_c by right multiplication such that the operation

$$(SL_2(C_K), S_c)$$

is a principal homogeneous space for every $c \in C_K$ and

$$S = \coprod_{c \in C_K} S_c.$$

This is the coset decomposition $GL_2(C_K)/SL_2(C_K)$. We started from the principal homogeneous space

$$(GL_2(_K), S)$$

and arrived at the smaller principal homogeneous spaces

$$(SL_2(C_K), S_c)$$

just as in the case of algebraic equations.

Let us come back to the general linear differential equation (3) defined over the differential field K. The field $L := K(Y) = K(y_{ij})_{1 \le i,j \le n}$ is closed under the derivation so that $K(Y)/K$ is a differential field extension, which is called a Picard-Vessiot extension. We can show that the Galois group G is isomorphic to the automorphism group $\operatorname{Aut}(L/K)$ of differential field extension. So the Galois group depends only on the differential field extension L/K and we set $G = G\,(L/K)$. In Picard-Vessiot theory we have Galois correspondence.

Theorem 3.3. — *Let L/K be a Picard-Vessiot extension with Galois group G. If the field C_K of constants of the base field K is algebraic closed, then the mappings in theorem (2.2) give a 1:1 correspondence between the elements of two sets.*

1. *The set $Field\,(L/K)$ of differential intermediate fields of the Picard-Vessiot extension L/K.*
2. *The set of closed algebraic subgroups of the Galois group G defined over C_K.*

For an differential intermediate field M, the following two conditions are equivalent.

1. *The extension M/K is Picard-Vessiot.*
2. *The corresponding algebraic subgroup $N = G(M/K)$ of $G(L/K)$ is normal.*

When these equivalent conditions are satisfied, then we have a natural isomorphism

$$G/N \simeq G\,(M/K).$$

Remark 3.4. — As the form of the linear differential equation shows, Picard-Vessiot theory is a Galois theory on the general linear group $GL_n(C)$ and its closed subgroups. Such algebraic groups are called linear algebraic groups. Hence we can say that Picard-Vessiot theory is a Galois theory on a linear algebraic group. We can construct a similar theory on algebraic groups in general other that the linear algebraic groups. This generalization was already known in the 19-th century and later worked out thoroughly by Kolchin.

4. Non-linear differential equation

4.1. Idea of mathematicians of the 19-th century. — Let $(K,\, d)$ be an ordinary differential field of meromorphic functions over a complex domain so that the derivation d of the differential field K is the derivation d/dx with respect to the independent variable x. One of the simplest examples is the field $K = (\mathbb{C}(x), d/dx)$ of rational functions.

We want to define Galois group of a non-linear algebraic differential equation with coefficients in K. To simplify the situation, we assume that the given algebraic differential equation is solved by $y^{(n)}$. Namely,

$$y^{(n)} = A(x,\, y,\, y',\cdots,y^{(n-1)}), \tag{6}$$

where $A \in K(y,\, y',\cdots,\, y^{(n-1)})$ is a rational function of

$$y,\, y',\cdots,\, y^{(n-1)}$$

with coefficients in K.

Definition 4.1. — *A meromorphic function*

$$F(X,\, Y,\, Y',\cdots,\, Y^{(n-1)})$$

of $(n+1)$-variables is a first integral of the algebraic differential equation (6) if for every solution $y(x)$ of the algebraic differential equation (6),

$$F(x,\, y(x),\, y'(x), \cdots,\, y^{(n-1)}(x))$$

is a constant, i.e., independent of x.

The following proposition is well-known.

Proposition 4.2. — *The following conditions are equivalent.*

1. *F is a first integral of the algebraic differential equation (6).*
2. *F satisfies a linear partial differential equation*

$$LF = 0, \tag{7}$$

where

$$L := \partial/\partial X + Y'\partial/\partial Y + \cdots + Y^{(n-1)}\partial/\partial Y^{(n-2)} + A(X, Y, Y', \cdots, Y^{(n-1)})\partial/\partial Y^{(n-1)}.$$

Let

$$\mathbf{F} = (F_1, F_2, \cdots, F_n)$$

be an ordered set of n-independent first integrals of (6). Namely

$$F_1, F_2, \cdots, F_n$$

be n-first integrals meromorphic over a sub-domain of $\mathbb{C}^{n+1}$ such that the Jacobian

$$\frac{J(F_1,\, F_2,\, \cdots,\, F_n)}{J(Y^{(0)},\, Y^{(1)},\, \cdots,\, Y^{(n-1)})} = \det[\partial F_i/\partial Y^{(j)}]_{1\le i\le n,\, 0\le j\le n-1} \neq 0. \tag{8}$$

Given a set of constants $\mathbf{c} = (c_0, c_1, \cdots, c_{n-1}) \in \mathbb{C}^n$, we can solve by implicit function theorem $y_i(x, \mathbf{c})$ for $0 \le i \le n-1$ such that

$$F_i(x, y_0(x, \mathbf{c}), \cdots, y_{n-1}(x, \mathbf{c})) = c_{i-1}$$

for $1 \le i \le n$. The assumption (8) that the Jacobian $\neq 0$ implies that we have

$$\partial^j y_0(x, \mathbf{c})/\partial x^j = y_j(x, \mathbf{c})$$

and

$$\partial^n y_0/\partial x^n = A(x,\, y_0,\, y_0', \cdots,\, y_0^{(n-1)})$$

so that $y_0(\mathbf{c},\, x)$ is a solution of the algebraic differential equation (6).

Theorem 4.3. — *Modulo implicit function theorem, which is transcendental by nature, the following procedures are equivalent.*

1. *To find a general solution $y(x,\, \mathbf{c})$ of the algebraic differential equation (6).*
2. *To find independent solutions F_i of linear partial equation $LF_i = 0$ for $1 \le i \le n$.*

Proof. — We have shown above how the second condition implies the first condition. Conversely, given a general solution $y(x,\,\mathbf{c})$ of (6) so that the Jacobian

$$\frac{J(y(x,\,\mathbf{c}),\,y'(x,\,\mathbf{c}),\,\cdots,\,y^{(n-1)}(x,\,\mathbf{c}))}{J(c_1,\,c_2,\,\cdots,\,c_n)} \neq 0. \tag{9}$$

We can express the constants c_i's as a function of the independent variable x and $y(x,\,\mathbf{c}),\,y'(x,\,\mathbf{c}),\,\cdots,y^{(n-1)}(x,\,\mathbf{c})$ by implicit function theorem so that the constants c_i's are independent first integrals of (6) for $1 \le i \le n$. □

4.2. Advantage of passing from non-linear ordinary to linear partial

What is the advantage of passing from the non-linear ordinary differential equation (6) to the partial linear differential equation (7)? We explain that the partial linear equation (7) reveals the hidden symmetry of the algebraic differential equation (6) that is indispensable for construction of a Galois theory. To this end we begin with the following trivial fact. Let

$$G_1,\,G_2,\,\cdots,\,G_m$$

be first integrals of (6) holomorphic on a domain W of $\mathbb{C}^{(n+1)}$ and φ be a holomorphic function on a domain U of $\mathbb{C}^m$ containing

$$G_1(W) \times G_2(W) \times \cdots \times G_m(W) \subset \mathbb{C}^m$$

so that we can compose functions to get

$$\varphi(G_1,\,G_i,\,\cdots,\,G_m) \tag{10}$$

which is a holomorphic function on the domain $W \subset \mathbb{C}^{n+1}$. Then it follows from the definition that the composite holomorphic function (10) is a first integral. Let us apply this to a set of n-independent first integrals and coordinate transformation of n-variables. Let $\mathbf{F} := (F_1, F_2, \cdots, F_n)$ be independent first integrals such that every F_i is holomorphic on a domain W of $\mathbb{C}^{n+1}$ for $1 \le i \le n$. and let

$$\mathbf{u} := (u_1, u_2, \cdots, u_n) \mapsto \Phi(\mathbf{u}) = (\varphi_1(\mathbf{u}), \varphi_2(\mathbf{u}), \cdots, \varphi_n(\mathbf{u}))$$

be a coordinate transformation of n-variables. To be more precise

$$\Phi : U \to V$$

is a biholomorphic isomorphism of two non-empty open subsets $U,\,V$ of $\mathbb{C}^n$. If

$$\mathbf{F}(W) \subset U \subset \mathbb{C}^n,$$

then we can consider the composite function $\varphi_i(\mathbf{F})$, which are holomorphic on W, for every $1 \le i \le n$ so that

$$\Phi(\mathbf{F}) = (\varphi_1(\mathbf{F}), \varphi_2(\mathbf{F}), \cdots, \varphi_n(\mathbf{F}))$$

is an ordered set of independent first integrals holomorphic on W.

Let us set

$$S := \{(F_1, F_2, \cdots, F_n) |\, \text{The } F_i'\text{s are independent first integrals for } 1 \le i \le n\ \}$$

and denote the set of all the coordinate transformations of n-variables by Γ_n. Let us try not to be too nervous about the domains of definition because our aim is to understand the idea of mathematicians of the 19-th century. With respect to the composition of two coordinate transformations, Γ_n is not a group but almost a group. Indeed, we can not necessarily compose two local isomorphisms unless the second transformation is regular on the image of the first transformation. Γ_n is an example of Lie pseudo-groups. A similar problem arises if we say that the Lie pseudo-group Γ_n operates on S. Anyhow Γ_n is almost a group, a Lie pseudo-group and it almost operates, pseudo-operates, on S such that

$$(\Gamma_n, S)$$

is almost a principal homogeneous space. Now we are very close to the situations in the Galois theory of algebraic equations and Picard-Vessiot theory. We replace Lie pseudo-group by formal group for our personal taste (cf. Subsections 4.6, 4.7, ..., 4.9).

Let us choose a set of n-independent first integrals

$$\mathbf{F} = (F_1,\, F_2,\, \cdots,\, F_n).$$

If there are constraints among the partial derivatives

$$\partial^m F_i/\partial X^a \partial Y^b, \qquad a \in \mathbb{N},\, b \in \mathbb{N}^n,\, m = a + |b|,\, 1 \le i \le n,$$

they would determine Galois group of the algebraic differential equation (6) as a Lie pseudo-subgroup of the Lie pseudo-group Γ_n. *This is a beautiful idea of the mathematicians of the 19-th century!*

4.3. Criticism to the idea. — The idea explained above is remarkable. Yet there are problems if we examine it closely.

1. After R. Dedekind, the Galois group is not attached to an algebraic equation but to the field extension that the algebraic equation determines. We start from a base field K for the non-linear ordinary differential equation (6). In the transition from ordinary to partial, choice of the partial base field M is not clear.
2. Even if we can properly choose the partial base field M, then the partial differential field extension

   $$M < \mathbf{F} > /M$$

   depends on the choice of a set of n-independent first integrals

   $$\mathbf{F} = (F_1, F_2, \cdots, F_n).$$

   Here we denote by $M < \mathbf{F} >$ the partial differential field with derivations

   $$\{\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \cdots, \frac{\partial}{\partial Y^{(n-1)}}, \}$$

   generated by $F_1,\, F_2,\, \cdots,\, F_n$ over M.

Namely let $\mathbf{F}'$ be another set of n-independent first integrals. The general picture is that they are related through a coordinate transformation. There exists a coordinate transformation Φ of n-variables such that

$$\mathbf{F}' = \Phi(\mathbf{F}).$$

Since the transformation Φ is transcendental or it involves power series, the field extension $M < \mathbf{F} > /M$ is totally different from

$$M < \Phi(\mathbf{F}) >= M < \mathbf{F}' > /M.$$

Remark 4.4. — In one of the last versions of his book, Dirichlet's Vorlesungen über Zahlentheorie, R. Dedekind arrived at the distinguished idea of attaching the Galois group to the field extension that a given algebraic equation defines. Galois theory is rich and has many aspects so that there are other interpretations than working in the framework of field extensions. Our view point in [**14**] is a Galois theory of differential field extensions, whereas B. Malgrange [**11**] proposes a Galois theory of foliations.

4.4. How do we overcome these difficulties?— We were inspired by an idea of Vessiot in one of his last articles published in 1946. We considered algebraic differential equation (6). This means we are working with a differential equations over an algebraic variety. The space of initial conditions at $x = x_0$ is an algebraic variety X_0 and the differential equation describes a movement over an algebraic variety so that algebraic rational functions on the space X_0 of initial conditions are considered as natural first integrals. Let K be a base field. Let us treat the convergent case where we assume, as we did previously, that K is a differential field of meromorphic functions over a complex domain $U \subset \mathbb{C}$ so that $C_K = \mathbb{C}$. We work in a slightly more general situation than the algebraic differential equation (6). The most general setting is the following. Let L be a differential field extension of the base field K. We assume that L is of finite type over K as an abstract field extension so that

$$L = K(z_1, z_2, \cdots, z_m).$$

Hence, we have

$$z_i' = F_i(z_1, z_2, \cdots, z_m), \tag{11}$$

with

$$F_i(z_1, z_2, \cdots, z_m) \in K(z_1, z_2, \cdots, z_m)$$

for $1 \leq i \leq m$. By localization, we may assume that

$$F_i(z_1, z_2, \cdots, z_m) \in K[z_1, z_2, \cdots, z_m$$

for $1 \leq i \leq m$. Now, we consider a general solution

$$z_i(c_1, c_2, \cdots, c_m : x) \quad \text{for } 1 \leq i \leq m$$

of equation (11) depending on the parameters c_i associated with the initial conditions

$$z_i(\mathbf{c} : x_0) = c_i$$

at a general point x_0 fixed once for all. In particular, we have an isomorphism of differential fields

$$\begin{aligned} L &= K(z_1, z_2, \cdots, z_m) \\ &\simeq K(z_1(\mathbf{c} : x), z_2(\mathbf{c} : x), \cdots, z_m(\mathbf{c} : x)). \end{aligned}$$

We have to be careful. Since the field extension L/K was first given, the generators $z_1, z_2, \cdots, z_m$ of L over K are not always algebraically independent over K. Hence, we can not choose the constants c_i's arbitrarily for $1 \leq i \leq n$. We illustrate the idea mainly in the following particular case. Indeed, what is essential is involved in this particular case and understanding this particular case allows us to write down a general theory in the language of algebraic geometry. See Example 3 below.

Instructive Case (IC). — *We assume that the following conditions are satisfied.*

1. $K = \mathbb{C}(x)$.
2. $L = K(z_1, z_2, \cdots, z_n)$ *and the* z_i*'s are algebraically independent over* K *for* $1 \leq i \leq n$.
3. $z_i' = F_i(z_1, z_2, \cdots, z_n)$ *with* $F_i(z_1, z_2, \cdots, z_n) \in \mathbb{C}[x, z_1, z_2, \cdots, z_n]$ *for* $1 \leq i \leq n$.

Under these assumptions, the system of ordinary differential equation in condition 3 of (IC) describes a dynamical system on the affine space $\mathbb{A}^n$. We notice that the algebraic differential equation (6) is a particular instance satisfying these conditions if A is a polynomial in $\mathbb{C}[x, y, y', \cdots, y^{(n-1)}]$. Now, we consider the partial derivatives

$$\partial^m z_i(\mathbf{c} :\, x)/\partial x^j \partial \mathbf{c}^I, \qquad j \in \mathbb{N},\, I \in \mathbb{N}^n,\, m = j + |I|,\, 1 \leq i \leq n$$

with respect to the independent variable x and the initial conditions $c_1, c_2, \cdots, c_n$. Since we can eliminate the derivation $\partial/\partial x$ by virtue of the differential equation (11), we have to consider only the derivatives

$$\partial^{|I|} z_i(\mathbf{c}, x)/\partial \mathbf{c}^I, \qquad I \in \mathbb{N}^n,\, 1 \leq i \leq n \tag{12}$$

with respect to the initial condition $\mathbf{c}$. If there is no algebraic relations or if there is no constraints among the derivatives (12) with coefficients in the field $K(\mathbf{c})$ of rational functions, then the Galois group of the differential field extension

$$L/K$$

is the full Lie pseudo-group Γ_n of all the coordinate transformations of the space $\mathbb{A}^n$ of initial conditions. We are soon going to replace the Lie pseudo-group by an automorphism group. So let us set

$$\text{(13)}\quad G\text{-}Gal(L/K) := \{\text{Transformations } \mathbf{c} \mapsto \Phi(\mathbf{c}) \text{ leaving all the constraints invariant}\}.$$

Now, we can clarify the transition from non-linear ordinary to partial linear in terms of differential field extension. We start from the ordinary differential field extension L/K with derivation $d = d/dx$ and arrived at the partial differential field extension

$$K(\mathbf{c}, \partial^{|I|} z_i/\partial \mathbf{c}^I)_{I \in \mathbb{N}^n}/K(\mathbf{c}) \tag{14}$$

with derivations

$$d = \partial/\partial x, \text{ and } \partial/\partial c_i,\, 1 \leq i \leq n.$$

We shall see later that we have to replace more correctly the partial differential *field* extension (14) by a partial differential *algebra* extension (cf. Remark 4.8, (1)).

4.5. Examples

Example 1.— Let us take the simplest example of linear ordinary equations

$$z' = z \tag{15}$$

over the base field $K = \mathbb{C}(x)$ with derivation d/dx. So in terms of differential field extension, we consider a differential field extension $L = K(z)/K$ with $z' = z$, z being transcendental over K. So this is a particular example of Instructive Case (IC) of Subsection 4.4. The elements of K are meromorphic over $U = \mathbb{C}$ and we choose the reference point $x_0 = 0 \in \mathbb{C}$. Let now $z(c : x)$ be the solution of (15) with initial condition

$$z(c : x_0) = c, \tag{16}$$

where c is a parameter, so that

$$\partial z(c : x)/\partial x = z(c : x). \tag{17}$$

We can express concretely

$$z(c : x) = c \exp x. \tag{18}$$

and hence we have a constraint

$$c \partial z(c : x)/\partial c = z(c : x). \tag{19}$$

We notice here that we can obtain (19) without knowing the explicit form (18). In fact, taking the partial derivative with respect to c of (17), we get

$$\partial \left(\partial z(c : x)/\partial c\right)/\partial x = \partial z(c : x)/\partial c,$$

i.e., $\partial z(c:\, x)/\partial c$ also satisfies the differential equation (15). Since both $z(c:\, x)$ and $\partial z(c:\, x)/\partial c$ satisfy (15),

$$\partial \left(z(c:\, x) \left(\partial z(c:\, x)/\partial c\right)^{-1} \right) /\partial x = 0$$

or

$$z(c:\, x) \left(\partial z(c:\, x)/\partial c\right)^{-1}$$

is independent of x. So, there exists a function $\phi(c)$ of c such that

$$z(c:\, x) = \phi(c)\partial z(c:\, x)/\partial c. \tag{20}$$

Substituting x_0 for x and using (16), we get $\phi(c) = c$, hence (19) as promised. Now, the new base field is a partial differential field

$$(K(c), \{\partial/\partial x, \partial/\partial c\})$$

and we consider the partial differential field extension

$$K(c)(z(c:x))/K(c).$$

So, an element of the Galois group *G-Galois* (L/K) is a coordinate transformation

$$c \mapsto \varphi(c)$$

of the space $\mathbb{C}$ of initial condition leaving the left hand side of

$$z(c:\, x) \left(\partial z(c:x)/\partial c\right)^{-1} = c$$

invariant, c being an element of the partial base field $K(c)$. Namely,

$$\varphi'(c)^{-1}\varphi(c) = c$$

or

$$c\varphi'(c) = \varphi(c).$$

Consequently, $\varphi(c) = \lambda c$, λ being a non-zero complex number. This means that the coordinate transformation $c \mapsto \varphi(c)$ is $c \mapsto \lambda c$. Hence, it follows from (13) that the Galois group

$$\textit{G-Galois}\,(\mathbb{C}(x,\, z)/\mathbb{C}(x)) \qquad \text{with } z' = z$$

is $G_m = \mathbb{C}^*$. We have, moreover,

$$\textit{G-Galois}\,(\mathbb{C}(x,\, z)/\mathbb{C}(x)) \simeq \mathrm{Aut}(\mathbb{C}(x,\, c,\, z(c:x))/\mathbb{C}(x,\, c)) \simeq \mathrm{Aut}(\mathbb{C}(x,\, z)/\mathbb{C}(x)),$$

where the middle term is the group of $K(\mathbf{c})$-automorphisms of the partial differential field $K(c,\, z(c:x))$ with derivations $\partial/\partial x$, $\partial/\partial \mathbf{c}$.

Example 2. — The argument of Example 1 allows us to show that for a Picard-Vessiot extension L/K, Galois group G-Galois (L/K) coincides with the Galois group $G(L/K)$ of the Picard-Vessiot extension L/K. To be more precise, let K be a ordinary differential field of meromorphic functions over a domain U of $\mathbb{C}$ with $C_K = \mathbb{C}$. Given an $n \times n$ square matrix $A \in M_n(K)$, we consider a linear differential equation

$$Y' = AY. \tag{21}$$

Replacing the domain U by a subdomain if necessary, we may assume that we can find a solution $Y(x)$ of the linear differential equation (21) meromorphic over the domain U with $\det Y \neq 0$. Now, we choose a reference point $x_0 \in U$ and let $Y(\mathbf{c} : x)$ be a solution containing the full parameters taking an appropriate initial conditions at the reference point x_0. Then, the argument in Example 1 shows that there exists an $n \times n$ square matrix $C = (c_{ij})$ with $c_{ij} \in \mathbb{C}(\mathbf{c})$ for $1 \leq i, j \leq n$ and $\det C \neq 0$ such that

$$Y(\mathbf{c} : x) = Y(x)C. \tag{22}$$

It follows from the equality (22) that the partial differential field

$$K(\mathbf{c}) < Y(\mathbf{c} : x) >$$

with derivations $\partial/\partial x$, $\partial/\partial \mathbf{c}$ generated by $Y(\mathbf{c}, x)$ over $K(\mathbf{c})$ coincides with the field $K(\mathbf{c}, Y(x))$. In terms of differential field extension, we start from the ordinary differential field extension $K(Y(x))/K$, which is a Picard-Vessiot extension, and pass to the partial differential field extension

$$K(\mathbf{c}) < Y(\mathbf{c} : x) >= K(\mathbf{c}, Y(x))/K(\mathbf{c}) \text{ with derivations } \frac{\partial}{\partial x}, \frac{\partial}{\partial \mathbf{c}}. \tag{23}$$

So, it follows form (13) that

$$G\text{-Galois}\,(K(Y(x))/K)$$

consists of the transformations $\mathbf{c} \mapsto \varphi(\mathbf{c})$ of the space of initial conditions leaving all the constraints invariant. Now, the argument of the previous Example shows that the group G-Galois $(K(Y(x)/K))$ coincides with the automorphism group of the partial differential field extension (23) and consequently to the Galois group of the ordinary differential field extension $K(Y(x))/K$:

$$\begin{aligned} G\text{-Galois}\,(K(Y(x))/K) &\simeq \operatorname{Aut}(K(\mathbf{c}) < Y(\mathbf{c}, (x) > /K(\mathbf{c})) \\ &= \operatorname{Aut}(K(\mathbf{c}, Y(x))/K(\mathbf{c})) \simeq \operatorname{Aut}(K(Y(x))/K). \end{aligned}$$

Example 3. — Let us apply this idea to the first Painlevé equation. Let us take as the base field $\mathbb{C}(x)$ which we denote by K. Let us consider the first Painlevé equation

$$y'' = 6y^2 + x. \tag{24}$$

This means in terms of field extension that we consider a differential field extension $K(y, y')/K$ such that y, y' are transcendental over K and such that the derivatives of y and y' satisfy $d(y) = y'$ and $d(y') = 6y^2 + x$. So, this is a particular case of the Instructive Case (IC) of Subsection 4.4. We choose a reference point $x_0 \in \mathbb{C}$ and consider a solution $y(c_1, c_2 : x)$ of the first Painlevé equation (24) regular around x_0 with initial conditions

$$y(c_1, c_2 : x_0) = c_1, \qquad y'(c_1, c_2 : x_0) = c_2. \tag{25}$$

We show that the Jacobian

$$\frac{J(y(c_1, c_2 : x), y'(c_1, c_2 : x))}{J(c_1, c_2)} = 1. \tag{26}$$

In fact, denoting the left hand side of (26) by $F(\mathbf{c} : x)$, we have

$$\begin{aligned}
\frac{\partial F(\mathbf{c} : x)}{\partial x} &= \frac{\partial}{\partial x}\begin{vmatrix} \dfrac{\partial y(\mathbf{c} : x)}{\partial c_1} & \dfrac{\partial y(\mathbf{c} : x)}{\partial c_2} \\ \dfrac{\partial y'(\mathbf{c} : x)}{\partial c_1} & \dfrac{\partial y'(\mathbf{c} : x)}{\partial c_2} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial y'(\mathbf{c} : x)}{\partial c_1} & \dfrac{\partial y'(\mathbf{c} : x)}{\partial c_2} \\ \dfrac{\partial y'(\mathbf{c} : x)}{\partial c_1} & \dfrac{\partial y'(\mathbf{c} : x)}{\partial c_2} \end{vmatrix} + \begin{vmatrix} \dfrac{\partial y(\mathbf{c} : x)}{\partial c_1} & \dfrac{\partial y(\mathbf{c} : x)}{\partial c_2} \\ \dfrac{\partial y''(\mathbf{c} : x)}{\partial c_1} & \dfrac{\partial y''(\mathbf{c} : x)}{\partial c_2} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial y(\mathbf{c} : x)}{\partial c_1} & \dfrac{\partial y(\mathbf{c} : x)}{\partial c_2} \\ 12y(\mathbf{c} : x)\dfrac{\partial y(\mathbf{c} : x)}{\partial c_1} & 12y(\mathbf{c} : x)\dfrac{\partial y(\mathbf{c} : x)}{\partial c_2} \end{vmatrix} \\
&= 0.
\end{aligned}$$

So, $F(\mathbf{c} : x)$ is independent of x. It follows from (25) $F(\mathbf{c}, x) = F(\mathbf{c} : x_0) = 1$ proving (26). Hence, the Galois group G-Galois(L/K) is a Lie pseudo-subgroup of coordinate transformations

$$(c_1, c_2) \mapsto (\varphi_1(c_1, c_2), \varphi_2(c_1, c_2))$$

leaving the left hand side of (26) invariant. Namely

$$\frac{J(y(\varphi_1(c_1, c_2) : x), y'(\varphi_2(c_1, c_2) : x))}{J(c_1, c_2)} = \frac{J(y(c_1, c_2 : x), y'(c_1, c_2 : x)}{J(c_1, c_2)}. \tag{27}$$

Substituting x_0 for x in (27), we get

$$\frac{J(\varphi_1, \varphi_2)}{J(c_1, c_2)} = 1. \tag{28}$$

Conversely, if (28) is satisfied, since the both sides of (27) is independent of x, the condition (27) is equivalent to condition (28). So G-Galois$(K(y, y')/K)$ is a Lie pseudo-subgroup of the Lie pseudo-group consisting of all the transformations

$$(c_1, c_2) \mapsto (\varphi_1(c_1, c_2), \varphi_2(c_1, c_2))$$

satisfying (28) or leaving the area invariant (cf. Question 5).

4.6. Technical refinement. — We started from an ordinary differential field extension L/K and constructed a partial differential field extension (14). We call reader's attention to the fact that the partial differential field

$$K(\mathbf{c}, \partial^{|I|} z_i / \partial \mathbf{c}^I)_{I \in \mathbb{N}^n}$$

depends on the reference point $x = x_0$. Hence, we set

$$\mathcal{L}|[x_0] := K(\mathbf{c}, \partial^{|I|} z_i / \partial \mathbf{c}^I)_{I \in \mathbb{N}^n}$$

to show clearly its dependence on the reference point x_0. We remark here two points. First, Examples 1 and 2 show that in those cases the partial differential field $\mathcal{L}|[x_0]$ is independent of the reference point x_0. Second, in Examples 1 and 2, the Galois group G-Galois$\,(L/K)$ is the automorphism group of the partial differential field extension (14). In Example 3, however, besides the fact that it is not clear that the partial differential field $\mathcal{L}|[x_0]$ is independent of the reference point x_0, the Galois group is not the automorphism group of the partial differential field extension $\mathcal{L}|[x_0]/K(\mathbf{c})$ but it is a set of transformations leaving the area invariant. So, it is not a group but a Lie pseudo-group. What about considering the automorphism group of the partial differential field extension $\mathcal{L}|[x_0]/K(\mathbf{c})$ in general? It is not a bad idea but it means that since a differential field automorphism of $\mathcal{L}|[x_0]$ is given by a birational transformation $\mathbf{c} \mapsto \varphi(\mathbf{c})$ of the space of initial conditions, we look for algebraic transformations leaving the constraints invariant or satisfying a system of partial differential equations such as (28). In the case of Example 3, we have sufficiently many solutions of (28) in the birational transformation group of the plane, the Cremona group of 2-variables. In general, however, we do not always have sufficiently many algebraic solutions to the system of partial differential equations. In other words, the automorphism group $\mathrm{Aut}(\mathcal{L}|[x_0]/K(\mathbf{c}))$ of the partial differential field extension might be too small (cf. Remark 4.5 below). Hence, we can not limit ourselves to algebraic solutions but we have to look for analytic solutions of the system of partial differential equations of constraints. In the general case where the field of constants is not the complex number field $\mathbb{C}$, we can not speak of convergence so that we consider formal solutions to the system of partial differential equations or we consider the continuous differential automorphism group of a completion of $\mathcal{L}|[x_0]$ with respect to a certain topology.

Remark 4.5. — We examined the idea of considering a subgroup that is defined by a system of partial differential equations, of the birational automorphism group of the space of initial conditions. The birational automorphism group of an algebraic variety

V defined over $\mathbb{C}$, which is the $\mathbb{C}$-automorphism group of the function field $\mathbb{C}(V)$ is small. In fact, let C be a non-singular projective curve defined over $\mathbb{C}$ of genus g. We know

1. If $g = 0$, then $\mathrm{Aut}(\mathbb{C}(C)/\mathbb{C})$ is isomorphic to $PGL_2(\mathbb{C})$.

2. If $g = 1$, then $\mathrm{Aut}(\mathbb{C}(C)/\mathbb{C})$ is an algebraic group whose connected component of the unit element 1 is isomorphic to the elliptic curve C.

3. If $g \geq 2$, then $\mathrm{Aut}(\mathbb{C}(C)/\mathbb{C})$ is a finite group of order d, where
$$d = 84(g-1),\, 48(g-1),\, 40(g-1),\, 36(g-1), \cdots .$$

4.7. Infinitesimal automorphism group. — Now, we choose a point $\mathbf{c}_0$ in the space of initial conditions or we choose a particular value

$$\mathbf{c}_0 = (c_{0\,1},\, c_{0\,2},\, \cdots,\, c_{0\,n}) \in \mathbb{C}^n$$

of $\mathbf{c}$ and we expand analytic functions of x and $\mathbf{c}$ around $(\mathbf{c}_0,\, x_0)$ into power series with respect to local parameters

$$\underline{\mathbf{c}}_0 := \mathbf{c} - \mathbf{c}_0 = (c_1 - c_{0\,1},\, c_2 - c_{0\,2},\, \cdots,\, c_n - c_{0\,n}) \in \mathbb{C}^n, \qquad \underline{x}_0 := x - x_0.$$

In the sequel, when we consider the Taylor expansion of an analytic function at a point, we say that we Taylor expand the function at the point. If there is no danger of confusion, we omit suffix 0 and denote $\underline{\mathbf{c}}_0$ and $\underline{x}_0$ respectively by $\underline{\mathbf{c}}$ and $\underline{x}$. In particular the solution $z_i(\mathbf{c} : x)$ of the ordinary differential equation (11) that is regular at $(\mathbf{c}_0,\, x_0)$, is Taylor expanded into a power series of $\underline{\mathbf{c}}$, $\underline{x}$. We have so far realized the partial differential field $\mathcal{L}|[x_0]$ as a partial differential subfield of the field of of Laurent series:

$$\mathcal{L}|[x_0] \to \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!][\underline{\mathbf{c}}^{-1},\, \underline{x}^{-1}].$$

So, we may write $y(\mathbf{c},\, x) = y(\underline{\mathbf{c}},\, \underline{x})$. We denote the image of the partial differential field $\mathcal{L}|[x_0,\, c_0]$ in $\mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!][\underline{\mathbf{c}}^{-1},\, \underline{x}^{-1}]$ by $\mathcal{L}|[\mathbf{c}_0,\, x_0]$. We consider the completion $\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]$. of the partial differential field $\mathcal{L}|[\mathbf{c}_0,\, x_0]$ with respect to the $\underline{\mathbf{c}}$-adic topology. We can show that the completion $\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]$ coincides with the closure, with respect to the $\underline{\mathbf{c}}$-adic topology, of the field $\mathcal{L}|[x_0\, c_0]$ in the field

$$\mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!][\underline{\mathbf{c}}^{-1},\, \underline{x}^{-1}]$$

so that

$$\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0] \subset \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!][\underline{\mathbf{c}}^{-1},\, \underline{x}^{-1}].$$

We would define the Galois group of the ordinary differential field extension L/K by

$$G\text{-Galois}\,(L/K)[\mathbf{c}_0,\, x_0] := \mathrm{Aut}(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]/K(\mathbf{c})),$$

where Aut means the group of continuous $K(\mathbf{c})$-automorphisms of the partial differential field. We notice here that in the definition of the Galois group

$$G\text{-Galois}\,(L/K)[\mathbf{c}_0,\, x_0],$$

we may replace the base field $K(\mathbf{c}) = K(\underline{\mathbf{c}})$ by its completion $\widehat{K(\underline{\mathbf{c}})} = K[\![\underline{\mathbf{c}}]\!][\underline{\mathbf{c}}^{-1}]$ so that we have

$$G\text{-Galois}\,(L/K)[\mathbf{c},\, x] := \mathrm{Aut}(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})}).$$

Let

$$\Phi \in G\text{-Galois}\,(L/K)[\mathbf{c},\, x] := \mathrm{Aut}(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]/K(\mathbf{c})).$$

Identifying the solution $z_i(c:\, x) \in \mathcal{L}|[x_0]$ with its image $z(\underline{\mathbf{c}} : \underline{x})$ in

$$\mathcal{L}|[c_0, x_0] \subset \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!][\underline{\mathbf{c}}^{-1},\, \underline{x}^{-1}],$$

we may denote $z_i(\mathbf{c} : x)$ by $z(\underline{\mathbf{c}} : \underline{x})$. Since topologically and differentio-algebraically the topological partial differential field $\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]$ is generated over $K(\mathbf{c})$ by the $z_i(\underline{\mathbf{c}} : \underline{x})$'s for $1 \le i \le n$, the continuous automorphism Φ is determined by the images $\Phi(z_i(\underline{\mathbf{c}} :\, \underline{x}))$ that are elements of

$$\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0] \subset \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!][\underline{\mathbf{c}}^{-1},\, \underline{x}^{-1}].$$

Since the $z_i(\underline{\mathbf{c}} : \underline{x})$'s and $\Phi(z_i(\underline{\mathbf{c}} : \underline{x}))$'s, which are elements of the field

$$\mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!][\underline{\mathbf{c}}^{-1},\, \underline{x}^{-1}]$$

of Laurent series, are solutions of the ordinary differential equation (11), they would differ by the initial conditions. There would exist a formal coordinate transformation

$$\mathbf{c} \mapsto (\varphi_1(\mathbf{c}),\, \varphi_2(\mathbf{c}),\, \cdots,\, \varphi_n(\mathbf{c})) \tag{29}$$

such that

$$\Phi(z_i(\underline{\mathbf{c}} :\, \underline{x})) = z_i(\varphi(\underline{\mathbf{c}}) :\, \underline{x}) \qquad \text{for } 1 \le i \le n.$$

The transformation $\mathbf{c} \mapsto \varphi(\mathbf{c}) = (\varphi_1(\mathbf{c}),\, \varphi_2(\mathbf{c}),\, \cdots,\, \varphi_n(\mathbf{c}))$ should satisfy a system of partial differential equations so that

$$z_i(\underline{\mathbf{c}} :\, \underline{x}) \mapsto z_i(\varphi(\underline{\mathbf{c}}) : \underline{x}) \qquad (1 \le i \le n)$$

determines a continuous $K(\mathbf{c})$-automorphism of the partial differential field $\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]$. This intuitive argument is almost correct but not rigorous and we need a technical refinement. We regret that this procedure of justification makes the theory less accessible.

The above argument contains two problems. The first problem comes from the fact that our guess that the transformation (29) is regular or equivalently it is given by a set of formal power series is false. In fact they are formal Laurent series. So to have a correct picture, we must restrict ourselves to formal coordinate transformations. Hence, we set

$$\mathrm{Aut}_0(\hat{\mathcal{L}}|[\mathbf{c}_0, x_0]/\widehat{K(\underline{\mathbf{c}})}) := \{\Phi \in \mathrm{Aut}(\hat{\mathcal{L}}|[\mathbf{c}_0, x_0]/\widehat{K(\underline{\mathbf{c}})}) \mid \Phi \text{ is induced by a regular formal transformation (29)}\}.$$

To obtain more natural definition of $\mathrm{Aut}_0(\hat{\mathcal{L}}|[\mathbf{c}_0, x_0]/\widehat{K(\underline{\mathbf{c}})})$, we must replace the partial differential field extension by a partial differential algebra extension. See Remark 4.8.

To illustrate the second problem that we encounter, we consider the differential equation (11) for $n = 1$. Suppose that in the differential equation (11) we have no constraints. This happens in the most general case. Then the above argument gives us if it were correct,

$$\mathrm{Aut}_0\,(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})}) = \{\varphi \in \mathbb{C}[\![c]\!] | \varphi'(0) \neq 0\}$$

The left hand side is a group by composition of maps but the right hand side is not a group. In fact, let $\varphi(c)$ and $\psi(c)$ be two formal power series with coefficients in $\mathbb{C}$, then we can not always consider the composite $\varphi(\psi(c))$. If we calculate formally for two formal power series

$$\varphi(c) = \sum_{i=0}^{\infty} a_i c^i, \qquad \psi(c) = \sum_{i=0}^{\infty} b_i c^i \in \mathbb{C}[\![c]\!]$$

the composite, we get

$$\varphi(\psi(c)) = a_0 + a_1 b_0 + a_2 b_0^2 + \cdots (a_1 b_1 + 2a_2 b_0 b_1 + 3a_3 b_0^2 b_2 + \cdots)c + \cdots \tag{30}$$

that does not have any sense in the formal power series ring $\mathbb{C}[\![c]\!]$ in c with coefficients in $\mathbb{C}$. The error of the argument comes from the fact that in general $z(\varphi(\underline{c}) : \underline{x})$ does not belong to the field $\hat{\mathcal{L}}|[c_0,\, x_0]$. To remedy this, we consider only infinitesimal deformations of the identity automorphism of the partial differential algebra or in terms of coordinate transformations we consider only those coordinate transformations that are infinitesimally close to the identity.

For a commutative $\mathbb{C}$-algebra A, we denote by $N(A)$ the ideal of all nilpotent elements of A. Let

$$\varphi(c) = \sum_{i=0}^{\infty} a_i c^i, \qquad \psi(c) = \sum_{i=0}^{\infty} b_i c^i \in A[\![c]\!]$$

such that $\varphi(c)$ and $\psi(c)$ are congruent to the identity or to the power series c modulo $N(A)$. More concretely,

$$a_0,\, a_1 - 1,\, a_2,\, \cdots,\, b_0,\, b_1 - 1,\, b_2,\, \cdots \in N(A).$$

Then, the composition $\varphi(\psi(c))$ in (30) is a well-determined element of $A[\![c]\!]$.

4.8. Formal groups and group functors. — Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ be variables over a commutative ring R. We denote formal power series rings

$$R[\![x_1, x_2, \cdots, x_n]\!], \qquad R[\![x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n]\!]$$

respectively by $R[\![x]\!]$, $R[\![x, y]\!]$. A formal group of n-variables defined over R is an n-tuple

$$F(x, y) = (F_1(x, y), F_2(x, y), \cdots, F_n(x, y))$$

of formal power series $F_i(x, y) \in R[\![x, y]\!]$ of $2n$-variables for $1 \leq i \leq n$ satisfying the following conditions.

$F(x, 0) = F(0, x) = 0.$

$F(F(x, y), z) = F(x, F(y, z))$ for three sets of n-variables x, y, z.

For a formal group $F(x, y)$ of n-variables, there exists a unique n-tuple

$$\phi(x) = (\phi(x_1), \phi(x_2), \cdots, \phi(x_n))$$

of formal power series $\phi_i(x) \in R[\![x]\!]$ of n-variables for $1 \leq i \leq n$ such that $\phi(0) = 0$ and such that

$$F(x, \phi(x)) = F(\phi(x), x) = 0.$$

Here are examples of formal groups of 1-variable.

$$F(x, y) = x + y,$$
$$F(x, y) = x + y + xy.$$

More generally, let G be a complex Lie group. Writing the group law $G \times G \to G$ locally at the unit element 1, we get a formal group. The above examples are particular case of taking $G = \mathbb{C}, \mathbb{C}^*$.

Let $F = F(x, y)$ be a formal group of m-variables and $G = G(u, v)$ a formal group of n-variables both defined over R. A morphism $\varphi : F \to G$ of formal groups is an n-tuple

$$\varphi(x) = (\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x))$$

of formal power series $\varphi_i(x) \in R[\![x]\!]$ of m-variables such that $\varphi(0) = 0$ and such that

$$\varphi(F(x, y)) = G(\varphi(x), \varphi(y)).$$

There is an elegant way of associating a group functor to a formal group. Let F be a formal group of n-variables defined over R. We set

$$\mathbb{F}(A) = N(A)^n$$

and define a group structure on $\mathbb{F}(A)$ by putting

$$(a_1, a_2, \cdots, a_n) \cdot (b_1, b_2, \cdots, b_n) := F(a, b).$$

Since a_i's and b_i's are nilpotent elements of the commutative R-algebra A, $F(a, b)$ is a well determined element of $\mathbb{F}(A)$. This composition law defined on the set $\mathbb{F}(A)$ a group structure. Indeed, the composition law is associative by the second condition in the definition of formal group, 0 is the unit element and the inverse a^{-1} of an element a of $\mathbb{F}(A)$ is given by $\phi(a)$. We constructed a group functor $\mathbb{F}$ on the category (Alg/R) of commutative R-algebras.

$$\mathbb{F} : (Alg/R) \to (Grp) := \text{Category of groups.}$$

We can prove the following

Proposition 4.6. — *The functor associating to a formal group F the group functor $\mathbb{F}$ is fully faithful. Namely, for formal groups $F = F(x, y)$, $G(u, v)$ defined over R, we have*

$$\mathrm{Hom}_R(F, G) \simeq \mathrm{Hom}\,(\mathbb{F},\, \mathbb{G}),$$

where Hom *in the right hand side is the set of morphisms of group functors.*

4.9. Lie pseudo-group and Lie-Ritt functor. — Let

$$\varphi(x) = a_0 + (1 + a_1)x + a_2x^2 + \cdots, \qquad \psi(x) = b_0 + (1 + b_1)x + b_2x^2 + \cdots$$

be two formal power series in 1-variable x. Assuming that $a_1, a_2, \cdots, b_1, b_2, \cdots$ are variables, let us calculate the composite power series $\varphi(\psi(x))$ formally so that we get

$$\varphi(\psi(x)) = a_0 + b_0 + a_1b_0 + a_2b_0^2 \cdots + (1 + a_1 + b_1 + a_1b_1 + 2b_0(1 + b_1)b_2 + \cdots)x + \cdots.$$

Setting formally the composite

$$\varphi(\psi(x)) := H_0(a, b) + (1 + H_1(a, b))x + H_2(a, b)x^2 + \cdots,$$

we have

$$\begin{aligned} H_0(a, b) &= a_0 + b_0 + a_1b_0 + a_2b_0^2 \cdots \\ H_1(a, b) &= a_1 + b_1 + a_1b_1 + 2b_0(1 + b_1)b_2 + \cdots \\ &\cdots. \end{aligned}$$

We can prove easily

$$H_i(a, b) \in \mathbb{Z}[\![a, b]\!] = \mathbb{Z}[\![a\ 0, a_1, a_2, \cdots, b_0, b_1, b_2, \cdots]\!].$$

with no constant term, i.e., $H_1(0, 0) = 0$ for $i = 0, 1, 2, \cdots$. Upon writing

$$H(a, b) = (H_0(a, b), H_1(a, b), \cdots),$$

we have

$$H(H(a, b), c) = H(a, H(b, c)).$$

$c = (c_0, c_1, \cdots)$ being another of variables. So we can consider $H = H(a, b)$ as a formal group of infinite dimension defined over $\mathbb{Z}$ and a fortiori over $\mathbb{C}$. We denote this infinite dimensional formal group H by Γ_1. The suffix 1 means that we deal with transformations of 1-variable. We can associate a group functor

$$\mathbf{\Gamma}_1 : (Alg/\mathbb{Z}) \to (Grp)$$

to the formal group Γ_1. It follows from the definition of the associated group functor

$$\mathbf{\Gamma}_1\,(A) = \{\varphi(x) \in A[\![x]\!] \mid \varphi(x) \equiv x \text{ modulo the ideal } N(A) \text{ of nilpotent elements of } A\ \}.$$

Here, the group law is the composite of power series that are congruent to the identity modulo the ideal $N(A)$ of nilpotent elements. This is the group functor that we introduced in Subsection 4.7. So far, we studied the 1-variable case. We can treat the n-variable case similarly to get the infinite dimensional formal group $\Gamma_n\,(a,\, b)$ of n-variable transformations and the group functor $\mathbf{\Gamma}_n$ associated to it. We consider not only the group functor $\mathbf{\Gamma}_n$ but also subgroup functors of $\mathbf{\Gamma}_n$ defined by a system of partial differential equations. We call such group functors, or formal groups, Lie-Ritt functors. So we replace a Lie pseudo-group by a Lie-Ritt functor.

Proposition 4.7. — *We define a group functor*

$$\mathit{Inf\text{-}aut}\,(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})}) : (\mathit{Alg}/\mathbb{C}) \to (\mathit{Grp})$$

in the following manner. For a commutative $\mathbb{C}$-algebra A, we set

$$\begin{aligned}\mathit{Inf\text{-}aut}\,(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})})(A) = \{\Phi \in \mathrm{Aut}_0\,(\hat{\mathcal{L}}[\mathbf{c}_0,\, x_0]\hat{\otimes}_{\mathbb{C}[\![\underline{\mathbf{c}}]\!]}A[\![\underline{\mathbf{c}}]\!]/K(\underline{\mathbf{c}})\hat{\otimes}_{\mathbb{C}[\![\underline{\mathbf{c}}]\!]}A[\![\underline{\mathbf{c}}]\!]\\ \mid \textit{Continuous differential automorphism } \Phi \textit{ is induced}\\ \textit{by a formal power series } \varphi \in A[\![\underline{\mathbf{c}}]\!]\\ \textit{congruent to the identity automorphism modulo } N(A)\ \}.\end{aligned}$$

In other words,

$$\begin{aligned}&\mathit{Inf\text{-}aut}\,(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})})(A) =\\ &\{\varphi \in \mathbf{\Gamma}_n(A) \mid z_i(\underline{\mathbf{c}} : \underline{x}) \mapsto z_i(\varphi(\underline{\mathbf{c}}) : \underline{x})\,(1 \le i \le n) \ \ \textit{defines}\\ &\textit{a continuous differential algebra automorphism of}\\ &\hat{\mathcal{L}}[\mathbf{c}_0,\, x_0]\hat{\otimes}_{\mathbb{C}[\![\underline{\mathbf{c}}]\!]}A[\![\underline{\mathbf{c}}]\!]/K(\underline{\mathbf{c}})\hat{\otimes}_{\mathbb{C}[\![\underline{\mathbf{c}}]\!]}A[\![\underline{\mathbf{c}}]\!]\}\end{aligned}$$

Here, in the right hand side, the completion is taken with respect to the $\underline{\mathbf{c}}$-adic topology and Aut *denotes the group of continuous differential automophisms. Then, the group functor $\mathit{Inf\text{-}aut}\,(\hat{\mathcal{L}}[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})})$ is a Lie-Ritt functor.*

We denote the Lie-Ritt functor $\mathit{Inf\text{-}aut}\,(\hat{\mathcal{L}}[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})})$ by $\mathrm{Infgal}(L/K)[\mathbf{c}_0,\, x_0]$ and call it the infinitesimal Galois group of the differential field extension L/K with respect to the point $(\mathbf{c}_0,\, x_0)$. We explained how we replace a Lie pseudo-group by a formal group (of eventually infinite dimension) or by the Lie-Ritt functor that it defines.

Remarks 4.8

(1) In the definition of $\mathrm{Aut}_0(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0])$ and the Lie-Ritt functor

$$Inf\text{-}aut\,(\hat{\mathcal{L}}[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})}),$$

we restricted ourselves to the infinitesimal regular formal transformations, which does not seem natural. We can carry out this procedure more naturally if we use a differential subring, a model whose quotient field coincides with the given differential field. Let us illustrate this for Instructive Case (IC) of Subsection 4.4. In this case, we take $R = \mathbb{C}[x]$, $S = R[z_1,\, z_2,\, \cdots, z_n]$ so that they are closed under the derivation and their quotient field is respectively K and L. In other words, R and S are respectively a model of K and L. In place of the partial differential field extension (14), we define a partial differential subalgebra $\mathcal{S}$ of $\mathcal{L}$ by

$$\mathcal{S}|[x_0] := R[\mathbf{c},\, \partial^{|I|+l} z_i/\partial x^l \partial \mathbf{c}^I]_{l\in\mathbb{N},\, I\in\mathbb{N}^n,\, 1\le i\le n}$$

and we introduce a partial differential subalgebra $\mathcal{R} := R[\mathbf{c}]$ of $K[\mathbf{c}]$ so that we have a partial differential algebra extension $\mathcal{S}|[x_0]/\mathcal{R}$. Then, we Taylor expand them with respect to the local parameters $\underline{\mathbf{c}}$ so that we have a morphism

$$\mathcal{S}|[x_0] \to \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!].$$

We denote the image of $\mathcal{S}|[x_0]$ by $\mathcal{S}[c_0, x_0]$ so that $\mathcal{L}|[\mathbf{c}_0,\, x_0]$ is the quotient field of

$$\mathcal{S}|[\mathbf{c}_0,\, x_0] \subset \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!] \subset \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!][\underline{\mathbf{c}}^{-1}, \underline{x}^{-1}].$$

We introduce the $(\underline{\mathbf{c}})$-adic completion in $\mathcal{S}|[\mathbf{c}_0,\, x_0]$ as in $\mathcal{L}|[\mathbf{c}_0,\, x_0]$, the partial differential algebra extension $\mathcal{S}|[\mathbf{c}_0,\, x_0]/R[\underline{\mathbf{c}}]$ defines a Lie-Ritt functor

$$Inf\text{-}aut\,(\hat{\mathcal{S}}[\mathbf{c}_0, x_0]/\widehat{R[\underline{\mathbf{c}}]}).$$

Namely, for a commutative $\mathbb{C}$-algebra A, we set

$$\begin{aligned} Inf\text{-}aut\,(\hat{\mathcal{S}}[\mathbf{c}_0, x_0]/\widehat{R[\underline{\mathbf{c}}]})(A) := \{\Phi \in \mathrm{Aut}(\hat{\mathcal{S}}[\mathbf{c}_0,\, x_0]\hat{\otimes}_{\mathbb{C}[\![\underline{\mathbf{c}}]\!]} A[\![\underline{\mathbf{c}}]\!]/R[\underline{\mathbf{c}}]\hat{\otimes}_{\mathbb{C}[\![\underline{\mathbf{c}}]\!]} A[\![\underline{\mathbf{c}}]\!]) \\ \mid \Phi \text{ is congruent to the identity automorphism modulo } N(A)\ \}. \end{aligned}$$

Then, we can show that that the infinitesimal automorphism Φ in the right hand side is induced by a regular transformation so that we have

$$Inf\text{-}aut\,(\hat{\mathcal{L}}|[\mathbf{c}_0,\, x_0]/\widehat{K(\underline{\mathbf{c}})})(A) = Inf\text{-}aut\,(\hat{\mathcal{S}}[\mathbf{c}_0, x_0]/\widehat{R[\underline{\mathbf{c}}]})(A)$$

for $A \in Alg\,(\mathbb{C})$. Consequently, we have

$$Inf\text{-}aut\,(\widehat{\mathcal{S}}[\mathbf{c}_0, x_0]/K\hat{[\underline{\mathbf{c}}]}) = \mathrm{Infgal}(L/K)[\mathbf{c}_0, x_0].$$

(2) We worked over the ordinary differential field extension (11)

$$L = K(z_1,\, z_2, \cdots, z_n)/K$$

under the assumption that the z_i's are algebraically independent. Modifying the argument slightly, we can drop this assumption. So, we can attach a Lie-Ritt functor $\mathrm{Infgal}(L/K)[c_0,\, x_0]$ to a general ordinary differential field extension (11).

(3) An important property of the Galois group Infgal is that it is big enough. We can express this fact by saying that if an element of L which is a subset of $\mathcal{L}|[\mathbf{c}_0, x_0]$ on which the Galois group acts, is left invariant by the Galois group, then it is algebraic over the composite field $K(C_L)$ of the base field K and the field C_L of constants of L. In fact, a principal homogeneous space with group $\mathrm{Infgal}(L/K)[\mathbf{c}_0,\, x_0]$ is hidden. See Remarks 4.16.

4.10. Galois group at the generic point. — For an ordinary differential field extension L/K, we defined the Galois group $\mathrm{Infgal}(L/K)[\mathbf{c}_0,\, x_0]$, which is a Lie-Ritt functor over $\mathbb{C}$, in grosso modo an algebraic group over $\mathbb{C}$. The Lie-Ritt functor $\mathrm{Infgal}(L/K)[\mathbf{c}_0,\, x_0]$ depends on the chosen reference point $(\mathbf{c}_0,\, x_0)$ of the space of initial conditions. We can expect that it is independent of the point $(\mathbf{c}_0,\, x_0)$. See Questions 2 and 3.

Following the argument of Subsection 4.9 at the generic point of the space of initial conditions, we get the Galois group $\mathrm{Infgal}(L/K)$ that is a Lie-Ritt functor over the field $L^\natural$. Here $L^\natural$ is the underlying field structure of the differential field L. The Galois group is canonically constructed but it is defined over $L^\natural$.

In fact, in the definition of $\mathrm{Infgal}\,(L/K)[\mathbf{c}_0,\, x_0]$, we chose a point x_0 that is called a $\mathbb{C}$-valued point in the language of algebraic geometry and consider the Taylor expansion around the reference point $x_0 \in \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[x]$. Let us carry it out at the generic point. This is done by the universal Taylor expansion, which we are going to explain. In Example of §5, we show the procedure concretely for the Instructive Case (IC). Let $(R,\, d)$ be an ordinary differential algebra over $\mathbb{Q}$.

Definition 4.9. — *Let A be a commutative $\mathbb{Q}$-algebra. A Taylor morphism is a differential algebra morphism*

$$(R,\, d) \to (A[\![X]\!],\, d/dx).$$

When the differential ring $(R,\, d)$ is fixed, among the Taylor morphisms

$$(R,\, d) \to (A[\![X]\!],\, d/dx),$$

there exists the universal one. Namely we consider a map

$$i : R \to R^\natural[\![X]\!]$$

sending an element $a \in R$ to its formal Taylor expansion.

$$i(a) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n(a) X^n. \tag{31}$$

We can check that i is a ring morphism and compatible with derivations d, d/dX. So i is a Taylor morphism. The following Proposition is a consequence of the definition of the morphism i.

Proposition 4.10. — *The Taylor morphism*

$$i : R \to R^{\natural}[\![X]\!]$$

is universal among the Taylor morphisms. Namely for a commutative $\mathbb{Q}$*-algebra* A*, we have a bijection*

$$\mathrm{Hom}_{Diff\text{-}ring}(R,\, A[\![X]\!]) \simeq \mathrm{Hom}_{Ring}(R^{\natural},\, A).$$

Proof. — In fact, there exists a natural correspondence between the elements of the two sets. We denote the ring morphism $A[\![X]\!] \to A$, $g(X) \mapsto g(0)$ of taking the constant term by f_0. For a Taylor morphism $\varphi : R \to A[\![X]\!]$, we associate the ring morphism $f_0 \circ \varphi : R^{\natural} \to A$, which gives a map from the left hand side to the right hand side. Conversely, given a ring morphism $\varphi : R^{\natural} \to A$, then φ naturally induces a differential algebra morphism $\varphi_X : R^{\natural}[\![X]\!] \to A[\![X]\!]$ and hence a Taylor morphism $\varphi_X \circ i : R \to A[\![X]\!]$. □

To understand the universal Taylor morphism, let us take the differential field $L = \mathbb{C}(x,\, z)$ of Example 1 so that $d(z) = z$. Then, since $d^n(z) = z$ for $n = 0,\, 1\, 2,\, \cdots$, it follows from definition (31) of the universal Taylor morphism, the image

$$i(z) = \sum_{i=0}^{\infty} \frac{1}{n!} d^n(z) X^n = z \exp\, X.$$

So, the image of z is the general solution of the differential equation $z' = z$ containing initial condition z at $X = 0$.

Now, let us come back to the general setting. Let L/K be an ordinary differential field extension such that L is finitely generated over K as an abstract field. Let

$$z_1, z_2, \cdots, z_n \in L$$

be a set of generators of the abstract field $L^{\natural}$ over $K^{\natural}$. Let

$$i : L \to L^{\natural}[\![X]\!]$$

be the universal Taylor morphism. The image of an elements of L is Taylor expanded as in the convergent case. In particular the images $i(z_i)$'s of the generators z_i's are Taylor expanded. They contain parameters, initial conditions. We differentiate the generators $i(z_i)$ with respect to the initial conditions to generate a partial differential subalgebra. To this end let us take a transcendental basis

$$u_1, u_2, \cdots, u_l$$

of the abstract field extension $L^{\natural}/K^{\natural}$. The partial derivations

$$\partial/\partial u_i \in Der\,(K(z)^{\natural}/K^{\natural})$$

are uniquely extended to the derivations of $L^{\natural}/K^{\natural}$ which we denote by the same symbol $\partial/\partial u_i$ so that

$$\partial/\partial u_i \in Der\,(L^{\natural}/K^{\natural}).$$

As it is well-known, the derivations $\partial/\partial u_i$'s form a basis of the $L^\natural$-vector space $Der\,(L^\natural/K^\natural)$.

Definition 4.11. — *We denote the partial differential field*

$$(L^\natural,\, \partial/\partial u_1,\, \partial/\partial/\partial u_2, \cdots, \partial/\partial u_l)$$

by $L^\sharp$.

Now, we add in the power series ring $L^\natural[\![X]\!]$, the partial derivations $\partial/\partial u_i$'s operating on the coefficients of power series. In other words, we introduce $L^\sharp[\![X]\!]$ whose derivations are the derivations $\partial/\partial u_i$'s and d/dX for $1 \le i \le d$. We interpret the universal Taylor morphism i as $i : L \to L^\sharp[\![X]\!]$.

Definition 4.12. — *The partial differential algebra generated by* $i(L)$ *and* $L^\sharp$ *in* $L^\sharp[\![X]\!]$ *with derivations* $\partial/\partial u_i$*'s and* d/dX *will be denoted by* A_L. *We also introduce the partial differential algebra* A_K *generated by* $i(K)$ *and* $L^\sharp$ *in the partial differential algebra* $L^\sharp[\![X]\!]$.

Remark 4.13. — Since the partial derivations $\partial/\partial u_i$'s form a basis of the $L^\natural$-vector space $Der\,(L^\natural/K^\natural)$ and since we added $L^\sharp$ in the construction of A_L and A_K, the partial differential algebra A_L is independent of the choice of the transcendence basis $u_1,\, u_2 \cdots, u_l$.

We would like to make the parameters or the initial conditions explicit so that the generators $i(z_i)$'s are expressed as power series with respect to the parameters as in the local convergent case studied in Subsection 4.4. As in the ordinary case, we have the universal Taylor morphism for a partial differential $\mathbb{Q}$-algebra. Let $j : L^\sharp \to L^\natural[\![U_1, U_2, \cdots, U_l]\!]$ be the universal Taylor morphism for the partial differential field $L^\sharp$ so that we have

$$a \mapsto \sum_{m=(m_1,m_2,\cdots,m_l)\in\mathbb{N}^l} \frac{1}{m_1!m_2!\cdots m_l!}\,\frac{\partial^{|m|}a}{\partial u_1^{m_1}\partial u_2^{m_2}\cdots \partial u_l^{m_l}} U_1^{m_1},\, U_2^{m_2}\cdots U_l^{m_l} \tag{32}$$

for an element $a \in L^\sharp$. So the morphism j is compatible with two sets of derivations

$$\{\partial/\partial u_1,\, \partial/\partial u_2, \cdots, \partial/\partial u_l\} \text{ and } \{\partial/\partial U_1,\, \partial/\partial U_2, \cdots, \partial/\partial U_l\}.$$

Thanks to the universal Taylor morphism j, we Taylor expand the coefficients of power series to get a differential algebra morphism

$$L^\sharp[\![X]\!] \to L^\natural[\![U_1,\, U_2,\, \cdots, U_l]\!][\![X]\!].$$

Hence, we get partial differential algebras

$$A_K \subset A_L \subset L^\natural[\![U, X]\!]$$

with derivations

$$\{\partial/\partial U_1,\, \partial/\partial U_2, \cdots, \partial/\partial U_l,\, \partial/\partial X\}.$$

Now, we have arrived at our goal.

Definition 4.14. — *We denote the quotient field of A_K A_L respectively by $\mathcal{K}$, $\mathcal{L}$ that are partial differential subfields of $L^\natural[\![U, X]\!][U^{-1}, X^{-1}]$.*

Lemma 4.15. — *The partial differential subalgebras $\mathcal{K} \subset \mathcal{L}$ of $L^\natural[\![U, X]\!][U^{-1}, X^{-1}]$ are contained in a smaller differential subalgebra $L^\natural[\![U, X]\!][X^{-1}]$.*

Proof. — In fact, the differential subalgebras A_K and A_L are subalgebras of the field $L^\sharp[\![X]\!][X^{-1}]$ of Laurent series so that we can construct their quotient fields $\mathcal{K}'$, $\mathcal{L}'$ in the field $L^\sharp[\![X]\!][X^{-1}]$. Then, the images of $\mathcal{K}'$ and $\mathcal{L}'$ under the Taylor expansion morphism of coefficients

$$L^\sharp[\![X]\!][X^{-1}] \to L^\natural[\![U, X]\!][X^{-1}] \to L^\natural[\![U, X]\!][U^{-1}, X^{-1}]$$

are respectively $\mathcal{K}$ and $\mathcal{L}$. □

Thanks to Lemma 4.15, we have an inclusion

$$\mathcal{K} \subset \mathcal{L} \subset L^\natural[\![U, X]\!][X^{-1}].$$

The completions $\hat{\mathcal{K}}$, $\hat{\mathcal{L}}$ with respect to the (U)-adic topology coincide with their closure in $L^\natural[\![U, X]\!][X^{-1}]$ and consequently they define Lie-Ritt functor.

Using the partial differential field extension

$$\mathcal{K} \subset \mathcal{L} \subset L^\natural[\![U, X]\!][U^{-1}, W^{-1}],$$

we can argue as we did in Subsection 4.9 with partial differential subfields

$$\mathcal{K}|[\underline{\mathbf{c}}, \underline{x}] \subset \mathcal{L}|[\underline{\mathbf{c}}, \underline{x}] \subset \mathbb{C}[\![\underline{\mathbf{c}}, \underline{x}]\!][\underline{\mathbf{c}}^{-1}, \underline{x}^{-1}]$$

to get the infinitesimal Galois group Infgal(L/K), which is a Lie-Ritt functor defined over $L^\natural$. for the given ordinary differential field extension L/K.

Remarks 4.16

(1) For the Galois group Infgal(L/K), we can not expect Galois correspondence. Indeed, whereas Infgal(L/K) is in general infinite dimensional, the field L is finitely generated over K. We can show, however, that for a differential intermediate field $K \subset M \subset L$ we have a canonical surjective morphism

$$\text{Infgal}(L/K) \to \text{Infgal}(M/K),$$

which will play an important role for irreducibility questions (cf. Theorem (5.14), [**14**]). For other properties of Infgal, see Theorem (5.16), [**14**].

(2) A principal homogeneous space of the group functor Infgal(L/K) is hidden (cf. Theorem (5.10), [**14**]).

PART II
QUESTIONS

5. Fundamental questions on Galois theory

In Subsection 4.10, we defined Galois group $\mathrm{Infgal}(L/K)$ of which the construction is canonical, depending only on the given differential field extension L/K. The Lie-Ritt functor $\mathrm{Infgal}(L/K)$ is, however, defined over $L^\natural$ which is evidently too big.

***Question 1* (*).** — *Can we descend the Galois group* $\mathrm{Infgal}(L/K)$ *that is defined over* $L^\natural$*, to* C_K*?*

As we have no idea to answer this Question, we propose a remedy (cf. Questions 2 and 3). Let us assume that the field L is finitely generated over the field C_K of constants of K. Using the notation of Subsection 4.10, we can find subalgebras

$$R \subset K, \quad S \subset L$$

closed under the derivations

$$d,\, \partial/\partial u_i \in Der(L^\natural/K^\natural) \quad (1 \leq i \leq l)$$

such that $R \subset S$, the fields K and L are respectively the quotient field of R and S and such that the algebras $R^\natural$ and $S^\natural$ are of finite type over C_K. (Example below will help the reader to understand what we do.) We can apply the argument of Definition 4.12 and what follows in Subsection 4.10, where we introduced the partial differential algebras A_L and A_K, for the differential field extension L/K to the differential algebra extension S/R so that we get partial differential algebras A_R and A_S that are partial differential subalgebras of $S^\natural[\![U,\, X]\!]$ with derivations

$$\{\partial/\partial U_1,\, \partial/\partial U_2, \cdots, \partial/\partial U_l,\, \partial/\partial X\}.$$

Namely, the partial differential algebra generated by $i(S)$ and $S^\sharp$ in $S^\sharp[\![X]\!]$ with derivations $\partial/\partial u_i$'s and d/dX will be denoted by A_S. Similarly the partial differential algebra A_R is differentially generated by $i(R)$ and $S^\sharp$ in the partial differential algebra $S^\sharp[\![X]\!]$. Here $S^\sharp$ denotes the partial differential algebra

$$(\, S,\, \{\partial/\partial z_1, \partial/\partial z_2, \cdots, \partial/\partial z_l\}).$$

and $i : S \to S^\sharp[\![X]\!]$ is the universal Taylor morphism. Using the universal Taylor morphism

$$j : S^\sharp \to S^\natural[\![U_1,\, U_2,\, \cdots, U_l]\!],$$

we Taylor expand the coefficients of power series to get a differential algebra morphism

$$S^\sharp[\![X]\!] \to S^\natural[\![U_1,\, U_2,\, \cdots, U_d]\!][\![X]\!]$$

We identify the differential subalgebras $A_R,\, A_S \subset S^\sharp[\![X]\!]$ with their images in $S^\natural[\![U,\, X]\!]$ to get partial differential algebras

$$A_R \subset A_S \subset S^\natural[\![U,\, X]\!] \subset L^\natural[\![U,\, X]\!]$$

with derivations

$$\{\partial/\partial U_1,\, \partial/\partial U_2, \cdots, \partial/\partial U_l,\, \partial/\partial X\}.$$

We denote A_R and A_S respectively by $\mathcal{R}$ and $\mathcal{S}$. So we obtained a partial differential algebra extension

$$\mathcal{S}/\mathcal{R}$$

and Galois group

$$\mathrm{Infgal}(S/R) = \mathrm{Aut}(\mathcal{S}/\mathcal{R})$$

that is a Lie-Ritt functor defined over the ring $S^\natural$, as in Subsection 4.10. We have by Lemma (4.5), [**14**]

$$\mathrm{Infgal}(S/R) \otimes_{R^\natural} L^\natural \simeq \mathrm{Infgal}(L/K).$$

See also Remarks 4.8 (1). By Hilbert's Nullstellensatz the set of $\bar{C}_K$-valued points is dense in the algebraic variety $\mathrm{Spec}\, S^\natural$.

Question 2. — *Does there exist a non-empty Zariski open subset $W \subset \mathrm{Spec}\, S$ such that for every $\bar{C}_K$-valued point P of W, the reduction*

$$\mathrm{Infgal}(S/R) \otimes_{S^\natural} \bar{C}_L(P)$$

is independent of P?

We could answer it affirmatively by the following argument. First, reduce to the case $C_L = \mathbb{C}$ by Lefschetz' principle and use analytic continuation.

Example. — Let us understand concretely what happens by the Instructive Case (IC) of Subsection 4.4. The reader will realize that the argument above at the generic point is very close to that of Remarks 4.8 (1). In fact, since $K = \mathbb{C}(x)$ and since $L = K(z_1,\, z_2,\, \cdots, z_n)$, we can take $R = \mathbb{C}[x]$ and $S = R[z_1,\, z_2,\, \cdots, z_n]$, the derivations being d and $\partial/\partial z_i \in Der\,(L^\natural/K^\natural)$ for $1 \le i \le l = n$. Let us denote the image of the z_i's under the composite morphism

$$S \to S^\sharp[\![X]\!] \to S^\natural[\![U,\, X]\!]$$

of the universal Taylor morphisms by $Z_i(U,\, X)$ for $1 \le i \le n$. The z_i's being a solution of the system of ordinary differential equations of condition 3 in Instructive Case (IC), the power series $Z_i(U,\, X)$'s in the U_j's and X with $1 \le j \le n$, satisfy the system of ordinary differential equation of condition 3, (IC) with respective to the derivation $\partial/\partial X$. In other words, the $Z_i(U,\, X)$'s are a solution of the system of ordinary differential equations of condition 3, (IC) containing the parameters U_j's. Let us clarify their initial conditions. To this end, let

$$i : S \to S^\natural[\![X]\!]$$

be the universal Taylor morphism so that

$$i(z_i) = z_i + F_i(z)X + \cdots \in S^\natural[\![X]\!] \tag{33}$$

by (31) and the system of ordinary differential equations of condition 3, (IC). We notice also

$$i(x) = x + X \in S^\sharp[\![X]\!] \subset S^\natural[\![U,\, X]\!] \tag{34}$$

by (32). The equality (34) shows that the universal Taylor expansion (33) is the formal Taylor expansion of the analytic function z_i in x at the generic point x or at the reference point $x_0 = x$. Let

$$j : S^\sharp \to S^\natural[\![U]\!]$$

be the universal Taylor morphism so that

$$j(z_i) = z_i + U_i \in S^\natural[\![U]\!] \tag{35}$$

by (32). Then, it follows from the definition of Z_i, (33) and (35)

$$Z_i(U,\, X) = z_i + U_i + j(F_i(z))X + \cdots \in S^\natural[\![U,\, X]\!]. \tag{36}$$

So, the $Z_i(U,\, X)$'s are a solution of the system of ordinary differential equations of condition 3, (IC) with respect to the derivation $\partial/\partial X$ with initial conditions

$$Z_i(U,\, 0) = z_i + U_i \in S^\natural[\![U]\!].$$

Therefore,

$$\begin{aligned} \mathcal{R} &= \mathbb{C}[x][x+X][z_1+U_1,\, z_2+U_2, \cdots, z_n+U_n] \subset S^\natural[\![U,\, X]\!], \\ \mathcal{S} &= \mathbb{C}[x][x+X][z_1+U_1,\, z_2+U_2, \cdots, z_n+U_n][\partial^I Z_i(U,\, X)/\partial U^I]_{1\le i\le n,\, I\in\mathbb{N}^n} \\ &\qquad\qquad \subset S^\natural[\![U,\, X]\!]. \end{aligned}$$

Points

$$\mathbf{c}_0 = (c_{01},\, c_{02}, \cdots, c_{0n}) \in \mathbb{C}^n, \qquad x_0 \in \mathbb{C}$$

being given, we have an $\mathbb{C}$-algebra morphism

$$S^\natural = \mathbb{C}[x,\, z_1,\, z_2, \cdots, z_n]^\natural \to \mathbb{C} \tag{37}$$

that sends x to $x_0 \in \mathbb{C}$ and z_i to $c_{0i} \in \mathbb{C}$ for $1 \le i \le n$. The morphism (37) induces a morphism

$$S^\natural[\![U,\, X]\!] \to \mathbb{C}[\![U,\, X]\!] \simeq \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!]. \tag{38}$$

The latter isomorphism identifies U_i with $\underline{c_i} = c_i - c_{0i}$ for $1 \le i \le n$ and X with $\underline{x}$. The image of $Z_i(U,\, X)$ by the morphism (38) is nothing but

$$z_i(\underline{\mathbf{c}},\, \underline{x}) \in \mathcal{S}|[\mathbf{c}_0,\, x_0] \in \mathbb{C}[\![\underline{\mathbf{c}},\, \underline{x}]\!].$$

So

$$\begin{aligned} \mathcal{R} \otimes_{\mathbb{C}[x,\, z_1,\, z_2, \cdots, z_n]^\natural} \mathbb{C} &= \mathcal{R} \otimes_{S^\natural} \mathbb{C} \simeq \mathcal{R}|[\mathbf{c}_0], \\ \mathcal{S} \otimes_{\mathbb{C}[x,\, z_1,\, z_2, \cdots, z_n]^\natural} \mathbb{C} &= \mathcal{S} \otimes_{S^\natural} \mathbb{C} \simeq \mathcal{S}|[\mathbf{c}_0,\, x_0]. \end{aligned}$$

Consequently, we have

$$\mathrm{Infgal}(R/S)\otimes_{S^\natural}\mathbb{C}\simeq \mathrm{Infgal}(L/K)[\mathbf{c}_0,x_0].$$

The equality

$$\mathrm{Infgal}(R/S)\otimes_{S^\natural}L^\natural\simeq \mathrm{Infgal}(L/K)$$

holds too.

Question 3. — *If C_L is $\mathbb{C}$, then is there a canonical isomorphism*

$$\mathrm{Infgal}(S/R)\otimes_{S^\natural}\mathbb{C}\simeq \mathrm{Infgal}(L/K)[\mathbf{c}_0,\,x_0]?$$

Above, we have affirmatively answered Question 3 for the Instructive Case, where $P=(\mathbf{c}_0,\,x_0)\in \mathrm{Spec}\,S^\natural$.

It is very natural to ask how Malgrange's Galois theory [**11**] of foliations and ours of differential field extensions are related.

First of all, comparison requires assumptions under which both theories work. So, we propose to clarify how his idea and ours are related. Let L/K be an ordinary differential field extension such that the field $L^\natural$ is finitely generated over $K^\natural$. Then we have $\mathrm{Infgal}(L/K)[\mathbf{c}_0,\,x_0]$ as we introduced in Subsection 4.9. On Malgrange's theory side, we need an analytic space and a foliation on it. In his theory, a particular attention is paid to get not only a Lie algebra but also a global Lie pseudo-group. For a comparison with our theory, however, we need only Lie algebra. Hence, the question is local. To make explanation simple, let us limit ourselves to the Instructive Case (IC) of Subsection 4.4. We use the notation of the previous Example after Question 2. We have on the algebraic variety

$$\mathrm{Spec}\,S=\mathrm{Spec}\,\mathbb{C}[x]\times_{\mathbb{C}}\mathrm{Spec}\,\mathbb{C}[z_1,\,z_2,\,\cdots,z_n]\simeq \mathbb{A}^1\times\mathbb{A}^n$$

a foliation F defined by the system of ordinary differential equations of condition 3, (IC). Let Y be a ringed space whose underlying topological space is the space $\mathbb{C}\times\mathbb{C}^n$ with the usual topology and whose structure sheaf is the sheaf of rings of rational functions regular on a given open set. Let X be the similar ringed space constructed from $\mathrm{Spec}\,\mathbb{C}[x]$. So we have the projection morphism $p: Y=\mathbb{A}^1\times\mathbb{A}^n\to X=\mathbb{A}^1$ of ringed spaces. Let $(x_0,\mathbf{c}_0)$ be a point of $Y=\mathbb{C}\times\mathbb{C}^n$. We choose a neighborhood U of the point $(x_0,\,\mathbf{c}_0)$. So we have $p|U: U\to X$. The foliation F on Y induces a foliation $F|U$ on U. We can speak of the Lie groupoid which we shall here denote by $Mgal\,(L/K)$, associated with the foliation $F|U$ on the ringed space U. This is, by definition, the smallest Lie groupoid defined over the ringed space U whose Lie algebra contains the vector fields of leaves of the foliation $F|U$. Our question in a concrete form is

Question 4. — *Do we have an isomorphism of Lie algebras*

$$\mathrm{Lie}\,((Mgal\,(L/K)[\mathbf{c}_0,\,x_0])\simeq \mathrm{Lie}\,(\mathrm{Infgal}(L/K)[\mathbf{c}_0,\,x_0])?$$

Here Lie *means the Lie algebra.*

We indicate briefly the reason why we can expect this isomorphism. The projection $p : Y \to X$ defines a transversal structure in the sense of Malgrange, 5.2, [**11**]. In the definition of $\mathrm{Infgal}(S/R)[\mathbf{c}_0, x_0]$, we take all the algebraic relations among the partial derivatives

$$\partial z_i^{|I|}(\underline{\mathbf{c}}, \underline{x})/\partial \underline{\mathbf{c}}^I \text{ for } I \in \mathbb{N}^n$$

over $K[\mathbf{c}]$. So, they are the richest transversal structure defined over the ringed space X (cf. loc. cit.).

It is easy to formulate this question in a general differential field extension L/K.

Question 5. — *Behavior of* Infgal *under specialization.*

It would be sufficient to express logically the following fact. If we specialize an equation, we will have more constraints so that the Galois group would be smaller.

6. Questions related with an application of Galois theory

***Question 6* (*)**. — *Using the notation of Example 3, calculate the Galois group* $\mathrm{Infgal}(K(y, y')/K)$ *for a general solution y of the first Painlevé equation. We can ask the similar question for the other Painlevé equations.*

The Galois group $\mathrm{Infgal}(K(y, y')/K)$ is conjectured since almost 100 years [**4**]. Namely it is the Lie pseudo-group of transformations on the plane leaving area invariant.

$$u = (u_1, u_2) \mapsto (\varphi_1(u), \varphi_2(u))$$

with the Jacobian

$$J((\varphi_1(u), \varphi_2(u))/(u_1, u_2)) = 1.$$

We can formulate this in terms of Lie-Ritt functor without difficulty. (cf. Example 3, Subsection 4.5.) It does not seem easy to prove the conjecture. Maybe, it requires a new idea.

A paper of J. Drach written in 1914 is quite original. He asserts the equivalence of the following two conditions for a function $\lambda(t)$.

(i) $\lambda(t)$ satisfies the sixth Painlevé equation.

(ii) The dimension of the Galois group of the non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}$$

is finite.

In the second condition, the Galois group of general algebraic differential equation is involved. Namely the second condition depends on his infinite dimensional differential Galois theory, which has been an object of discussion since he proposed it in his thesis in 1898.

We proved that the first condition (i) implies the second (ii).

Theorem 6.1. — *Let $\lambda(t)$ be a function of t satisfying the sixth Painlevé equation. Let $K = \mathbb{C}(t, \lambda(t), \lambda'(t))$ which is a differential field with derivation d/dt. Let $L = K(y)$ be a differential field extension of K such that y is transcendental over K and such that y satisfies*

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}.$$

Then the Galois group $\mathrm{Infgal}(L/K)$ *is at most of dimension* 3.

Our proof depends on R. Fuchs' system. Looking at our proof, B. Malgrange showed it using the Jimbo-Miwa system [**8**](cf. [**16**]). .

Question 7 (*). — *To give a solution of P6 such that*

$$\dim \mathrm{Infgal}(L/K) \leq 2.$$

More generally, to classify such solutions.

Maybe,

$$\dim \mathrm{Infgal}(L/K) \leq 1$$

for Hitchin's algebraic solutions λ of P6 related with the dihedral groups [**7**]. It is natural to ask the following

Question 8. — *The notation being as in the previous question, is the extension L/K not embeddable in a strongly normal extension?*

Noumi and Yamada [**12**] introduced a new Lax pair associated with $\widehat{\mathfrak{so}}(8)$ that defines P6. This system seems more natural than Fuchs' or Jimbo-Miwa's [**8**]. Indeed, in the Noumi-Yamada system, all the Bäcklund transformations arise from gauge transformations. So, we expect an affirmative answer to the following

Question 9. — *Can we use the Noumi-Yamada system to prove Theorem 6.1 or to answer Question 7?*

We can expect it.

The Noumi-Yamada system describes a monodromy preserving deformation of a linear system that has an irregular singular point. So, the Galois group of the linear system remains invariant.

Question 10. — *What is the deformation invariant Galois group G of the Noumi-Yamada system? Is it small? In other words, is the Lie algebra* $\mathrm{Lie}\, G$ *isomorphic to a Lie subalgebra of sl_2?*

Question 11 (*). — *To develop the idea of Drach or to clarify what Drach meant by the converse of Theorem 6.1.*

G. Casale [**2**] pointed out that we could not expect the converse. He also proposes to prove that if $\mathrm{Infgal}(L/K)$ is finite dimensional, then λ has no movable singular point.

7. Question on infinite dimensional Galois theory of difference equation

It is a mixed theory in the following sense. We start from a difference equation and we get a Lie algebra or a formal group of infinite dimension in general. Or we start from what is discrete and get a continuous invariant. The idea is simple. In the definition of Infgal at the generic point in Subsection 4.10, we just replace the universal Taylor morphism by the universal interpolation morphism, which we will explain. Let us briefly sketch the idea.

A difference ring is a commutative ring R with operation of the additive group $\mathbb{Z}$ on the ring R. Let us denote the automorphism $R \to R$ sending an element $a \in R$ to $1 \cdot a \in R$ by φ. Since the automorphism $\varphi : R \to R$ determines the operation of the additive group $\mathbb{Z}$ on R, we denote the difference ring R with operation of $\mathbb{Z}$ by (R, φ). When there is no danger of confusion of the operation of $\mathbb{Z}$, we denote (R, φ) by R. When we emphasize that we consider the commutative ring R, we use the notation $R^{\natural}$. A morphism of difference rings is a morphism of rings compatible with the operations of $\mathbb{Z}$.

Definition 7.1. — *For a commutative ring A, we set*

$$F(\mathbb{Z}, A) := \{f : \mathbb{Z} \to A\}$$

that is the ring of A-valued functions on $\mathbb{Z}$.

The commutative ring $F(Z, A)$ has a natural difference ring structure. Namely, for a function $f(x) \in F(\mathbb{Z}, A)$, we define $(\varphi f)(x) = f(x + 1)$ for $x \in \mathbb{Z}$.

Definition 7.2. — *Let A be a commutative ring. We call a difference morphism $(R, \varphi) \to F(\mathbb{Z}, A)$ an interpolation morphism.*

Let (R, φ) be a difference ring. Then we have a canonical interpolation morphism $i : (R, \varphi) \to F(\mathbb{Z}, R^{\natural})$ sending an element $a \in R$ to the function $f(x)$ such that $f(x) = \varphi^x(a)$ for $x \in \mathbb{Z}$. We call the canonical morphism the universal interpolation morphism. A similar argument as for the universal Taylor morphism allows us to show the following assertion.

Lemma 7.3. — *The universal interpolation morphism is universal among the interpolation morphisms. In other words, for a commutative algebra A, we have a natural bijection*

$$\mathrm{Hom}_{\mathbb{Z}}((R, \varphi), F(\mathbb{Z}, A)) \simeq \mathrm{Hom}_{alg}(R^{\natural}, A).$$

Let now L/K be a difference field extension such that the field $L^\natural$ is finitely generated over $K^\natural$. We attach to the extension L/K a Galois group $\mathrm{Infgal}\, D\,(L/K)$. Let $i : L \to F(\mathbb{Z}, L^\natural)$ be the universal interpolation morphism. Let us denote by $L^\sharp$ the partial differential field $(L^\natural, \{d_1, d_2, \cdots, d_l\})$, where the d_i's $(1 \le i \le l)$ form a basis of the $L^\natural$-vector space $Der\,(L^\natural/K^\natural)$ of $K^\natural$-derivations of $L^\natural$. Now $\{d_1, d_2, \cdots, d_l\}$ operates on the values of functions, or we can consider $F(\mathbb{Z}, L^\sharp)$. Hence, we have now $i(L)$, $L^\sharp \subset F(\mathbb{Z}, L^\sharp)$. Here, we regard $L^\sharp$ as the set of constant functions on $\mathbb{Z}$. Let us set $DA_L :=$ the subalgebra of $F(\mathbb{Z}, L^\sharp)$ generated by $i(L)$ and $L^\sharp$ closed under the set $\{d_1, d_2, \cdots, d_l\}$ of derivations and $\mathbb{Z}$-difference operator φ of $F(\mathbb{Z}, L^\sharp)$. Similarly, $DA_K :=$ the subalgebra of $F(\mathbb{Z}, L^\sharp)$ generated by $i(K)$ and $L^\sharp$ closed under the set $\{d_1, d_2, \cdots, d_l\}$ of derivations and $\mathbb{Z}$-difference operator φ of $F(\mathbb{Z}, L^\sharp)$. We expand elements of $L^\sharp$ by the universal Taylor morphism $j : L^\sharp \to L^\natural[\![U]\!]$. So, we have

$$DA_K \subset DA_L \subset F(\mathbb{Z}, L^\sharp) \to F(\mathbb{Z}, L^\natural[\![U]\!])$$

We define the (U)-adic completions $\widehat{DA_K}$ and $\widehat{DA_L}$ respectively of DA_K and DA_L. So finally we can define the Lie-Ritt functor $\mathrm{Infgal}\, D\,(L/K)$ on the category $Alg\,(L^\natural)$ of $L^\natural$-algebras.

***Question 12* (*)**. — *Can we calculate* Infgal *for the discrete sixth Painlevé equation qP6 of Jimbo-Sakai* [**9**]*?*

The discrete Painlevé equation qP6 has the conventional sixth Painlevé equation as a continuous limit. In general, what is discrete is more difficult. Yet we might have a chance.

8. Arithmetic questions on Painlevé equations

Today, about 100 years after the discovery, no one can doubt that the Painlevé equations define special functions. Here is a list of reasons why they deserve to be so considered [**15**].

(i) They are irreducible to the classical functions.
(ii) They involve hypergeometric functions and their confluents.
(iii) They have combinatorial features. Particularly they are related with combinatorics of Young diagrams in substantial way.

We call reader's attention to the arithmetic nature of the Painlevé equations that is not yet widely recognized. We use the notation of Noumi and Yamada [**12**] for the sixth Painlevé equation. Let ε_1, ε_2, ε_3, ε_4 be variables over the ring $\mathbb{C}(x)$. We set

$$\alpha_0 = 1 - \varepsilon_1 - \varepsilon_2, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3,$$
$$\alpha_3 = \varepsilon_3 - \varepsilon_4, \quad \alpha_4 = 1 - \varepsilon_3 + \varepsilon_4.$$

The relation with the traditional notation is

$$\alpha_0 = \kappa_t, \qquad \alpha_1 = \kappa_\infty, \qquad \alpha_2 = \kappa_\rho, \qquad \alpha_3 = \kappa_1, \qquad \alpha_4 = \kappa_0.$$

Let p, q be variables over the ring $\mathbb{C}(x)[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]$ so that

$$\mathbb{C}(x)[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4][p, q]$$

is a polynomial ring with coefficients in $\mathbb{C}(x)[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]$. We extend the differential algebra structure

$$(\mathbb{C}(x)[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4], d/dx)$$

to the overring $\mathbb{C}(x)[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4][p, q]$ by

$$\left\{ \begin{aligned} \frac{dq}{dx} &= \frac{\partial H}{\partial p}, \\ \frac{dp}{dx} &= -\frac{\partial H}{\partial q}, \end{aligned} \right. \tag{39}$$

where

$$H := \frac{1}{x(x-1)} \left[p^2 q(q-1)(q-x) - p\left((\alpha_0 - 1)q(q-1) \right.\right.$$
$$\left.\left. + \alpha_3 q(q-x) + \alpha_4(q-1)(q-x)\right) + \alpha_2(\alpha_1+\alpha_2)(q-x) \right].$$

We know that the Hamiltonian system (39) is equivalent to the sixth Painlevé equation. So, we denote the Hamiltonian system (39) by P6$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$. When the variables $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ take particular values e_1, e_2, e_3, e_4 respectively, we denote the corresponding Hamiltonian system by P6(e_1, e_2, e_3, e_4).

***Question 13* (*)**. — *Is every rational or more generally algebraic solution* $(q(x), p(x))$ *of the sixth Painlevé equation P6*(e_1, e_2, e_3, e_4) *defined over the field* $\mathbb{Q}(e_1, e_2, e_3, e_4)$*?*

A priori, there is no reason why they are rational over the field $\mathbb{Q}(e_1, e_2, e_3, e_4)$. It seems, however, that no counter-example is known so far (cf. Boalch [**1**]). For a logical formulation of Question (13), see the argument below.

A more plausible and weaker setting is as follows. Let e_1, e_2, e_3, e_4 be complex numbers. Let $q(x)$, $p(x)$ be an algebraic solution of P6(e_1, e_2, e_3, e_4). Let R be the Riemann surface of the algebraic functions $q(x)$, $p(x)$ so that since the sixth Painlevé equation has no movable singular point, we have a covering structure $\pi : R \to \mathbf{P}^1_{\mathbb{C}}$ unramified over $\mathbf{P} \setminus \{0, 1, \infty\}$. The field of meromorphic functions $\mathbb{C}(R)$ has a differential field structure coming from the covering map π. Let us formulate the question rigorously. Ring theoretically we have a $\mathbb{C}(x)$-differential algebra morphism

$$f : \mathbb{C}(x)(e_1, e_2, e_3, e_4)[p, q] = \mathbb{C}(x)[p, q] \to \mathbb{C}(R) \tag{40}$$

sending $q \mapsto q(x)$, $p \mapsto p(x)$. By Belyi's theorem, the Riemann surface R is defined over $\bar{\mathbb{Q}}$. Namely, there exists a non-singular projective curve C over $\bar{\mathbb{Q}}$ and a $\bar{\mathbb{Q}}$-morphism $\psi : C \to \mathbf{P}^1_{\bar{\mathbb{Q}}}$ such that we have an isomorphism

$$R \simeq C \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$$

over $\mathbf{P}^1_{\mathbb{C}}$.

We ask

Question 14 (*). — *Does the differential ring morphism f in (40) descend over*

$$\bar{\mathbb{Q}}(e_1,\, e_2,\, e_3, e_4)?$$

Namely, does there exist a $\bar{\mathbb{Q}}(e_1,\, e_2,\, e_3, e_4)$-differential algebra morphism

$$f_0 : \bar{\mathbb{Q}}(x)(e_1,\, e_2,\, e_3, e_4)[P,\, Q] \to \bar{\mathbb{Q}}(e_1,\, e_2,\, e_3, e_4)(C)$$

such that

$$f = f_0 \otimes_{\bar{\mathbb{Q}}(e_1,\, e_2,\, e_3, e_4)} \mathbb{C}?$$

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Remarks added on October, 30th 2006

This article was written in 2004. Here are recent developments in this branch.

(i) As for Question 4, the author proved that his theory is equivalent to Malgrange's. The result will appear in a note in preparation.

(ii) With regard to Question 6, G. Casale succeeded in calculating the Galois group of the first Painlevé equation (G. Casale, Groupoïde de Galois de P1 et son irréductibilité, to appear in *Commentarii Mathematicii Helvetici*). He also determined the Galois group of the Picard solution of the sixth Painlevé equation. We know that in general, or to be more precise if it is not algebraic, the Picard solution is not classical. Yet its Galois group is finite dimensional after Casale. We can recognize this phenomenon only through general differential Galois theory, illustrating how useful the theory is.

(iii) The following paper of Casale replaces reference [**2**] of the original version of our article. G. Casale, A note on Drach's conjecture, in preparation, which we find as well as his other papers in his home page (`http://www.perso.univ-rennes1.fr/guy.casale/Article.htm`).

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